Multi-Criteria Decision Making and Applications
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Week 02
Lecture 10

Welcome back my dear students and participants, a very good morning, good afternoon, good evening to all of you for this course titled multi criteria decision making and my good name is Raghunandan Sengupta from the IME department at IIT Kanpur and if you just recollect even though I have mentioned this time and again. This is a 12 week course which total consist of 60 lectures and each week has 5 lectures of half an hour each and after each week which is 5 lectures we have an assignment. So, we are nothing to do with the assignment I am just mentioning the coverage. This is the 10th lecture under the 60 lecture series and which is the last lecture in the second week. So, still the broad outline is we have covered definitions consist of utility theory and we did consider lot of theories, and there would be many concepts. So, I am just mentioning the broad one safety first principle stochastic dominance and coverage for this 10th one even though many points are written whatever is covered you would definitely understand that when you go through the video recording.

We will cover rational choice, properties of utility theory, the concepts I mentioned in the third point of risk aversion is neutrality and risk seeking property, marginal utility, absolute risk aversion, relative risk aversion. Even though many points are mentioned, let me see how the coverage proceeds and there are other concepts also I am mentioning that, but do not worry they would not be missed they would be covered as you proceed accordingly. This is the broad area and for each course depending on how the flow goes and how the explanation what I am doing as goes we may cover a much shorter portion, but the main idea is to convey the idea. Now, corresponding to the concept of utility theory if you remember main idea was to find out the expected value and the expected value formula which I am again writing.

So, expected value of the utility U(W) being the utility E expected value that is equal to summation of U(W) which is utility multiplied by the probability of utility and where the probability, if you remember, was given, I am using a different colour, was given by N(W), which is number of occasions where for wealth W, W can be W_1, W_2, W_3. So, W agrees divide with the total $\sum$ N(W) total outcomes. So, this is just a concept of simple relative frequency of probability and obviously W is continuous idea can be replaced. So, this summation would be replaced by integration sign between minimum to maximum limits based on the utility function which is also dependent on W. So, it would be given by, I am only using U, which is the utility function and multiplied by the corresponding function, f (U), U is now a random variable dU and this U value only U, U basically be coming up from W.

So, given W or wealth or some concept other ideas you will find out the probability of distribution of W then find it out for U and then do the calculations to find out the expected value of the utility. So, I am only using E, for the expected value of the utility. So, remember that in general utility functions cannot be negative, but many functions may give negative values, but utility functions in many of the decisions can be negative like if you are considering the total output and based on utility net worth of pollution for a factory it is negative effect is negative or any decision where the values are the known net value is negative it may be considered, but for our simple cases we will we try to consider the utility functions which is 0 or positive we will try to. For analysis to make the problem simple we may consider the values to be 0 as I said even though in actuality they are negative. So, the utility functions as I mentioned even if negative we will consider and replace them with 0. Now, let us consider a second example. We will have lot of examples in utility functions.

So, consider an example where a single individual is facing the same set of outcomes at any instant of time, but we try to analyze his or her expected value by using two different utility functions. So, the person is going to analyze the same situations with two different utility functions. What are the utility functions? I will mark the first one with red. So, which is linear one $W_1 + 1$. So, this concept of W_1 or W_2 which you also see on the slide are only the corresponding variable differentiation actually it is wealth and they will take the same set of wealth values, but they are just a way of differentiation in order to denote.

So, this is $W_1 + 1$ which is linear and with second one I will use a different color like blue this is a quadratic which is $W_2^2 + W_2$ and in this problem we are given the outcomes. Outcome values are given which is in the first column they start from 15, 20, 25, 10, 5, 25. Again this value if you see there are two values of 25 here. So, why did I write two values I could have clubbed the issue is that if you remember in one of the examples I did mention the wealth values can be different, but the outcomes are also tending to be the same depending on the utility function value. So, for the first utility which will continue using the red color so the wealth values are given 1.5 till 5 and the corresponding probability is based on the outcomes which is based on the first column they are given.

So, these are the probabilities, probabilities are given again I am marking 0.15 to 0.25 and the corresponding utility value for the first utility is given which is the third column which I am marking. So, how do I find out the utility 1.5 + 1 is 2.5, 2 +1 for the utility is 3, 2.5 +1 3.5, 3 + 1 4, 0.5 +1 1.5, 5 +1 6 and the corresponding values outcomes are found out. So, if I calculate the total value so outcome values are given in numbers so 15 + 25 is basically 40, 50, 60, 70, 75, 100 and based on that total numbers being 100 the corresponding values of the probabilities for both the cases outcomes are same: utility function 1 and utility function 2. So, mark with yellow 15 /100 is 0.15, 20 /100 is 0.2, 25/100 is 0.25, 10/100 is 0.1, 5/100 is 0.05, 25 /100 is 0.25. Now based on that I need to find out what is the expected value for the utility function 1. So, I will continue using the red color so I am using E only which is expected value and the red color would denote it is the utility function 1 which is linear.

So, it will be the values of utility multiplied by the probabilities which is $2.5 \times 0.15 +$

utility function value for the second is 3.0 × by 0.2 + 3.5 × 0.25 + 4.0 × probability of 0.1 + 1.5 × 0.05 final value being final which needs to be at 6 × 0.25. If you find the total value for the utility function comes out to be 3.825 which is given here. So, this is the total value for the utility function 1.5 × 0.25 ×0.25 ×0.25 ×0.25 × 0.25. Now if I consider the second utility function which I marked in blue which is quadratic.

So, I will erase this again I repeat the expected value for the utility function 1 would be multiplied by corresponding probabilities and utility which are given in the third column which is utility and the fourth column which is basically probability. Now if I go to the corresponding values for utility function 2. So, I am going to erase this. So, let this part left side be there. So, in order to have some semblance in the discussion. So, and if I consider based on utility function 2. So, values of the wealth are given as same wealth if you remember interesting I have taken it the same. So, the wealth values are given 1.5, 2, 2.5, 3, 3.5. So, fifth column which I am marking here exactly would match the second column wealth I am taking the same wealth and based on the fifth column utility of the wealth I find of the utility which is based on the quadratic function. So, if the utility function for 1.5 would be what 1.5 × 1.5 square it is 2.25 + 1.5 comes out to be 3.75. 2^2 is 4 + 2 is 6, $(2.5)^2$ + 2.5 is 8.75, 3^2 is 9, 9 + 3 is 12, $(0.5)^2$ is 0.25 + 0.5 is 0.75, 5^2 is 25, 25 + 5 is 30. So, that gives us the calculation for the utility and the probability is the same even though I mark it in yellow blue, but you should remember the probabilities are corresponding to the outcomes divided by the total values which is 100. So, 0.15, 0.2, 0.25, 0.1, 0.05, 0.25 remain the same because the number outcomes for both case 1, case 2 which is utility function 1 and utility function 2 remains the same.

Now, if I do the calculation which is I am using blue color to denote utility function 2 the corresponding expected value would be given by utilities 3.75 × 0.15 this is the first value + 6 × 0.20 + 8.75 × 0.25 + 12.00 × 0.1 + 0.75 × 0.05 I am just multiplying the corresponding values of the last 2 columns + 30 × 0.25 if I calculate the value comes up to be 12.69 which is here.

Now, why did I do all this calculation? Was it to compare the utility functions? No, it was to show that depending on the utility function for the same type of outcomes the values may be totally different. One was basically the linear part for which the value came out to be 3.825 which I have just circled and the value for using the second utility for the same outcome concept the value comes out to be 12.69. Now, consider a decision where there is a yes-no type of decisions. So, here there all the outcomes were possible for both the utility functions. Now, outcomes would be possible depending on yes and no for two different utilities or two different decisions. Let us consider this. Now, we have I will highlight the important fact we have two different utility functions used one at a time for two different decisions and the utilities are linear in this case and this is linear part is termed in such a way that you will find one interesting point and the second one is not a quadratic one, but not on non-linear one. So, first one is $W_1 - 5$ and second one is $2W_2 - (W_2)^{1.25}$. Now, this W_1 W_2 are just nomenclature of denoting, but the wealth remains the same. Now, let us again use the same different coloring concept to make things easy for us to understand. So, the outcomes are given here on the first column, but very interestingly decision A and decision B will have outcomes depending on which is possible which is not possible. So, I will use now different coloring scheme for all both different if you remember red and blue. So, for the case of decision A it is nothing to do with the utility is decision A. Decision A, yes when the outcome is 8, yes when the outcome is 8, yes when the outcome is 6, no no for 3 and 4, yes when the outcome is 9, no for 5. So, this I will mark with the red color which is corresponding with the first concept which I am using. And if I use the second concept for the utility function, again yes for 3, yes for 4, and yes for 5 other 3 which is 8, 6, 9 are no no. Now, the utility functions wealth are given wealth for both of them I will use the same color because wealth are same is 4, 5, 6, 7, 8, 9 and the utility function values based on this wealth for the first one are 0. Now, here is what is interesting which I mentioned.

So, $4 - 5$ is what? -1. So, for our simplicity we have considered minus values as 0 that is why it is the first value which I am just circling is 0 actually it is -1. Second value is $5-5$ 0, $6 - 5$ 1, $5 - 5$ 0, $6 - 5$ 1, $7 - 5$ 2, $8 - 5$ 3, $9 - 5$ 4. And if I consider the corresponding utility for the second utility function which is 2 into W 2, W 2 would values would be the second column 4, 5, 6, 7, 8, 9. So, the utility function is $2W_2 - (W_2)^{1.25}$ and I have put this values of 4, 5, 6, 7, 8, 9 correspondingly the values come out to be 2.34, 2.52, 2.6, 2.61, 2.54, 2.41. Now, I have to find out the utilities for case A and B based on utility function 1 and 2. So, now they are different combinations case A decision decision B, utility function 1, utility function 2, 1 being linear and another being non-linear. Now, let us put that in calculation. So, now the calculations are given in details you will understand. For utility function 1 utilizing that in decision A and decision B, this is the answer.

So, decision for utility function if I see for utility function 1, utility function for the utility function 1. So, the cases would be utility function 1 for decision 1 would be 0 is basically the corresponding utility multiplied by the corresponding probabilities. Similarly, the probabilities would be for the case where there are yes. So, yes is 8 and how many yeses are there 8, 6, 9. For the second case when I have a yes I will highlight this is 2. So, 2 outcome is 6 and how many yeses are there 8, 6, 9. For the third case I am still following utility function 1 for decision A. The utility is 3 for which outcome 9 and how many such yeses are there 8, 6, 9. So, now I multiply the corresponding utilities with this corresponding ratios. First ratio being $8 \div 8, 6, 9$. Second being $6 \div 8, 6, 9$, $8 + 6 + 9$. Third one being ratio being $9 \div 8 + 6 + 9$. Corresponding utilities being which are circled here if you can see the slide 0 to 3. So, let us see. So, utility function 0 multiplied by the corresponding ratio of the outcomes which is $8 \div 8 + 6 + 9$. Second utility as I mentioned value was $2 \div$ (the number outcomes for that) which is $6 \div 8 + 6 + 9$. Third utility is 3 again $\times$ the corresponding probability. The value comes out to be 1.69. Now, if I consider the decision B for utility function 1. So, I am using a different color if you see the blue one. So, just for case this blue color even though I denoted for utility function 2 will come back because there are too many coloring scheme it may confuse. So, I will use blue for the case B utility function 1. So, this is a yes here. So, I will use 0, utility function 1. So, this is true 0 output is true for the utility for utility function 1, 0 is true and 4 is true because if you see the yeses. And what are the corresponding relative frequency of the probabilities? So, the values for 0 is 3 total number of outcomes is $3 + 4 + 5$. So, $3 \div 3 + 4 + 5$. Second case when it is output utility is 1 it is $4 \div 3 + 4 + 5$. And the last case when the utility function is 4 the relative frequency for the outcomes is $5 \div 3 + 4 + 5$.

So, let us see if it comes out as mentioned. Yes it is. So, the first probability utility

0 × (the corresponding ratio of the relative frequency of the outcome). Second utility 1 multiplied by the relative frequency of the outcome probability outcome. Third value is 4 utility × (the corresponding relative frequency of the outcome). If you use that the value comes out to be 2. So, under the idea of utilizing the same utility for two different decisions again we are seeing they are different 1.69, 2.0. Now, let us switch our concentration on the case of utility function 2 for case A, case B. So, I will erase the color red and blue. So, again I use the same color, but I have erased that. So, that slate is clean for no confusion to occur. Case 1 again a decision A red color decision B blue color, but now it will function 2. Utility function 2 being the utilities are given based on the calculation which you have already done. So, for decision B and decision corresponding utilities are given and similarly for decision A, A is yes yes yes.

So, the corresponding utilities are 2.34, 2.61 and 2.54 and the corresponding ratio to find out the outcomes for the first case would be I am writing in the other example when we are discussing utility function 1 with A and B I just verbally told it. So, it will be 8÷8 +6 + 9 this would be 6, ÷ 8 + 6 + 9 the last one would be 9 ÷ 8 + 6 +9. So, these two would be multiplied and these two would be multiplied these two would be multiplied to find out the expected value of based on utility function 1 for decision A. So, let us see whether they come out. So, here you see 2.34 multiplied by the corresponding probability of outcome 2.61, 2.54 multiplied by the corresponding probability. So, the value comes out to be 2.5. Now, let us go back and find utilize the decision B for utility function 2. So, here again the coloring scheme remains same blue. So, it is yes here, yes here, yes here the corresponding utility functions are 2.52, 2.6, 2.41 and the ratios of the outcomes for decision B are given by 3 ÷3 + 4 + 5. The second case being 4 ÷3 + 4 + 5 and the third case being 5 ÷ 3 +4 + 5. So, these two get multiplied, these two get multiplied and these two get multiplied. So, to find out the overall expected value of the utility for 2 based on this case B.

So, here are the values 2.452 multiplied by the corresponding value of the ratio of the outcomes 2.6 multiplied by this. My mistake I think I should have changed it, my mistake my mistake there is. So, this is 4 and this is 5. So, let me there is an error in the red color also I will mark it in error in typing when expression was going fine. So, 2.41 into this, so the value comes out to 2.5. So, the errors which need to be marked here are this is 6, not 8 and this is 9 my apologies. Here it is fine 8, 6, 9, 3, 4, 5 it is 8, 6, 9 here also 3, 4, 5. So, based on that we calculate. So, for utility 1, 1.692 for case A, B and it is same for interestingly for case A and B for utility 2 with different outcomes the value comes out to be 2.5 in both the cases. That means, depending initially we saw for utility functions being different outcomes for the same outcomes. Now, with the same utility, but with the different sets of outcomes being managed you can have the same expected value. So, these combinations would be coming up later on also and more discussions will be done accordingly. Thank you very much and have a nice day.