Similitude And Approximations In Engineering,
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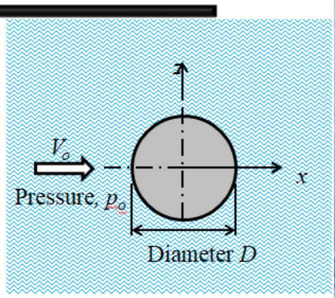
Week - 02
Lecture – 07

Welcome back. In the last lecture, we developed the non-dimensional form or the normalized form of the equations governing the fluid flow and determined that the three named pi numbers, Euler number, Froude number and Reynolds number control the similitude. In today's lecture, we will try to develop an understanding of the physical significance of these three main pi numbers of fluid mechanics. This is the equation that we developed for incompressible steady viscous flow of a fluid past a body with a uniform velocity V_0 far upstream and the pressure p_o far upstream. When we normalize the equation, we got this form where we had made the coefficient of the inertial term as 1. The number $\left(\frac{\rho V_o^2}{p_o}\right)$ is termed as the Euler number.

More about Euler number

$$\mathbf{V}^* \cdot \nabla^* \mathbf{V}^* = -\left(\frac{p_0}{\rho V_o^2}\right)\nabla^* p^* - \left(\frac{gD}{V_o^2}\right)\mathbf{k} + \left(\frac{\mu}{\rho V_0 D}\right)\nabla^{*2}\mathbf{V}^*$$

$$\mathbf{V}^* \cdot \nabla^* \mathbf{V}^* = -\left(\frac{1}{\mathrm{Eu}}\right)\nabla^* p^* - \left(\frac{1}{\mathrm{Fr}^2}\right)\mathbf{k} + \left(\frac{1}{\mathrm{Re}}\right)\nabla^{*2}\mathbf{V}^*$$



Note that it is not the pressure that appears in the governing equation, but its gradient. Therefore, all pressures may be measured with p_o as the datum, and we can work with gauge pressure, $p_g = p - p_o$.

This is a pressure difference and requires a characteristic pressure difference for normalization, or that we need to know the pressure at some other point in the flow field. However, since no other pressure is specified in this problem, let us assume that the characteristic pressure difference is $(\Delta p)_0$, as yet unknown.

The number $\frac{V_0}{\sqrt{gD}}$ is termed as the Froude number, and $\left(\frac{\rho V_0 D}{\mu}\right)$ as Reynolds number and is abbreviated by Re. Let us first talk about the Euler number. Note that it is not the pressure that appears in the governing equation, but its gradient. Therefore, all pressures may be measured with p_o as the datum.

That is, we work with differences in pressure rather than the pressure itself. We can work with a gauge pressure $p_g = p - p_o$. This is the pressure difference that requires a characteristic pressure difference for normalization, or that we need to know the pressure at some other point in the flow field. However, since no other pressure is specified in the problem, let us assume that the characteristic pressure difference is $(\Delta p)_0$, as yet unknown. We will discuss about what should the value of $(\Delta p)_0$ be later.

With this, the equation changes to this equation. $V^* \bullet \nabla^* V^* = -\left(\frac{(\Delta p)_0}{\rho V_o^2}\right)\nabla^*\left(p_g\right)^* - \left(\frac{1}{Fr^2}\right)k + \left(\frac{1}{Re}\right)\nabla^{*2}V^*$. And the pressure boundary condition becomes $p_g^* \rightarrow 0$ on $z^* = 0$ as $x^* \rightarrow \infty$, or minus infinity. p_o no longer appears in the mathematical equation of the problem, and therefore, is not a uniquity parameter. If the pressure forces within a flow field are to be significant, that is, they are not to be negligible, the pressure force terms in this equation should be of the same order as the inertial term.

So, $\left(\frac{(\Delta p)_0}{\rho V_o^2}\right)\nabla^*\left(p_g\right)^*$ must be of the same order as $V^* \bullet \nabla^* V^*$. The coefficient of the inertial term is 1. So, $\left(\frac{(\Delta p)_0}{\rho V_o^2}\right)\nabla^*\left(p_g\right)^*$ should also be of order 1. This is possible if the characteristic pressure difference $(\Delta p)_0$ is of the same order as ρV_o^2. So, this equation: $V^* \bullet \nabla^* V^* = -\frac{1}{2}\nabla^*\left(p_g\right)^* - \left(\frac{gD}{V_o^2}\right)k + \left(\frac{\mu}{\rho V_0 D}\right)\nabla^{*2}V^*$

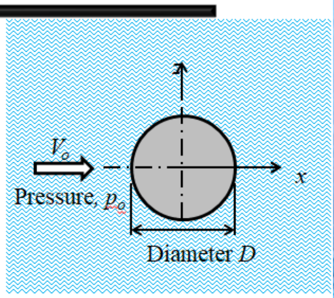
More about Euler number

$$\mathbf{V}^* \cdot \nabla^*\mathbf{V}^* = -\left(\frac{(\Delta p)_0}{\rho V_o^2}\right)\nabla^*\left(p_g\right)^* - \left(\frac{1}{\mathrm{Fr}^2}\right)\mathbf{k} + \left(\frac{1}{\mathrm{Re}}\right)\nabla^{*2}\mathbf{V}^*$$

It is customary to set $(\Delta p)_0 = \frac{1}{2}\rho V_o^2$.

With this, the governing equation becomes

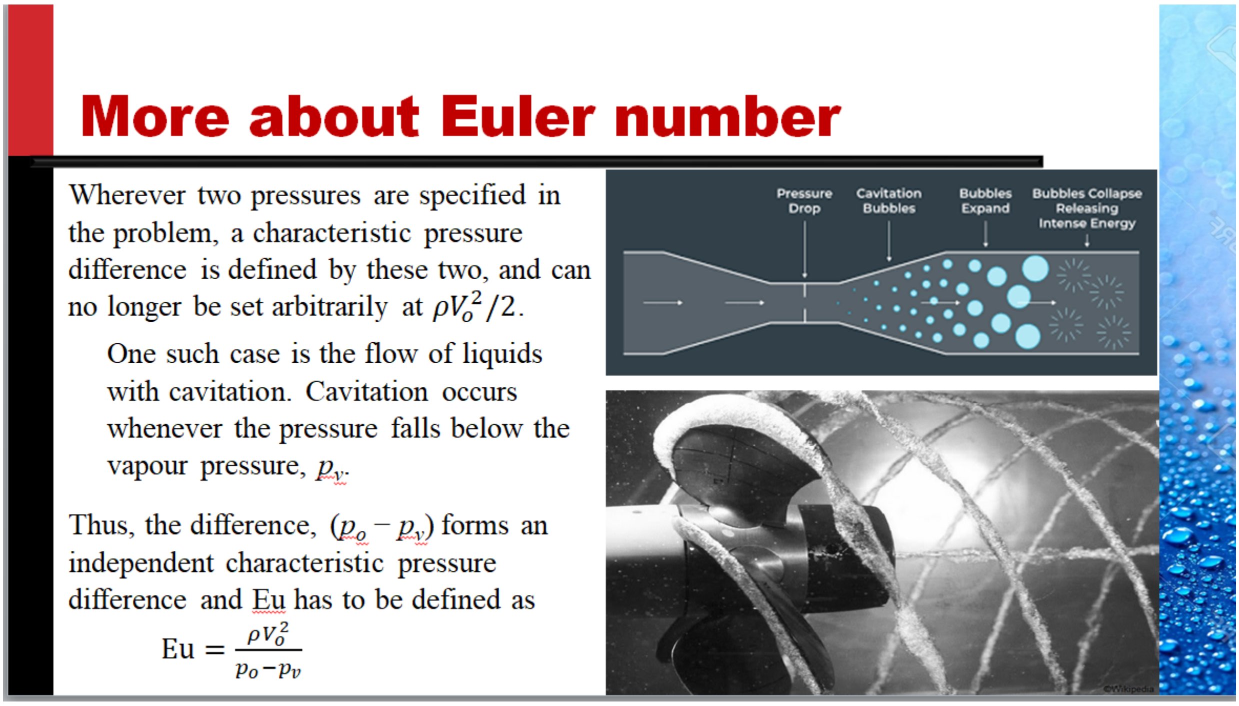
$$\mathbf{V}^* \cdot \nabla^*\mathbf{V}^* = -\frac{1}{2}\nabla^*\left(p_g\right)^* - \left(\frac{gD}{V_o^2}\right)\mathbf{k} + \left(\frac{\mu}{\rho V_0 D}\right)\nabla^{*2}\mathbf{V}^*$$



Then, we can write $\mathbf{V}^*(\mathbf{x}^*) = \mathbf{V}^*\left(\mathbf{x}^*; \frac{gD}{V_o^2}, \frac{\mu}{\rho V_0 D}, \text{geometry}^*\right)$

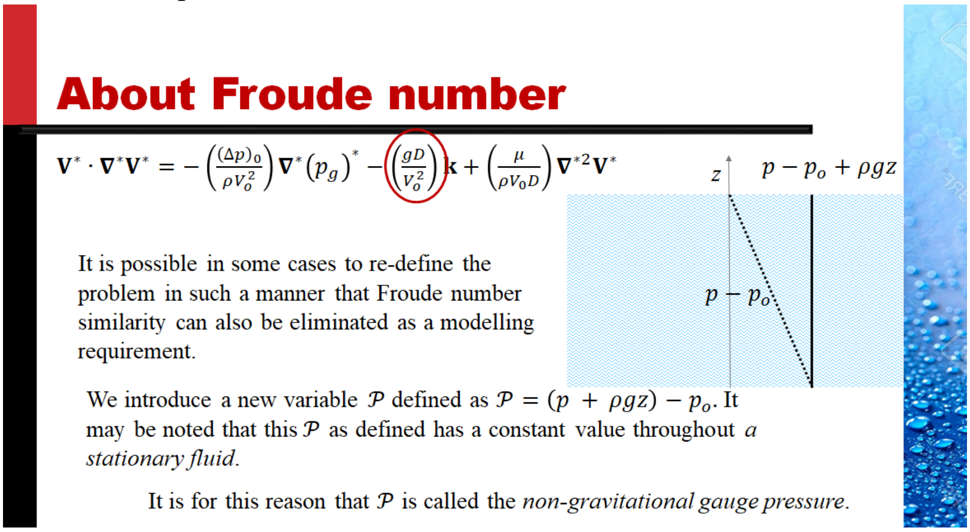
In this, it is customary to set $(\Delta p)_0$ as $\frac{1}{2}\rho V_o^2$. What is significance of this? You recall from Bernoulli equation $\frac{1}{2}\rho V_o^2$ is the pressure difference between the stagnation pressure and the static pressure in incompressible flow. So, that pressure difference which is also known as a dynamic pressure of the flow is taken as the characteristic pressure difference. And with this the coefficient of the pressure term becomes $-1/2$ and this is the resulting equation: $V^* \bullet \nabla^* V^* = -\frac{1}{2}\nabla^*\left(p_g\right)^* - \left(\frac{gD}{V_o^2}\right)k + \left(\frac{\mu}{\rho V_0 D}\right)\nabla^{*2}V^*$. Notice that in this equation the Euler number has disappeared.

So, that we can write the functional form of the velocity, the non dimensional normalized velocity, as function of $\left(x^*;\ \frac{gD}{V_o^2},\ \frac{\mu}{\rho V_0 D},\ geometry^*\right)$. Wherever two pressures are specified in the problem a characteristic pressure difference is defined by these two, and we can no longer set this to be $\frac{1}{2}\rho V_o^2 2$. The two given pressure in a problem would define the characteristic pressure difference. One such case is the flow of liquids with cavitation. Cavitation is a phenomenon where the liquid boils when the pressure in a flow field of liquid drops below the vapor pressure of that liquid.



Then the vapors are formed and they result in vapor bubbles. If the vapor pressure at the temperature of the flow is p_v, then p_o minus p_v is a characteristic pressure difference that should be used to normalize the pressure difference. We have shown here a converging diverging channel with a liquid flowing through this as the flow accelerates in the converging portion the pressure drops and that pressure drop, if it is large enough so that the pressure becomes lower than the vapor pressure of liquid, then the vapor bubble would form they would grow and as this moves in the diverging portion there would be an increasing pressure and this would tend to make the vapor bubble collapse. This picture shows the vapor trail in a propeller of a ship. These are water vapors trails that you see the pressures at the tip of the blades is very low, below the cavitation pressure, and the vapor bubble so form a nice beautiful picture as seen here.

Thus, the pressure difference $p_o - p_v$ forms an independent characteristic pressure difference and Euler number has to be defined now as $Eu = \frac{\rho V_o^2}{p_o - p_v}$. We can no longer take $\frac{1}{2}\rho V_o^2$ as the characteristic pressure difference.



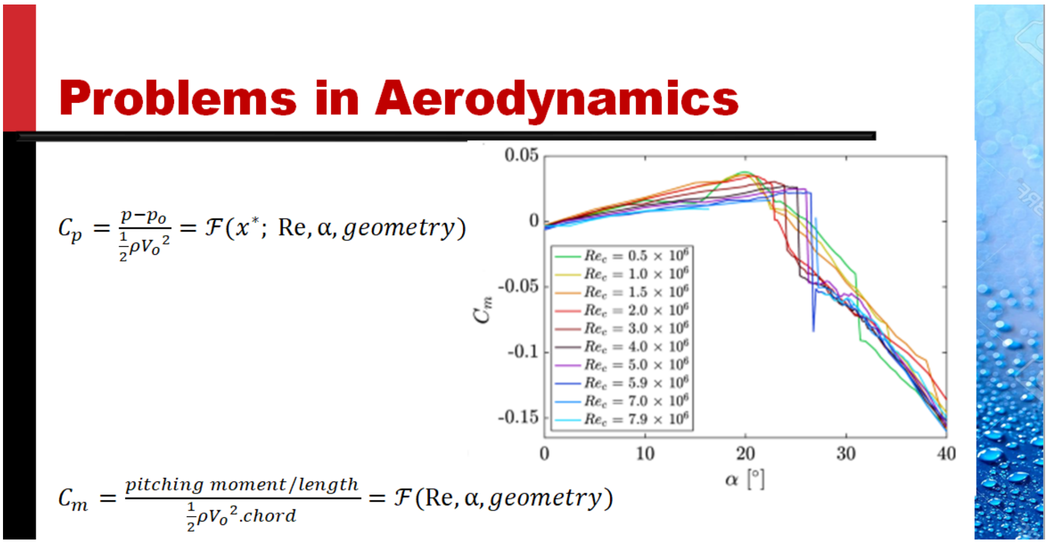
Now, let us talk about the Froude number. Froude number is defined as $\frac{V_0}{\sqrt{gD}}$. It is possible in some cases to redefine the problem in such a manner that the Froude number similarity can also be eliminated as a modeling requirement.

We introduce a new variable script p (P) defined as $P = (p + \rho gz) - p_o$. It may be noted that P is defined to have a constant value throughout a stationary fluid. For example, if we have a fluid in this region the pressure varies like this. This broken line shows the plot of the gauge pressure $p - p_o$ increasing with the depth in the fluid. If we subtract from this the value of ρgz and formulate the variable $(p + \rho gz) - p_o$, we observe this to be constant.

Thus, in a stationary fluid the pressure $(p + \rho gz) - p_o$ is constant. This is why this is termed as non gravitational gauge pressure. With this value of P we can write $p - p_o$ as $P - \rho gz$ and if we take the divergence of this with a negative sign, $-\nabla\left(p - p_o\right) = -\nabla P + \rho gk$. On normalizing the pressure difference by using $(\Delta p)_0$ and length by D, we get this equation. Now, here you notice these two terms $\nabla^* P^* - \left(\frac{\rho gD}{(\Delta p)o}\right)k$ become $\nabla^*\left(p_g\right)^*$ andand $-\left(\frac{(\Delta p)_0}{\rho V_o^2}\right)\nabla^*\left(p_g\right)^* = -\left(\frac{(\Delta p)_0}{\rho V_o^2}\right)\nabla^* P^* + \frac{gD}{V_o^2}k$ and we can write $V^* \bullet \nabla^* V^* = -\left(\frac{(\Delta p)_0}{\rho V_o^2}\right)\nabla^* P^* + \left(\frac{\mu}{\rho V_0 D}\right)\nabla^{*2} V^*$.

Notice that the gravity term has been absorbed in this term $\left(\frac{(\Delta p)_0}{\rho V_o^2}\right)\nabla^* P^*$ which represents non dimensionalized pressure, non dimensional modified pressure, non gravitational pressure. And we see that the Froude number has disappeared from the equation completely. This is the equivalent equation that we obtained in the last slide and here P has been normalized to P^* using $(\Delta p)o$. The pressure conditions now change to $P^* \rightarrow 0$ on $z* = 0$ as $x* \rightarrow -\infty$. Thus it is seen that the Froude number does not appear in the problem statement at all and only Reynolds number and Euler number have to be matched.

If in addition the flow conditions such that only one pressure p_o is specified in the problem statement, then we can use rho $\frac{1}{2}\rho V_o^2$ as the characteristic pressure difference. This condition will be obtained in flows where there is no cavitation.



Let us consider the application of this to problem in aerodynamics. We have an aerofoil at an angle of attack and we want to define or to determine the pressure distribution on the surface of the aerofoil. We can define a non dimensional pressure coefficient $C_p = \frac{p-p_o}{\frac{1}{2}\rho V_o^2}$.

We can use $\frac{1}{2}\rho V_o^2$ as a characteristic pressure difference because cavitation is not involved in problems of aerodynamics. And this dependent variable should be a function of independent variable x^*, the location on the surface of the aerofoil and the parameter Re, Reynolds number. Because Froude number as well as Euler number would not occur. And this will also depend upon alpha the angle of attack and the geometry of the aerofoil. Dependent parameter like the pitching moment coefficient which is defined as the pitching moment per unit length of the aerofoil divided by $\frac{1}{2}\rho V_o^2$ times the chord is equal to the function of Reynolds number and alpha and geometry.

Obviously, this is a dependent parameter. It does not depend upon the independent variable. This curve shows the plot of C_m versus the angle of attack alpha for various values of Reynolds number. It depends only on Reynolds number and not on the individual quantities involved within the Reynolds number that is ρ, μ length and velocity of the flow. From the independent parameters we can determine the value of the Reynolds number, and use the appropriate curve to find out the pitching moment variation with the angle of attack alpha. Let us discuss further about the Froude number.

More about Froude number

This can be recast as

$$\mathcal{P}^* = \frac{\rho V_o^2}{(\Delta p)_o} \frac{gL}{V_o^2} . z_f^* \quad \text{at } z^* = z_f^*$$

or $$\mathcal{P}^* = \frac{\text{Eu}}{\text{Fr}^2} \text{ at } z^* = z_f^*$$

Thus, though it vanishes in the equation of motion, the Froude number Fr appears in the boundary conditions, and needs to be matched

Such conditions occur for example in the motion of ships and other vessels close to the surface of oceans. Similarly, Fr in the models must be the same as Fr in the prototypes in flows over dams, weirs and in open channels. However, Fr may be ignored as a similarity requirement for a submarine operating at large depths

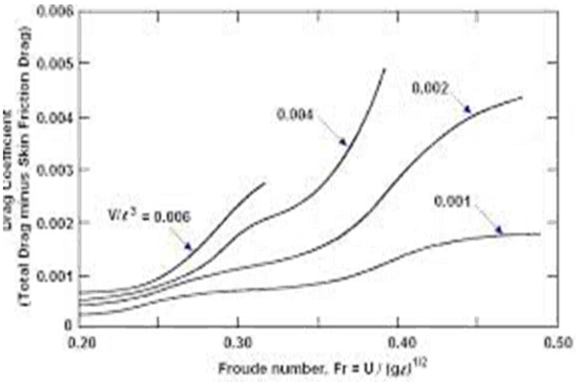
We have said that by defining the non dimensional, non gravitational pressure difference we can get rid of the Froude number in the problem. However, this simplification is not always possible. For examples in problems involving a free surface of a liquid the shape of it changes with the motion we cannot make the problem independent of the Froude number. At the free surface which is given as $z^* = z_f^*$, a function of x. This free surface between the liquid and the air for example, on the surface of a sea the pressure is atmospheric everywhere. So, at different values of z as a function of x the pressure is constant, atmospheric.

If this be so and we introduce the non gravitational pressure difference $(p + \rho g z) - p_o$, the pressure boundary condition now modifies to $P_g = \rho g z_f$ at $z = z_f$, which on non dimensional gives you $P^* = \frac{\rho V_o^2}{(\Delta p)_o} \frac{gL}{V_o^2} . z_f^*$ at $z^* = z_f^*$. Now the gravity and consequently the Froude number appear in the boundary conditions. So that the boundary conditions becomes $P^* = \frac{Eu}{Fr^2}$ at $z^* = z_f^*$, the free surface. Thus though the Froude number vanishes in the equation of motion the Froude number Fr appears in the boundary conditions and needs to be matched for similarity. Such conditions occur for example, in the motion of ships and other vessels close to the surface of oceans.

Similarly Froude numbers in models must be same as Froude numbers in the prototypes in flows over dams, weirs and open channels where free surface is important. However, Froude number may be ignored as a similarity requirement for submarines operating at large depths. Then the presence of the free surface is far away and we can neglect it. In wave drag on ships the drag coefficient C D defined as drag divided by one half rho minute squared into area is now a function of Froude number, Reynolds number, non-dimensional geometry. So wave drag depends upon Froude number and Reynolds number.

Wave drag on ships

$$C_D = \frac{Drag}{\frac{1}{2}\rho {V_o}^2 \cdot Area} = \mathcal{F}(\text{Fr, Re, geometry})$$



We can show that the dependence on Reynolds number is rather low. So the drag coefficient is a function of Froude number and the geometry. Here we have drag coefficients. On this side is Froude number and these various curves of of various geometry. Geometry is measured by the displaced volume divided by the length cubed, and the length here is the draft of the ship. So the wave drag changes with Froude number and also with the geometry.

Let us summarize the rules of thumb that we obtained so far for similarity. For a flow situation where there is no free surface, but there is cavitation. For example, in poorly designed pumps where the cavitation occurs, in siphons, and in torpedoes which operate at quite a bit of depth, we need to match the Euler number and Reynolds number only. The Froude number is not important because we are operating away from the free surface.

However, the dependence on Reynolds number is rather weak and the most important modeling rule is to match the Euler number. In the second flow situation that we discussed, we have free surfaces, but no cavitation like in ships, dams, harbors, offshore platforms, etcetera. Since there is a free surface, Froude number is important. Since there is no cavitation, Euler number is not important, and so we should match Reynolds number and Froude number. Here again in most cases the dependence on Reynolds number is rather weak, and so the most important modeling rule is the matching of Froude number.

The third case that we discussed there is no free surface, there is no cavitation, deep submarines, airplanes, enclosed close pipes, etcetera. Neither the Euler number nor the Froude number are important and we need to match only the Reynolds number. We will see later that if the Reynolds number is very high the dependence on Reynolds number also becomes unimportant, and the flow does not depend upon any of these parameters. It depends only on the geometry. The last case that we discuss is that we have free surface, we have cavitation like in high speed ships or torpedoes operating very close to the surface.

All the three numbers, Euler numbers, Reynolds number and Froude number need to be matched. Thank you.