Multi-Criteria Decision Making and Applications
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Week 03
Lecture 12

Welcome back my dear friends and students, a very good morning, good afternoon, good evening to all of you who are taking this course titled multi criteria decision making under MOOC and NPTEL. And my good name is Raghunandan Sengupta from the IME department at IIT Kanpur in India. And the broad outline for few sets of lectures which we have been continuing is in the area of definitions of MCDM concepts, utility theory, safety first principle and stochastic dominance. And today we will start the 12th lecture for this total series of 60 lectures which is spread over 12 weeks, each week we have 5 lectures half an hour each. What will be the coverage for this lecture? Even though many points are written we will cover it as you know that as you proceed nothing is left only that the overall broad line is given and we slowly proceed. We will consider the properties of utility function, concept of risk aversion, risk neutrality, risk seeking properties, what is the concept of marginal utility, what is the concept of absolute risk aversion, relative risk aversion, we will consider that there are two different topics. We will consider examples of utility theories and so and so forth we will be covering.

So, obviously this slide which you have seen we will cover later on, but this is just given as a precursor. So considering the property where we left the 11th lecture which was the first lecture in the third week. So, we did discuss the concept of first the concept of non-satiation property one, property two was basically property related to a person being risk loving, person being risk indifferent, person being risk hater. And we did discuss in details with the fair gamble with the outcomes increasing for the output as well as for the sure event and there person can decide whether he or she will go for the gamble or go for the certainty event. And as we have seen as the question was asked and as the stakes increases people do change their decision from being a person who is willing to take the risk to case where he is indifferent, she is indifferent and then being a person who is avoiding the risk.

So considering those three properties risk loving, risk indifferent, risk hater we will consider these three things. Number one is basically the person who is risk averse, number two being the person who is risk neutral and the number three is basically the person who is risk seeking. And based on the second derivative because the first derivative give you the concept of non-satiation, first derivative means the derivative of the utility function. So considering the second derivative < 0, so you have I had used different colors in order to come make a note of that. So it is the second derivative being < 0 gives you the risk aversion property, the second derivative being = 0 gives you the risk neutral property and the second derivative being > 0 gives you the risk seeking property and we will see that diagrammatically also. These are basically, this is the second derivative for the third case > 0, this is the second derivative I am just repeating it please bear with me, second derivative for the second case risk neutral and this being the second derivative for the first case which is risk averse. So if we draw the diagram and we should note down what is there in the X-axis, X-axis is the wealth here and Y-axis is the utility. So if you have this the property related to where obviously very interestingly in all these three diagrams the green one, the blue one and the red one, the property of non-satiation which is more I give you more you want which is the first derivative being greater than 0 always holds for all these three. All these three means for the red one yes, I am putting the color tick mark accordingly for the blue one yes and for the green one yes. Now coming back to the concept of double derivative which will denote whether the person wants to take risk is indifferent or is avoiding the risk. So obviously if I see the second derivative for the red one, the second derivative which is increasing at an increasing rate all three are increasing, but the red one is increasing at an increasing rate which means the red one would be the case where it is > 0. So, this is a person who wants to take risk is basically a risk lover or risk seeking person. So this is for the red one.

If I see the blue one second derivative is increasing at a constant rate, so it is 0. So is basically a risk neutral, so I would not write the word risk, so but I will only write the concept of neutral person. And for the green case, it increases increasing at a decreasing rate < 0 is basically the person who hates risk. I am not using the word risk is risk hater, risk neutral blue one and risk lover. So using the diagrammatic representation of the utility functions you can classify the human being as loving risk, hating risk and being risk neutral.

The color schemes are given accordingly. So the green one is basically for this, the blue one is for this one and the red one is basically for this one. So, marginal rate is basically the rate of increase. So, if I consider the marginal utility functions based on the fact what we just discussed in the last slide of the diagram. Here you will have the marginal rate is basically marginal utility function looks like a concave function, which is a risk averse person, which means that it is increasing at a decreasing rate. For the case where the marginal utility function looks neither concave nor convex, which is risk neutral person, which is wants to take risk or avoid risk both are same. And for the person where the utility function looks convex he or she is a risk taker, because it is increasing at an increasing rate. So, here $U''(W) > 0$ for the third bullet point, for the second bullet point $U''(W) = 0$, the first bullet point is $U''(W) < 0$. Then if I consider the marginal utility rates, which I have discussed there it becomes more detailed explanation, what I have been talking about verbally.

So, for the case where you have it is increasing at a decreasing rate is a risk averse person, which is this is marginal rate. It is increasing at a constant rate, which is risk neutral person, which is double derivative = 0. And if it is increasing at an increasing rate is a risk seeker, where the double derivative > 0. So, the diagrams basically portray it, so if I draw it accordingly here is same thing. So, the red one would be increasing, the blue one is constant, the green one is decreasing here.

So, the risk averse person is increase is happening but it is increasing at a decreasing rate, so if I draw the tangent, so for this point A and for this point B. So, the line if the line moves like this, so from my side if I am looking moves like this and slowly it decreases. So, the rate of change if I have drawn the angle, so this if I draw A here A point is here and B point is say for example, here. So, if I draw this is angle θ_A, this is angle θ_B, so obviously $\theta_A > \theta_B$ and so on and so forth. So, it is slowly becomes horizontal, so this is a risk averse person and obviously this should be true and which I am pointing out sorry for my repetition, you draw the utility functions along the Y-axis and the wealth along the X-axis.

If I consider the risk neutral person, so increasing at a constant rate again the weights wealths are denoted along the X-axis, utility along the Y-axis, so here you will basically have $U''(W) = 0$, it is constant $U'(W)$ is constant. If I go for the third diagram which is risk seeking person, so here if I consider again point A point B, so if I am able to draw it this is A and consider this is B and this is B. So, if it is θ_B it is θ_A, $\theta_A < \theta_B$, because it is becoming vertical.

Again for convenience we have drawn W's along the X-axis and U(W) along the Y-axis, so this is risk seeking person. Now, continuing with the other important concepts, if I consider risk aversion, risk neutrality and risk seeking person with based on the fact that we have considered a fair gamble, the situation along with the concept of U"(W) the table is made accordingly.

So, if I consider a risk averse person, so sloping curve going down. There obviously he or she would reject the fair gamble and take the sure event. So, reject the fair gamble take the sure event and obviously U"(W) < 0, because the graph which you have is sloping downwards. If I consider the risk neutral person, he or she would be indifferent to the fair gamble with respect to the sure event, there in that case U"(W) = 0 and the graph is a straight line, graph consequence and then finding of the second derivative. And finally, for the case when he or she is a risk seeking person would select the fair gamble and avoid and not take the sure event, the overall graph of U"(W) would be > 0, which means the graph is increasing like this. Now, we will consider the third property, third property is known as the absolute risk aversion property, which is property 3 and for the utility function and it can be proved we are not going to the detail proof that the concept of absolute risk aversion and the corresponding A'(W) we will also see is given by this.

A'(W) , A is basically the ratio which again I am saying it can be proved of the negative value of the double derivative in the numerator and the one derivative which is non-satiation property in the denominator. So, obviously we know that U'(W) is always > 0 and this property of U"(W) can be < 0, can be > 0 and can be = 0. So, based on that you will basically have the A'(W) property also which is known as the absolute risk aversion. What is the concept of absolute risk aversion qualitatively we will come to that. So, it can be proved that for an decreasing absolute risk aversion which is for the case when you have A'(W), not A, A'(W), as which is dA/dw < 0. So, A'(W) being < 0 it is given as the decreasing absolute risk aversion. So, the person's absolute risk aversion property is decreasing. So, mark the words risk aversion property is decreasing in the absolute chance. If somebody wants to consider A'(W) as 0 which is here then there is a constant absolute risk aversion property. And for the third case when A'(W) > 0 you have the increasing absolute risk aversion property and mark these concept which I said absolute risk aversion property, absolute risk aversion property.

And they as you see can be in decreasing constant and increasing based on that you have A'(W) being < 0 = 0 and > 0. What does it mean? This is what the slide says here. So, if I have the decreasing absolute risk aversion property where the fact is A'(W) < 0 then it means that as wealth increases, wealth means W increases the amount held in risky assets also increases. So, it is words decreasing absolute risk aversion. So, the risk aversion property is decreasing that means I am willing to take the risk.

So, that is why I said mark the words. So, it increases the amount the person increases the amount in quantum terms held in risky assets. If it is constant where A'(W) = 0 then as wealth increases the amount held in risky asset remains same. And if A'(W) > 0 it means the person is increasing absolute risk aversion and as wealth increases the amount held in risky asset decreases. So, these were from the point of view of absolute risk aversion property.

So, mark this. Decreasing here A'(W) < 0, constant here A'(W) = 0, increasing here A'(W) > 0. So, decreasing, constant, increasing absolute risk aversion would have these implications. Main thing is mark what is written in the second column which is the definitions. Now, we will come to the fourth property which is basically related relative risk aversion property and it is concept relative sense that was the absolute sense is the relative sense. The formula is given by R(W) is now.

So, we already had $U'(W) > 0$. True minus sign is also there also in A, A(W) here also it is there W is positive remember that wealth. And if you pay attention here $U''(W)$ can be > 0, $U''(W)$ can be $= 0$ and $U''(W)$ can be < 0. So, based on that you can find out how R(W) which is relative risk aversion property would behave and how it would look like. So, just remember with respect to A and R only the factor W has been multiplied in there in case of R. Other concepts even the same the numerator is $U''(W)$ denominator is $U''(W)$ there is a minus sign everything.

Now, let us consider the relative risk aversion property in its more detailed sense in the same way as we did for the absolute risk aversion property. So, whatever I am discussing we will do that in examples. So, if I consider the decreasing relative risk and the relative sense that was in the absolute sense then obviously $R'(W)$ as given here is < 0. If I consider in the constant sense relative risk aversion property then $R'(W)$ is $= 0$ and in the last case if I consider in the increasing sense relative risk aversion property then $R'(W)$ is > 0. So, those were with respect to absolute sense now in there in the relative sense.

What do they mean again? In the sense when you have decreasing relative risk aversion property which means $R'(W) < 0$ then as wealth increases the percentage held in this case it increases that was in the quantum sense absolute sense here percentage sense. Similarly, if I have the idea of constant relative risk aversion $R'(W) = 0$ then as wealth increases, increase is there in all these two definitions of A and R. But in one case you will consider the increase decrease and constant being with respect to in the absolute sense here in R which is relative case it is in the percentage sense. So, continuing the second point so as wealth increases percentage held in this case remains the same and finally, as for the increasing relative risk aversion property which is $R'(W) > 0$ as wealth increases percentage held in risk assets decreases.

So, this was basically true. So, we will consider very simply the ideas of some utility functions details I will come continue in the next class. Some utility functions which are important is basically the quadratic utility function. So, generally we will consider the quadratic utility function. So, of the form let me use a different color actually it is of the form A is this quadratic equation we know $ax^2 + bx + c$. So, this is basically the quadratic form and you have the quadratic form as given.

So, constant term can be there need not be there, but in general it is given. So, you will basically have $W - \frac{1}{2} c W^2$ is a quadratic and remember c is a positive constant we will see that it will come out to be true. In the exponential sense it is basically an exponential function of the form shown here: $-e^{-aw}$, a is again a positive constant it will give you the exponential utility function and if you have the power utility function I will use the. So, it is $cW\gamma$, so obviously c and γ values as they are given both are < 1. And in many of the cases very simple examples the utility functions can also be denoted by cW^c. So, there is a same parameter. So, the power utility function and this the value of γ will dictate how the power utility function will look like because c is a constant it can be multiplied and the last one is basically the logarithmic utility function. Ln(W) which will be log. And remember the properties of which we will discuss if you remember the property of non-satiation, property of risk averse, risk neutral, risk loving those this property 1, property 2 and corresponding also the concept of A, $A'(W)$, R, $R'(W)$ which is absolute risk aversion and its derivative, relative risk aversion and its derivative. All these things would be considered for all these four different utility functions and some other cases also and there you can find out what would be basically the corresponding values of the constant. So, for the quadratic equations we are more interested to find out value of c then for exponential values of a power utility function value of c and γ and logarithmic considering there is no parameter we will be interested in general characteristics. So, what is important for all these four cases is I am just writing property 1 is important to be checked, property 2 consider risk, property 3 consider A'(W) and property 4 concept of R'(W).

So, A'(W) and R'(W) are coming out from A and R. With this I will end this 12th lecture and considering details the concept of these four properties and for this four utility function which are there and discuss them in further details. Thank you very much and have a nice day.