Similitude And Approximations In Engineering,

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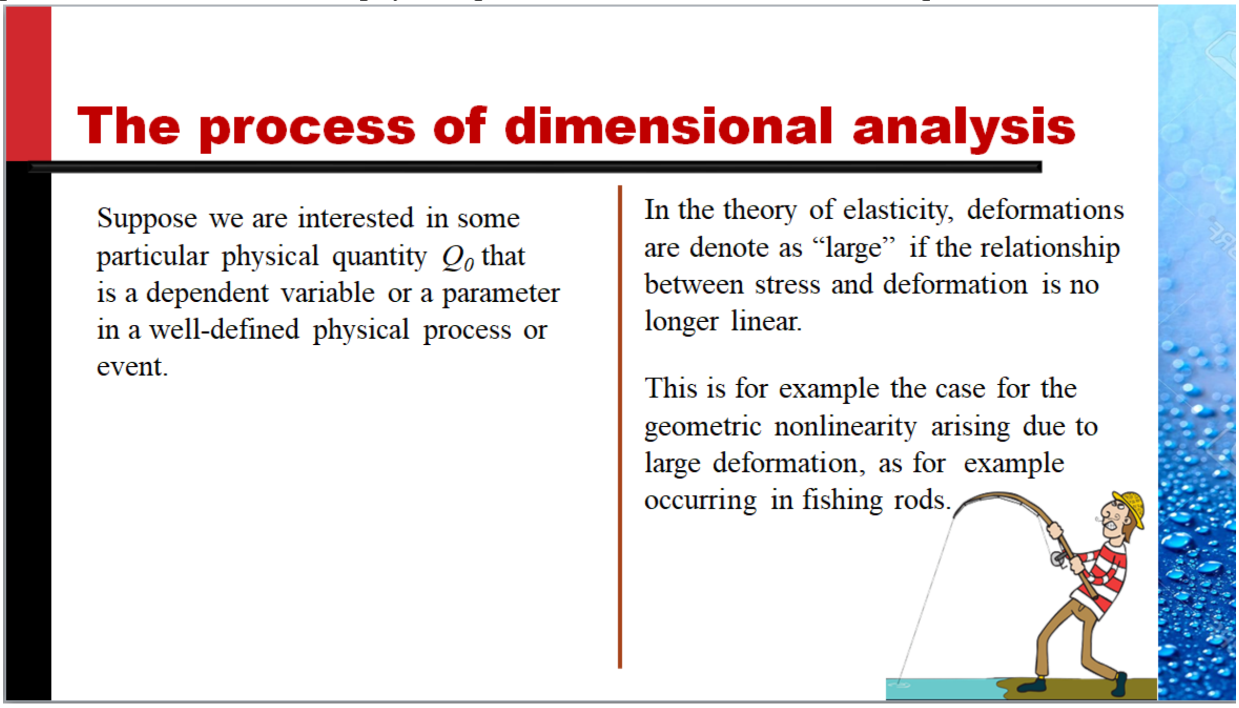
Week - 02
Lecture – 09

Welcome back. In today's lecture, we will learn the classical dimensional analysis as a technique of obtaining similarity rules. The premise of dimensional analysis is that the form of any physically significant equation must be such that the relationship between the actual physical quantities remains valid independent of the magnitude of the base units. So if the relationship between the actual physical quantities remain valid independent of the magnitude of base units that means the equation must be dimensionally homogeneous. Suppose we are interested in some particular physical quantity Q_0 that is a dependent variable or a dependent parameter in a well-defined physical process or event. I will do an example.



In the theory of elasticity, deformations are denoted as large if the relationship between stress and deformation is no longer linear. In your first course in strength of materials or properties of materials, you had done relations where the equation between stress and deformation is linear, that is valid only for what are termed as small deformations. So let us do an example where deformations are large. This is for example the case for the geometric non-linearity arising due to large deformation as for example occurring in a fishing rod.

The first and most important step in dimensional analysis is to identify a complete set of independent quantities Q_1, Q_2, ... , Q_n that determine the value of Q_0. So that we could write Q_0 as a function of Q_1, Q_2, etc. In the example we seek the deformation δ of the elastic rod under the influence of single force P applied at the end of the rod and forcing of it under its own weight. The general relationship between the deformation delta and the length L and the cross section area A of the rod, the force P that we apply and the modulus elasticity E, the density ρ of the rod material, and the gravitational acceleration g, is then of the form $\delta = f(P,\ L, A,\ \rho,\ g,\ and\ E)$. Thus, the value of n here is 6.

There are 6 independent parameters that are there in the problem. We have to determine the dependent parameter δ, the deflection at the end of the rod. The next step is to choose a system units MLT, FLT or even FMLT, as we discussed in the last class. Let us choose the MLT system. Write down the dimensions of all the variables and parameters dependent and independent in terms of the dimension of the base quantities.

Variable/ parameter	**M**	**L**	**T**
δ	0	1	0
P	1	1	−2
A	0	2	0
ρ	1	−3	0
L	0	1	0
g	0	1	−2
E	1	−1	−2

So, we have this table here. The first column lists the variables and parameters in this case only the parameters and then the dimension of mass, the dimension of length and the dimension of time MLT. δ, the deflection obviously has only the dimension of length. P the force is mass times acceleration. So, it is the dimension of $M\,L\,T^{-2}$.

Similarly, the dimension of the other quantities. E, the elasticity is like stress per unit strain. So, it has a dimension of stress which is force divided by area. So, the dimensions are $M\,L^{-1}\,T^{-2}$. We now pick from the complete set of physically independent variables and parameters. In our case only the parameters $Q_1, Q_2, \ldots, Q_n$.

A complete and dimensionally independent subset $Q_1, Q_2, \ldots, Q_k$ with k less than n. So, it is subset of Q_1 to Q_N, k less than equal to n. Dimensionally independent, what does that mean? A subset $Q_1, Q_2, \ldots, Q_k$ of the set Q_1 to Q_n is dimensionally independent. A none of the member has a dimension that can be expressed in terms of dimension of the remaining members. We will explain this with example.

Complete: the subset is complete if the dimension of all the remaining quantities Q_{k+1}, Q_{k+2}, Q_n, that is beyond Q_k of the full set can be expressed in terms of dimension of the subset $Q_1, Q_2, \ldots, Q_k$ only. In our example, the complete set of independent variables and parameters is P, L, A, rho, g and E. From this we choose a subset of three quantities. Since we are dealing with MLT system. With these three base quantities this subset can have only three quantities. Three quantities can provide you a complete and dimensionally independent subset.

If we choose four, then most probably the fourth quantity will not be dimensionally independent of the other three. Let there be L, g and E denoted in yellow tint in this table. Notice that these three quantities form a complete subset in the sense that we can express the dimension M from these three. In these three we can make a algebraic combination, a multiplicative combination, such that the dimension of that group would only be mass. Similarly we could have a combination which are dimension only of length.

For example L, the length of the rod has dimension L, and similarly, we can formulate a system which has a dimension only of time. Since equation for Q_0 is dimensionally homogeneous, the dimension of the dependent variable Q_0 is also expressible in terms of dimension $Q_1, Q_2, \ldots, Q_k$ the subset. So, the three parameters that we have chosen in the basic group L, g and E. The dimension of L is simply L, or g is L T^{-2}, the acceleration, and our E is like stress M L^{-1} T^{-2}. Now having chosen a complete dimensionally independent subset we express the dimension Q_0 and the remaining quantities k plus 1 to Q_n in terms of the dimension of $Q_1, Q_2, \ldots, Q_k$.

The process of dimensional analysis

We express $[Q_i] = [L]^{n_{i1}}[g]^{n_{i2}}.[E]^{n_{ik}}$
for of the each remaining variables.

$$[P] = ML^1T^{-2} = [L]^{\alpha}[g]^{\beta}[E]^{\gamma}$$

$$ML^1T^{-2} = [L]^{\alpha}[\mathrm{LT}^{-2}]^{\beta}[\mathrm{ML}^{-1}\mathrm{T}^{-2}]^{\gamma}$$

Solving, we get $\alpha = 2, \beta = 0$ and $\gamma = 1$

or, $[P] = [L]^2[E]$

	Variables		
	Basic group		
	L	g	E
Dimensions	L	LT^{-2}	$ML^{-1}T^{-2}$
	L^{α}	g^{β}	E^{γ}
	exponents		
δ	1	0	0
P	2	0	1
A	2	0	0
ρ	-1	-1	1

Thus they will have the form $[Q_1] = [L]^{n_{11}}[g]^{n_{12}}.\ [E]^{n_{1k}}$ and similarly Q_k raised to power n1k. So, in our example we would have δ would have a dimension of $[L]^{\alpha}[g]^{\beta}[E]^{\gamma}$. After taking out the variable L, g and E, the remaining variables are δ, P, A and ρ, which we have written on the left hand column at the end. We express $[Q_i] = [L]^{n_{i1}}[g]^{n_{i2}}.\ [E]^{n_{ik}}$ for each of the remaining variables. δ obviously would have simply the dimension L.

So, it can be written as L: alpha is equal to 1, beta is equal to 0. So, δ would have the same dimension as L. P on the other hand has a dimension M L^{-1} T^{-2}, and we want to represent $[L]^{\alpha}[g]^{\beta}[E]^{\gamma}$. We substitute the dimension of g: L T^{-2}, L T^{-2} raised to power β and we substitute the dimension of E: M L^{-1}1, T^{-2} raised to power γ.

Now dimensional homogeneity requires that the value of α should be 2. We can write the equation dimension of M would mean that M on the left hand side should be equal to on the right hand side M raised to power γ. So, gamma must be 1. Now let us look at the dimension of T. T is −2 on the left hand side, on the right hand side is −2γ −2β from the dimension of g, and −1γ from the dimension of E.

So, −2 on the left hand side power of T should be equal to −2γ −2β. Since γ is 1 determined already, β must be 0. Similarly we get α is equal to 2. So, P has the same dimension as L^2 E. So, we write the value of alpha as 2, write the value of beta is 0, and of gamma as 1 as a second line in this table against P.

Similarly A area has a dimension 2 dimension of length, and so it can be simple like 2 0 0, and the density ρ can be written as L^{-1} g^{-1} and E raised to power 1. Verify this would have the same dimension as that of density M L^{-3} . So, the next step is defining dimensionless forms of the n − k remaining independent variables by dividing each one of the product of the powers of Q_1, Q_2, Q_k which has the same dimension. So, $\Pi_i = \frac{Q_{k+i}}{=[Q_1]^{n_{i1}}[Q_2]^{n_{i2}}...[k]^{n_{ik}}}$ where i=1, 2,..., *n-k*, and similarly, a dimensionless form of the dependent variable $\Pi_o = \frac{Q_o}{=[Q_1]^{n_{o1}}[Q_2]^{n_{o2}}...[k]^{n_{ok}}}$

The process of dimensional analysis

We can write

$$\Pi_o = f(Q_1, Q_2, \ldots, Q_k, \Pi_1, \ldots \Pi_{n-k})$$

The values of the dimensionless quantities are independent of the sizes of the base units. The values of $Q_1 \ldots Q_k$, on the other hand, do depend on base unit size. They cannot be put into dimensionless form since they are (by definition) dimensionally independent of each other.

Therefore, $\Pi_o = f(\Pi_1, \ldots \Pi_{n-k})$

Variables			
Others			
P	*A*	ρ	δ
MLT^{-2}	L^2	ML^{-3}	L
			δ/L
P/L^2E			
	A/L^2		
		$\rho g L/E$	

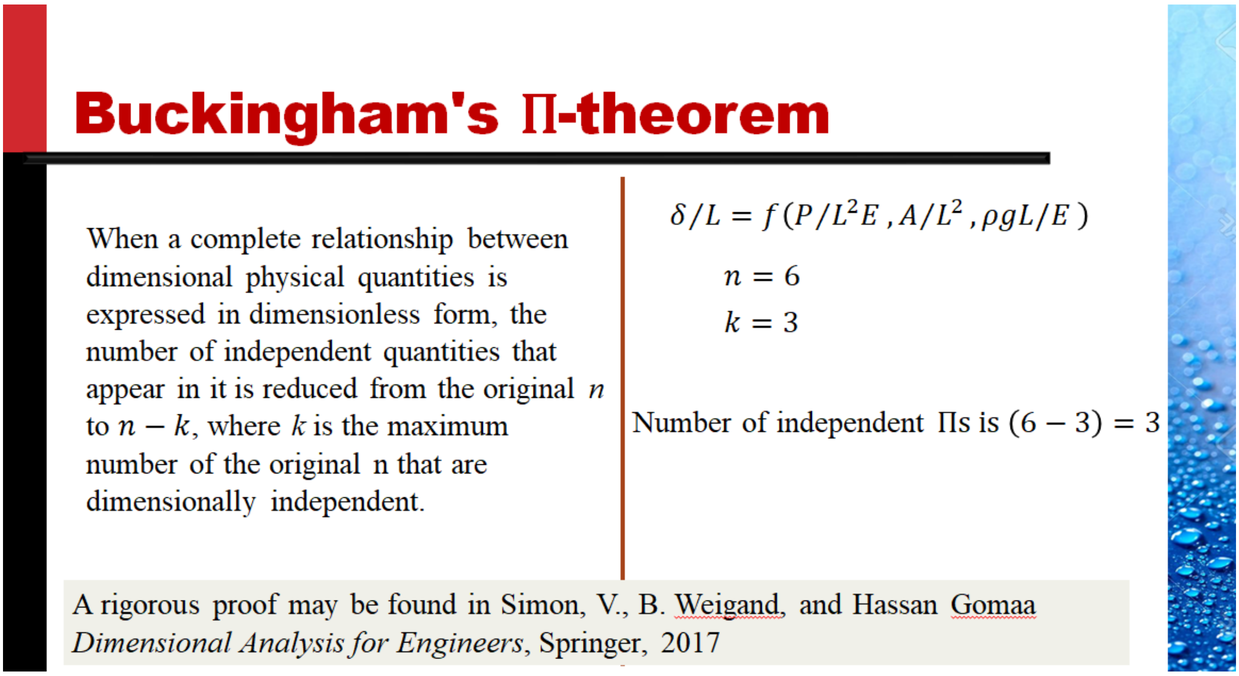
$$\delta/L = f(P/L^2E, A/L^2, \rho g L/E)$$

So, delta by L would be dimensionless. Similarly P has a dimension L square times E. So, P divided by L square E is dimensionless. A has a dimension L^2. So, A L^{-2} is dimensionless.

Similarly ρ was shown to have the dimension L^{-1} g^{-1} E, that is $\rho g L/E$ is dimensionless. Actually, $\frac{\frac{\rho}{E}}{Lg}$ which simplifies to $\rho g L/E$. So, this factor $\frac{\delta}{L}$ shown with the red background, dependent parameter, non dimensional dependent parameter, in terms of the basic group is a function of three non dimensional parameters $\frac{P}{L^2E}, \frac{A}{L^2}$ *and* $\frac{\rho g L}{E}$. This is how we can organize our calculations in this table. And this, then, we can write Π_o the dependent non dimensional parameters in terms of the basic group which in our case is $\frac{\delta}{L}$ is a function of Π_1, ... Π_{n-k}.

The values of the dimensionless quantity are independent of the sizes of the base units. Their value of Q_1 to Q_k, on the other hand, do depend upon base unit size. They cannot be put into dimensionless form since they are by definition dimensionally independent of each other. Therefore, Π_o can be written as function of $\Pi_1, \ldots \Pi_{n-k}$. So, if out of n basic quantities independent uniquity parameters, we need k to form the basic group the n − k pi's would be formed; n − k non dimensional groups of parameters would be formed.

And Π_o the dependent non dimensional parameter would be function of these n − k non dimensional pi's. So we can write $\frac{\delta}{L} = f\left(\frac{P}{L^2E}, \frac{A}{L^2}, \rho gL/E\right)$. When a complete relationship between dimensionless physical quantities expressed in dimensionless form, the number of independent quantities that appear in it is reduced from the original n to n − k, where k is the maximum number original n that were dimensionally independent. A rigorous proof of this theorem which is known as the Buckingham pi theorem can be found in this reference. Bridgman is another reference where you can find the proof.



So, this is the form of the deflection of the fishing rod that we have determined. Notice that originally there was 6 independent parameters. Now, the number of independent pi's is only 3, 6 minus 3 where k the number of basic groups is 3. Now, this change of the functional relationship between delta and the 6 independent parameters has now changed into a functional relationship between $\frac{\delta}{L}$ and 3 independent groups of non dimensional parameters. This decrease from 6 to 3 represents a substantial advantage to the researcher, to the person solving the problem.

We can construct scale models usingt modeling rules when we make scale models. All we have to ensure for similarity is that the values of these 3 non dimensional groups match between models and prototype. Thus, the first group $\frac{P}{L^2E}$ must have the same value in the model as the value of this in the prototype. Similarly, $\rho gL/E$ the value of this in the model must match the value of this in the prototype. $\frac{A}{L^2}$ in the model should match $\frac{A}{L^2}$ in the prototype.

This is an easier order than matching all the 6 quantities independently because then it will not be a model at all it will be the prototype itself. So, reduction in the numbers of matching parameters, in the number of parameters that are required to be matched, results in substantial convenience to the researcher. And once we do this, once these 3 parameters have matching values in model and prototype, then the dependent parameter delta by L in the prototype would have the same value as in the model. Thus, we can construct a scale model, and in this scale model if we measure delta by L, the value of delta by L of the prototype would be exactly the same value as you obtained in the model. So, if we could measure the value of delta in the model, we could predict the value of delta in the prototype.

This is what a major use or dimensional analysis is. It permits us to do scale model tests. Thank you.

Thank you.