Multi-Criteria Decision Making and Applications
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Week 03
Lecture 11

A very good morning, good afternoon, good evening to all the participants and the students for this MOOC course titled multi-criteria decision making and my good name is Raghunan Sengupta from the IME department at IIT Kanpur in India. So as you know this course is for 60 lectures which is for 12 weeks and we will be starting the next lecture which is the eleventh one that means we are starting the third week and the broad outline for this set of lectures which are happening is about definition, concepts, utility theory, safety-first principle, stochastic dominance. So under that we are proceeding slowly. The coverage for this eleventh lecture which is the first lecture in the third week will be utility theory examples, some properties. So obviously what I mentioned we will try to cover whatever is not covered obviously that will be mentioned in the slides when they are uploaded which I have already did point it out in one of the classes earlier and there would be point wise if you go to the details risk aversion, neutrality seeking properties, risk seeking properties, marginal utility, absolute risk aversion, relative risk aversion and these examples for certainty equivalent, geometric mean methods, safety first principle, stochastic dominance, hyperbolic absolute risk aversion functions and others. So let us start with an example. So, consider a venture capitalist, so he wants he or she wants to invest and there are two possibilities in front of him or her.

So the first alternative is to buy government bonds, treasury bills which cost 6 lakh that is the cost. I have not come into the concept of utility yet. The second alternative has three possibilities as noted in point number 3, and the cost for these three alternatives are respectively 10 lakhs, 5 lakhs and 1 lakh and with their respective probability of 0.2, 0.4, 0.4 and as you do note, the sum of the probabilities which have already been mentioned, should be double checked, of 0.2, 0.4 + 0.4 comes out to be 1 as it should be.

Now consider for the venture capitalist, the utility function is $W^{½}$ that is the utility function's value how you define. Now if I consider the first alternative which is a sure event in the sense there is only government bond. So obviously there is no risk, which means probability is 1. So if I want to find out the expected value for that first option, first part which is 6 lakhs to the power half is the utility for investing 6 lakhs $\times$ 1 which is the probability, the value comes out to be 775. Now if I consider the second alternative, I am going to use a different color.

So the utility would be there are three alternatives under that. So the first one for investment is 10 lakhs, $10^{1/2}$ lakhs, gives you the utility and what is the probability, if you remember it is $\frac{1}{2}$. For the second outcome, for the second alternative it is 5 lakhs. $5^{1/2}$ lakhs is the utility × its probability and the last one is $1^{1/2}$lakh, which is the utility function × 0.4 which is the probability that value comes out to be 609.

Now if I consider and if I compare the value of 609 to 775 obviously the expected value for the government investment is much more than the second alternative so obviously the decision maker based on the expected value of the utility will choose the first alternative which is the government one and not the second one where there were three possible outcomes. So I have just found out the probabilities multiplied them with their utility function and found out the expected value. Now just another important fact which I will mention which you would already would have understood but I am still mentioning it. So, I am going to erase all the colors and not make it too cluttered. So consider now due to some reason the utility function was not $W^{1/2}$ but it was only W, only W so that can be utility function also. So in that case the expected value of the utility for the first alternative which is the government one would now be 6×10^5 which is 6 lakhs and obviously it is W so there would not be any power. 6 lakhs is the utility × 1 that would be given as the expected value for the first one.

I am denoting E1 as for the government one and for the second case when is the second alternative again the values are same: 10 lakhs, 5 lakhs, 1 lakhs and the corresponding probabilities are also same 0.4, 0.2, 0.4, 0.4. The expected value for the second alternative would be 10 lakhs × 0.2, the second arm would be 5 lakhs × 0.4 which is the probability and the third one arm is 1 lakh × 0.4. So find out the expected value for the first case which was given by E1. The second case alternative which was for three such decisions considered is E2. Find out the expected value for this and at the end of the day you basically compare these values: E1 with respect to E2 which one is more you take the decision accordingly. So obviously utility function changing may give you definitely different values of expected value and you can find out the ranks accordingly. Simple question is would the above problem give a different answer if we use an utility function of the form $W^{1/2}$ + c. So, this is a constant. So adding that constant would do what? Only this constant is outside. So obviously if you multiply by the probabilities on the left hand side and right hand side, they would not affect the overall decision because in that case what you would have is for the government one it will be $[W^{1/2} + c] \times 1$. So, this will be E1 and for the second case which is the venture capital is investing it will be W1 that is the values which are given as 10 lakhs, 5 lakhs, 1 lakh. So W1 is the 10 lakhs + c × 0.2 + W2 which is $[5^{1/2}$ lakh + c] × 0.4 and the last value is W3 which is $1^{1/2}$ lakh again. $\frac{1}{2}$ means because the utility function is not changing + c × 0.4 and if you notice the red portion which is for the case when you have the government one. So this is, you take out is c × 1 which is c and if I consider the values of c × 0.2, c × 0.4, c × 0.4 so the value comes out to be as usual, so c will cancel each other from the left hand side and the right hand side. Now consider another example in utility theory. Consider in a span of 6 days the prices of security fluctuates and a person makes his or her transaction only at the falling prices and we assume the utility to be logarithmic. So there is a utility function logarithmic of that form.

The day numbers are given in the first column, the prices are given in rupees, dollars, euros, yen whatever it is and when you convert using this logarithmic utility function so ln1000, log of which is 6.91, ln 975 is 6.88 so on and so forth. So, the third column gives the utility based on the logarithmic utility function and the number of outcomes is given. What does the number of outcomes mean? Actually the day 1, let us not be confused so what it actually means is this. A price of 1000 occurs 35 number of times in the total number of samples or the observation which is done. So if I count the total number of days so obviously it is 100. So out of this 100, 35 number of instances are there where the price was 1000, 20 number of instances were there out of the 100 where it was 975 and so on and so forth. So based on the number of outcomes I find out the probabilities. If you remember probability was N(W) number of times for that price ∑N(W). So, ∑N(W) is basically 100 so for the first case 35/100 gives you 0.35, 20/100 gives you 0.1 and similarly for the last one 15/100 gives you 0.15. So given the utility which is there in the third column and given the probability which is the last column I need to find out or we need to find out the expected value so here it is calculated. So 6.91 the first value $\times$ probability 0.35, 6.88 $\times$ 0.2 then the third one is 6.86 $\times$ 0.1 and I keep multiplying the third column respective values with the last column probability. The value comes out to be 6.91 which is the expected value of the utility based on the prices and where the utility function is logarithmic in nature. Now let us change the utility function but the scenario remains the same. Consider now the utility function is $p^{0.25}$. So, if I find out those utility those values would be given like this need to erase the colors in order to make things visible to all of you. So, the utility function is $p^{0.25}$ and the utilities are accordingly. So, for the first price 1000, I find out the utility which is $1000^{1/4}$. For the second one which is 975 is $975^{1/4}$, third one is $951^{1/4}$, fourth one is $1050^{1/4}$, then the second last one is $925^{1/4}$ and the last one is $1025^{1/4}$. So, this should come here and interestingly the outcomes are same. So we already know the outcomes are given. The corresponding probabilities are already known. Multiply them so if you see 0.35 for the first probability 0.2, second case 0.1, third case 0.15 for the fourth case, 0.05 for the fifth case and 0.15 for the last case when you multiply them the corresponding expected value comes out to be 5.62.

So, using that you will see that, based on the fact that, there are two different utility functions for the same scenario you will get two different expected values. Now we will consider some of the important properties of utility function. So, there would be four properties we will go slowly and discuss them one by one. The first property is known as the property of non-satiation that means people are not satisfied. The first restriction placed on the utility function is that is consistent with more being always being preferred than less. So more I give you, more you want and it is always increasing. So which may not be true in many of the practical examples but it is considered as one of the important properties for utility function. This means that between two certain investments which are there we will take the one which has the largest outcome among them. So, if I consider that obviously W plus 1 which is one unit more of wealth is more than W. So we will consider the utility function of W + 1 is > U(W) which when translates in simple concept of mathematics means the first derivative for the utility function is always positive and this will be utilized in many of the cases later on.

So first derivative of utility function is positive. The second property is that we will consider investors or decision makers perception of risk and based on this concept of risk which would come out from property 2, any decision-maker any investor can be classified any one of these three categories which is the first one, mean people are risk averse. I am putting one tick mark there. The second characteristics under property 2 would be people are risk neutral that means they are neither here nor there increasing risk, decreasing risk does not matter. The first one is that averse means I want to avoid risk and the third one where I am putting three tick marks, is the case where people want to face risk and we will see that how property 2 using simple mathematics can be formulated accordingly. So, for that let us consider a very simple example.

The example is like this. In case 1 consider on the left hand side of yours there is a table which is mark 1 on the right hand side for yours there is a table which is marked 2 and in table 1 there is a coin which is an unbiased coin with a head and a tail and probability is obviously for a head is half tail is half and the overall investment which you do is if a head comes technically whatever investment means what is the outcome you are going to get whatever is the input you are going to give is immaterial now for the time being. So consider that with probability $\frac{1}{2}$ you get 2, and for probability $\frac{1}{2}$ for the tail, you get 0 none and for the table 2, it is a coin which is both head or both tail that means sure. So, for that, with probability 1 you will get 1. So the left one which I am circling now on table 1 is known as a fair gamble in the sense is values is exactly equal to the decision making of not investing. Fair gamble also consider the probabilities are $\frac{1}{2}$ and $\frac{1}{2}$.

If I consider the expected value of the fair gamble it will what 2 is the value I am getting or somebody is getting multiply the probability $\frac{1}{2}$ + 0 when a tail comes × probability $\frac{1}{2}$ the value comes out of 1 and for the sure event which is on table 2 the value which person gets is 1 × the probability 1 and is 1. So, both the values are same. So now let us change the values for the same example setup remains same but the values are changed. Consider four instances of the problem first case. In the first case for which is there for the gamble it is now not 2 it is 2000 and for the one which you are not getting it is still 0. So, the corresponding probabilities would be what? It will be 2000 multiplied by $\frac{1}{2} + 0 \times \frac{1}{2}$ gives you a value 1000 and for the sure one so this was on table 1. This is in table 2, so I mark it as TII, TI table I table II and for table II, the investment is now the value which you get is 1000 probability remains 1. So, when you multiply 1000×1 the value is 1000 again. If I follow want to follow the concept of expected value they are same. Consider the second instance of the problem again the same setup but now the investment values have increased by 100 times so from instance 1. So it is now 2 lakhs for table I which is changed, 0 remains same probabilities are $\frac{1}{2}$ and $\frac{1}{2}$ as usual unbiased coin and for the table II the investment of the values now increases to 1 lakh.

So if I want to find out the expected value it is 2 lakh $\times \frac{1}{2} + 0 \times \frac{1}{2}$ gives you a value of 1 lakh and similarly 1 lakh for the table II. So instance 2 again same expected values are same. So if I want to denote so this is expected value for table I the thing which is there on table I and this is for expected value for table II. So here table II's expected value which was 1000 exactly matches table I's investment equal and for this second instance problem also expected value for table II is 1 lakh and for table I, the value is also 1 lakh. Now go to the third instance. So, if I find if I check the value, is now 20 lakhs, again value is 0 for the tail probabilities are $\frac{1}{2}$ and $\frac{1}{2}$ and the sure event is 10 lakhs, probability is 1, the expected value as usual comes out to be 1 lakh for table I, 1 lakh for table II. Table I, table II are the games of the fair gamble and the certainty won you take.

So, table I expected value table 2 expected value is same. Similarly, when I increase it to 2000 lakhs or 200 million for instance 4 for the gamble and another arm is 0 the probabilities are also $\frac{1}{2}$ and $\frac{1}{2}$ and for the sure event, which is table II, is 100 million or 1000 lakhs and the probability is 1, as I have been mentioning. So if you find out the expected value which is given here which is 200 million $\times \frac{1}{2} + 0$ into $\frac{1}{2}$ gives you the value of 100 million and for table II which is the sure one 100 million $\times$ 1 which is the probability becomes also the same value of 100 million.

Here also expected value for instance 4 is exactly equal to expected value of table II for the same instance. Now why did I do instances 1, 2, 3, 4? If I ask you the question that for instance I which one will you take? The sure one or the gamble maximum of us will take the gamble. Very few would go for the sure event and some may go either for the gamble or the sure event. Now as the stakes increase, 2000 becomes 2 lakhs then it is almost certain that the number of people who were willing to take the gamble would now start thinking and they may switch to the sure event. That means the number of people who are also undecided may slowly shift to the sure event.

So people are shifting that means people are shifting from the case who were risk seeking slowly to risk neutral and slowly to the risk averse. That means I am trying to avoid the gamble which is there on table I. Third instance the stakes have really increased. Now they are 20 lakhs and 0 for the gamble and 10 lakhs for the sure event. So now a whole lot of people will be shifting for the sure event and for the case when it instance 4 they would be almost none who are willing to take the gamble which means depending on the output human means will slowly keep changing their risk properties or risk taking properties. So depending on the instance or what is the outcome people may move from risk seeking to risk neutrality to risk hating or averse and there can be instance where people can move the other way round. So, these are four instance of the same problem gives the idea that depending on the outcome people may change. Maybe if I consider a person salaried person who is working and who has basically a salary of about 15 lakhs per year he or she may go to the case of about instance 2 but it would be a rarity or a rare example if a person of that salary 15 lakhs per year would go for instance 3 or instance 4. So depending on your age, depending on your income, depending on risk taking abilities, people may take different decisions for the same problem but the values being different.

So, what I mean by that is as follows. If I consider the case as U(I) being the case for the gamble, one arm so I will consider the gamble I will use the red colour. So this is one arm, this is second arm, the probability is P_1 here, probability is P_2 here. What is P_1, P_2 I will denote. The values which are being invested is I_1 gives you a utility of U (I_1) is the amount which is basically a certain value of for W. I_2 gives you U(II_2) and on the other side there is a sure event that means there is an investment of 2 consider I_2 probability is 1.

So if somebody wants to balance equivalence, so if I am willing to take the gamble which is red colour that means I am a risk seeking person as you see the greater than sign here. If I am neutral equality sign, so I am risk neutral person and for the case when my overall value for the expected case where for the sure event is more which is greater here because the right hand side is the sure event then I am a risk averse person. So depending on the examples and the three bullet points you can classify the human beings accordingly. This example which we considered about the risk averse, risk neutral and risk seeking persons can be just considering the sure event which is marked in blue and the red one which is the gamble and based on that human beings can be classified as I am told in risk averse, risk neutral and risk seeking person and we will continue discussing this utilizing property 2 to analyse how a human being or a decision maker will behave. Thank you.