Multi-Criteria Decision Making and Applications
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Week 03
Lecture 13

A very good morning, good afternoon and good evening to all of you and all the participants and students for this course. The title of the course is multi criteria decision making, which is under the MOOC NPTEL course lectures and my good name is Raghunandan Sengupta from the IME department at IIT Kanpur. So, as I generally start the class with the broad outline what is the coverage and under which domain. So, actually we are covering the definitions, concept of utility theories, safety first principle, stochastic dominance and this is the thirteenth lecture, which is we are in the third week. The coverage would be as we are discussing the four different utility functions, the concept of quadratic, the concept of par utility function, the concept of logarithmic utility functions and the concept of exponential utility functions. We will cover them in details, then go into certainty equivalent what we mean by certainty equivalent, then go into geometric mean and this is the coverage we will do slowly do and obviously updation would happen accordingly.

So, first let us consider the case of quadratic utility function and the formula given is this. If you remember the quadratic utility or quadratic functions generally we can consider as $ax^2 + bx + c$. So, here considering the utility function as $W - \frac{1}{2}cW^2$. This half is just a constant in order to negate the two when you differentiate. So, if I find out U'(W), so this would be $1 - \frac{1}{2}c \times 2W$. So this is $(1 - cW)$ and if we consider U''(W) it comes out to be $-c$.

So, this is $1 - \frac{1}{2}c$ and U'(W) basically would become $-c$. Now, if I consider the concept of A(W), the formula for A(W) if I use A(W) was $-$ U''(W)/ U'(W), so minus sign remains U''(W) was c, so $- -$ becomes c and in the denominator you have $(1 - cW)$ and you need to differentiate to find out A'(W). So, A'(W) would be c is a constant, so it remains and this $(1 - cW)$ is in the denominator, so when you differentiate it, it will become $-(1 - cW)^2$ of a -2 obviously, because it will go into the denominator $\times$ $(1 - cW)$ that is differentiation again, so that will become $-c$, which is again c^2. So, if I see it becomes $c^2 / (1 - cW)^2$ and that is what the formula is. Now, here c is constant even if it is positive, so even if it is negative square of that will become the positive term, denominator is a square so is a positive term.

So, it can easily be deduced that A'(W) would basically be > 0 which is increasing absolute risk aversion property. Now, if I take the value of R(W), so I will erase all these things at least the green portion, this part also goes, so if I use a different color for the calculation, so we use a darker color. So, now R(W) is $-W \times$ U''(W)/ U'(W) which is $-W \times$ U''(W) is basically c, so $- -$ + c + $(1 - cW)$, I need to differentiate, so differentiation is basically two functions. So, it will be the second function $- \times$

the differentiation of first – first function × differentiation of second which will be $+c^2$ because − and ÷ by square of the second. So, if I expand it $-cWc^2W + c^2W$ − + + cancels, so this c here $(1 - cW)^2$, so this exactly matches the term which we have here.

So, this is $c > 0$ you will basically, I will erase this part $c > 0$ and the denominator $(1 - cW)^2 > 0$, so obviously you will have an increasing relative risk aversion property which $R'(W) > 0$. So, quadratic root functions with $c > 0$ has both increasing $A'(W)$ and increasing $R(W)$. So, how does the equations look like? So, I have tried drawing that using R- programming, so other diagrams also which you saw in the discussion all were build up on R- programming also. So, if I consider the quadratic root function with a value $(W - \frac{1}{2}cW)^2$ and c, I am taking as + 0.02. So, if I plot one by one U(W), next I plot A'(W) and third I plot R'(W). So, along the X-axis I have W as noted down here and along the Y-axis I have all these three values of U(W), A'(W), R'(W) drawn in the same graph scaling. So, that's why it will be difficult for us to understand now, but in the later graphs it will become clear. So, the utility function U(W) would be like this decreasing and A'(W) and R'(W) are shown here, they are not discernible, but they will become soon once we see them separately. So, if I draw them separately all these three diagrams they look like this.

Again wealth is plotted along the X-axis, utility only it is not the three things U(W), A'(W) and R'(W) are not being done together as in the last diagram, we are separately taking them. U(W) is plotted along the Y-axis and for the same quadratic utility function $W - \frac{1}{2}cW^2$ with the c value. Now, with the c value of 0.002 because in order because it is only $c \times 10^{-2}$ here. In the another case it was 0.02 now I shift two places is 0.02. So, that just a scaling factor, but the characteristics of graph remains same. So, if I plot and if I plot the diagram, so this is basically I use the color as given this is the quadratic utility function. Same type of characteristics of the last diagram here down the slope, this part, this is also same.

If I continue doing them separately for A'(W) and R'(W) the equations which we have derived. So, this is A'(W) here as shown this one is A'(W) for the quadratic utility function same thing and same value of c 0.002. The graph looks like this and the value of W along being plotted along the X-axis W(A'(W)) only not three together on the Y-axis. So, this green line which you see here is basically the red part here which is not discernible it looks like a straight line, but actually if you zoom in the graph would look like this for A'(W).

Finally, if I consider the R'(W), so dR(W)/dW is R'(W) for the same utility function with c value 0.002 the graph is also similar to A'(W). Similar in characteristics values may not be same and again we plot W along the X-axis, R'(W) along the Y-axis and this blue curve which you see here is the green one which almost look like a straight line. So, that is why I told that we are trying to draw them in the same graph paper and separately will give you lot of more information. Now, let us come to the case of exponential utility function.

The exponential utility function is given by minus e $^{-aW}$ and remember a > 0. So, if I want to derive I will use the red color of the first case U'(W) is equal to – – + e $^{-aW}$ and U''(W) is again – a × a^2 e $^{-aW}$. So, if I calculated A(W) which is
– U''(W)/ U'(W) would be equal to – – +. So, it becomes a^2 e $^{-aW}$/ U'(W), which is again + a e $^{-aW}$. So, e^{-aW} cancels from both numerator denominator. a cancels from numerator denominator the value is a.

So, when you differentiate A(W) capital A'(W), it becomes 0. So, hence it is constant absolute risk aversion property and as given A'(W) is 0. So, now let us come to the fact of R(W). So, I will remove this green portions the red portion which is U'(W) and U(W) continues to be there because we will be utilizing that. So, let us use the different color. So, R(W) is utilizing U'(W) and U(W) is –W U''(W) / U'(W) becomes –W U'(W) is
– – + a^2 e $^{-aW}$ / + ae $^{-aW}$. So, it becomes equal to – e $^{-aW}$ cancels from both numerator and denominator one value of a cancels from both nuAmerator and denominator of the value becomes aW. So, if I differentiate R(W) it becomes a only which is as given here which is increasing relative risk aversion property because R'(W) is equal to a constant. So, will be constant value. So, is it is constant value in the sense a > 0. So, it is increasing. So, for this case for exponential utility function U(A(W)) you have constant A(W) and increasing R(W). Now, let us do the same thing when we plot. When we plot the U'(W) curve is this U(W) curve sorry U(W) curve which utility is this and again the A'(W) and this is the U(W), A'(W) and R'(W) they look almost straight line they are not we will sAee that later W is given along the X-axis and all the three values U(W), A'(W), R(W) giving along the Y-axis. By the way for these graphs which I am drawing there is a reason for that because if you remember for quadratic utility function the values of R'(W) and A'(W) were increasing. So, if they are increasing obviously the graph should also replicate that which you saw we will again see that for all this utility function the second one being the exponential one.

You have A'(W) here almost a straight line and you have R'(W) also a straight line, but we will differentiate that and by the way this utility function is e $^{-aW}$ and the value chosen of a is + if you remember it was positive so + 0.01. Now let us draw the curves first is basically the utility function itself. So, again W along the X-axis only the utility along the Y-axis and if I plot this -e $^{-aW}$ with values of a is +1 +1 here only in order to this is a scaling factor so does not matter. So, the equation looks like this.

So, if I plot the values accordingly and what are the values so I am plotting A'(W) here for the exponential utility function which is – e $^{-aW}$ with the value of
a = + 0.1. We are plotting W along the X-axis and A'(W) along the Y-axis and the graph looks like this a straight line. If you remember and the straight line is with value of 0. So, if you remember we had derived as A'(W) as 0 and this is being very nicely replicated here that is why I told that once you solve if you draw the graph separately it become easier for us to understand.

For the next case if you remember R'(W) was a constant a was constant. So, let us see whether it is replicated. So, again this is R'(W) which is dR(W)/dW for the exponential utility function) – e $^{-aW}$ with the value of a as 0.1. We are plotting W along the X- axis and R(W) along the Y-axis and very interestingly this again a straight line, but this straight line is not 0 because if you remember the value of R'(W) was = a and what is a, 0.1 which is here. So, the value of A(W) which you see here and the value of a we started here actually corroborates the fact that the R'(W) calculations on the graphs are exactly the same. Let us go to the third example which is basically the case where we are discussing the power utility function. Power utility function is cW^{γ} and $c < 1$, $\gamma < 1$, this is the utility function. So, let us derive accordingly $U'(W) = c\,\gamma\,W^{\gamma-1}$ $U''(W) = c\,\gamma\,(\gamma-1)\,W^{\gamma-2}$ and remember $\gamma < 1$. So, let us derive A(W), A'(W) and R(W) R'(W).

So, we use a different color. So, now $A(W) = -U''(W)/U'(W)$, which is $-[c\,\gamma\,(\gamma-1)W^{\gamma-2}]/\,U'(W) = c\,\gamma\,W^{\gamma-1}$. So, c γ , c γ cancels – c remains, $-(\gamma-1)$ remains and here $W^{\gamma-2}$ $-(\gamma+1)$ basically will give you W^{-1}. So, if I differentiate, A'(W) becomes this – – becomes 1. So, this is $(\gamma-1)/W^2$ because it is W^{-2}. So, it comes down. So, this value exactly matches the decreasing absolute risk aversion property which is $A'(W) < 0$. So, why I am saying decreasing because W is wealth, wealth is positive even if it is consider any negative value square is positive. So, $1/W^2 > 0$, $\gamma < 1$. So, < 1 value – 1 would be < 0. So, it is basically $A'(W) < 0$. So, this is corroborated.

Now, let us erase it and do the problem for the case when we have the R(W) factor. So, let us again use a different color. So, consider R(W) is as per the formula $-(W/U''(W)) \times$ sorry, this is, let us not get confused, let us do the calculations correctly. So, it's $WU''(W)/\,U'(W)$. So, it will become it is – or this is – sign $-[W\,c\,\gamma\,W^{\gamma-1}]/\,c\,\gamma$, $\gamma-1$, did I make a mistake? Oh I should have taken the other way round.

So, $-[W\,c\,\gamma\,(\gamma-1)\,W^{-(\gamma-2)}]/\,U'(W) = c\,\gamma\,W^{\gamma-1}$. Now, let us see c γ, c γ cancels. So, you have and also very interestingly there is a W also. So, $W^1 \times W$ to the γ^{-2} in the numerator gives me $W^{\gamma-1}$. So, this is $W^{\gamma-1}$ on the denominator, also both of them cancel.

So, the end result is so basically it would be $(\gamma-1)$ with the – sign because this this this cancels these two cancels. So, it is $-(\gamma-1)$ minus. So, if I find out $R'(W) - 1$ is a constant. So, it is 0. Now, if you see here R'(W) is 0 which is constant R(W) value.

So, it corroborates the second factor also for the power utility function. Let us see the graphs. So, again all of them are three are plotted in the same graph first which is U(W), A'(W), R(W) along the Y-axis, W on the X-axis and the graphs are given accordingly. So, this is the power utility function for a fact of $cW^{-\gamma}$, with a c value of if you remember, < 0.9, $\gamma < 0.3$. So, W, U(W) is plotted. Then also is difficult to differentiate is basically A'(W) which seems like a straight line it would not we will check it later and again R'(W) which seems to be a straight line we will check it later. So, this is for R'(W), this is for A'(W) and we will corroborate that with the calculation just done. So, the first fact is only the utility function which is the power utility function $c\,W^{\gamma-1}$ with the c value of 0.9 γ of value 0.3 W being plotted along the X-axis only U(W) being plotted along the Y-axis and this is as the graph we have seen.

Now, come let us come to the other two derivatives A'(W) and R'(W). First A'(W). So, again W being plotted along the X-axis, A'(W) being plotted along the Y- axis and here you have dA(W)/dW which is A'(W) for the utility function which is cW^{γ} with c value 0.9 gamma 0.3 both are < 1 and if we plot the curve is this. So, you remember it was not 0 and let us recollect the value was $(\gamma - 1)/W^2$ as it is here and if you plot if you plot derived it was R(W) it was a different value we will see it later.

If I plot the value of R(W), R'(W), for the utility function cW^{γ} with c as 0.9 γ as 0.3 both < 1, W being plotted along the X-axis, R'(W) plotted along the Y- axis and interestingly this is a straight line and if you remember R'(W), was calculated because it was 0 why because R(W) value was $-(\gamma - 1)$, γ was constant. So, hence derivative is 0.

So, it is corroborates that. So, with this I will end this lecture and continue with the discussion with the last utility function discussion which will be the logarithmic one. Have a nice day and thank you very much for your patience. Thank you.