Similitude And Approximations In Engineering,
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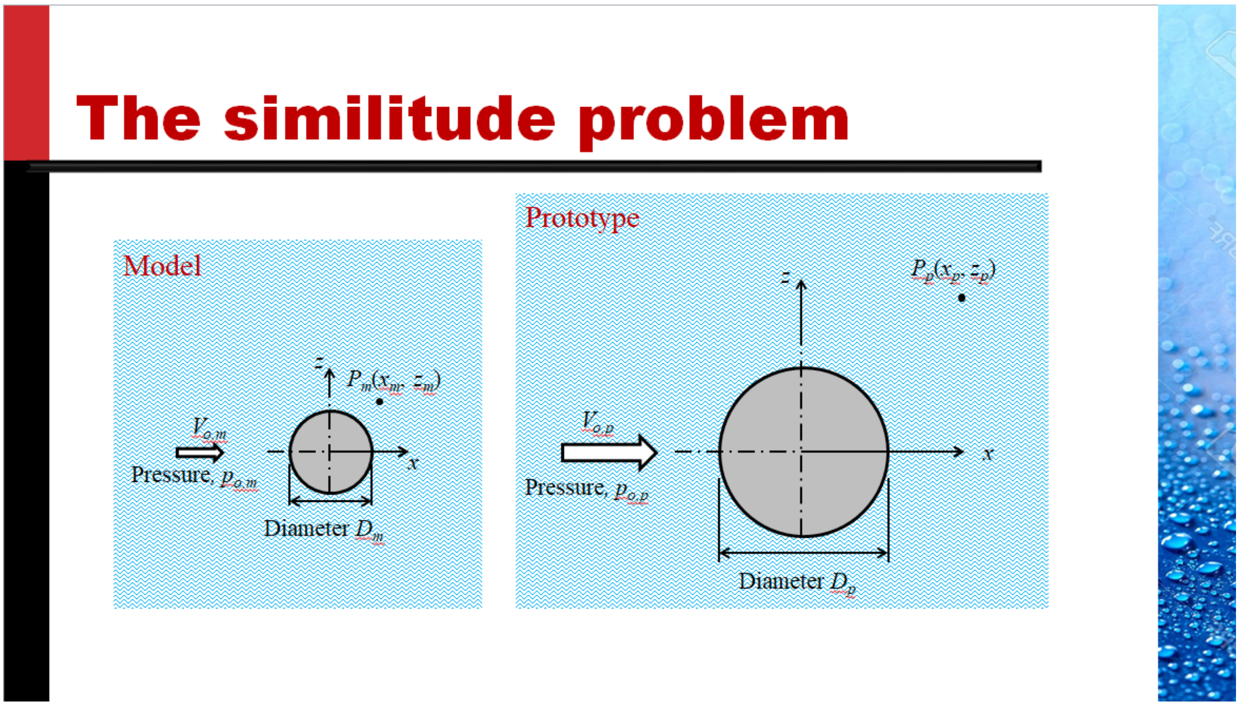
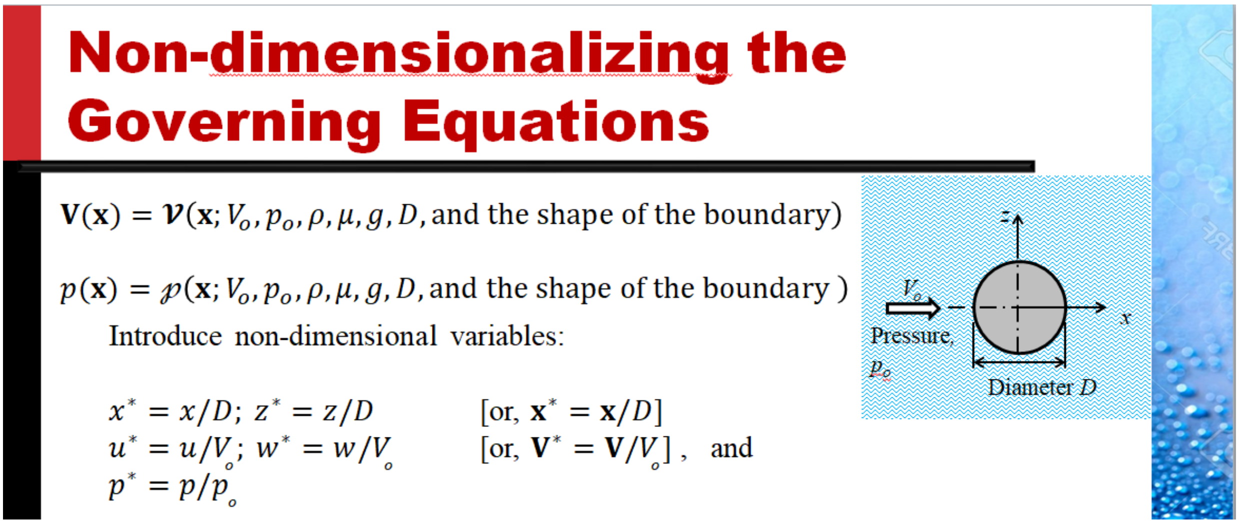
Week - 01
Lecture – 06

Welcome back. In this lecture, we will continue with what we did in the last lecture. We were explaining with more examples the process of obtaining modeling and prediction rules from the governing equations. As was stated earlier, the whole business of similitude is to determine the conditions under which we can predict the values of the dependent quantities for one set of independent parameters from those obtained from an experiment with different, though related, values of the independent parameters. We have done two examples of obtaining the similitude results, for the case of a mass damping spring vibrating system and for unsteady flow of heat from a slab. Let us now introduce the problems in fluid mechanics.



Fluid mechanics offer a very rich example for the development of principles of similitude because of the complicated equations and the varied behavior under different values of various uniquity parameters. The similitude problem is that if in two systems, one a model and a prototype, they are geometrically similar. How else would the two problems be similar such that results from a model experiment, experimental or computational, can be used to predict the result for the prototype. We are given the uniquity parameters for the prototype.

What should be the uniquity parameters for the model so that this similitude is obtained? And once this similitude is obtained, what should be the prediction rules that should be used to predict the results for the prototype from the model results. So, we will take this example of a flow past a circular cylinder, incompressible, steady flow. The far upstream velocity is V_o, the pressure is p_o, the cylinder diameter, two dimensional, is D. Clearly the vector velocity V would be a function of x, the location of the point in the flow field, the independent variables, and the list of parameters which include V_o, the velocity far upstream, p_o, the pressure far upstream, ρ and μ, the properties of the fluid, g the acceleration due to gravity, and D the size of the cylinder measured by its diameter, and the shape of the boundary, in this case cylindrical. Similarly, the dependent variable p as a function of x, the independent variable, is again a function of the independent variable x and the values of the same set of uniquity parameters.



We introduce the non dimensional variables, x^* is x divided by characteristic length, and the diameter is a valid characteristic length for this problem, z^* is z by D, or in the vector form is vector x^* is vector **x** divided by D. And the velocity component u and v as $u^*$is equal to u divided by V_o and w^*as w divided by V_o, and we introduce the non dimensional pressure p^*as p divided by p_o. These are the normalized independent and dependent variables. The quantities used for non dimensionalizing the various variables should be those that characterize the problem. That is why they are called characteristic quantities.

These are the scales that are natural to the problem. They do not depend upon artificial definition of meter foot or anything else. They are natural to the problem on hand. The diameter of the cylinder is a natural dimension for use in this problem. All the variables are dimensionless, and then expected to be order unity, except an isolated points.

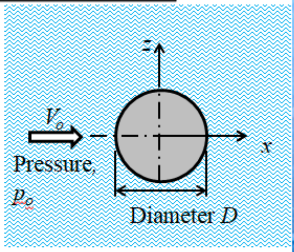
We will talk about that later, much later. And it is for this reason these are termed as normalized variables as well. These are the governing equations for incompressible flow when it is steady. These equations in normalized form would look like $\nabla^* \cdot V^* = 0$. This is the continuity equation, or mass balance equation, and for the steady flow the Navier Stokes equation in its normalized form reduces to $V^* \bullet \nabla^* V^* = -\left(\frac{p_0}{\rho V_o^2}\right)\nabla^* p^* - \left(\frac{gD}{V_o^2}\right)k + \left(\frac{\mu}{\rho V_0 D}\right)\nabla^{*2} V^*$.

The $\nabla^* = i\frac{\partial}{\partial x^*} + j\frac{\partial}{\partial y^*} + k\frac{\partial}{\partial z^*}$. And this problem is to solve with boundary condition that the non dimensional normalized velocity $V^* \to i$, or tends to 1 far away. $V^* = 0$ on the circular boundary which now become $x^{*2} + z^{*2} = 1/4$. This is when we have non dimensionalized x and z by the diameter D and $p^*\to 1$ far away. We notice that in this equation, all the uniquity parameters are concentrated in 3 groups of parameters.

Non-dimensionalizing the Governing Equations

$$\nabla^* \cdot \mathbf{V}^* = 0$$

$$\mathbf{V}^* \cdot \nabla^* \mathbf{V}^* = -\left(\frac{p_0}{\rho V_o^2}\right)\nabla^* p^* - \left(\frac{gD}{V_o^2}\right)\mathbf{k} + \left(\frac{\mu}{\rho V_0 D}\right)\nabla^{*2}\mathbf{V}^*$$



with the boundary conditions

$$\mathbf{V}^* \to \mathbf{i} \qquad \text{as } x^*, z^* \to \pm\infty$$

$$\mathbf{V}^* = \mathbf{0} \qquad \text{on } x^{*2} + z^{*2} = 1/4$$

$$p^* \to 1 \qquad \text{on } z^* = 0 \text{ as } x^* \to \infty$$

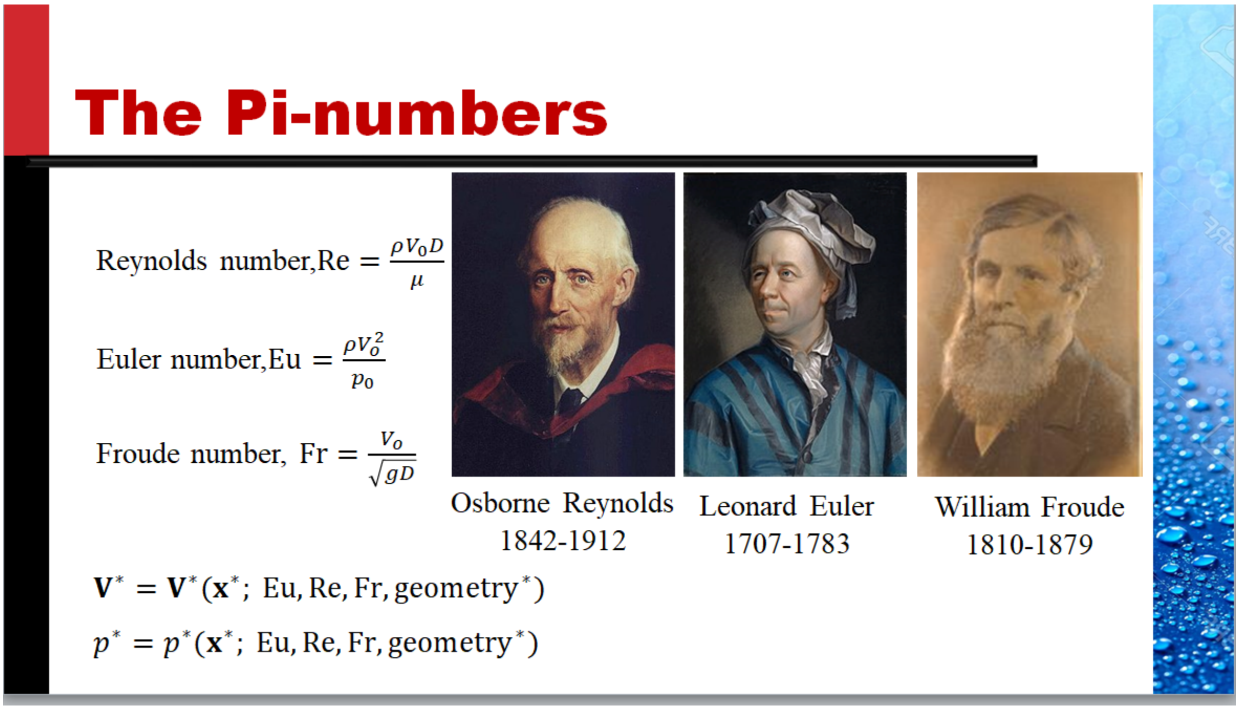
where $\nabla^* = \mathbf{i}\frac{\partial}{\partial x^*} + \mathbf{j}\frac{\partial}{\partial y^*} + \mathbf{k}\frac{\partial}{\partial z^*}$

So, these are now the 3 uniquity non-dimensional parameters that define the solution of the problem. There is no parameter within the boundary conditions. These are only these 3 groups. And that is why the non dimensional solution of velocity is a function of x^* the variable, and 3 non-dimensional groups, and the geometry. Similarly, p^*.

Geometry in the above list of parameters is a non dimensional geometry curve $x^{*2} + z^{*2} = 1/4$, which was seen at this place in the boundary conditions. So, instead of the original list of 6 plus 1 uniquity parameters we have 3 plus 1 uniquity groups of parameters and these groups are non dimensional. These 3 groups are termed as pi numbers for the problem of fluid flow past a circular cylinder. Thus, non dimensionalization reduced the number of independent parameters from 6 plus geometry to only 3 plus the geometry. This is a significant improvement in that if we were developing a database for solutions to this problem, we would not need to vary the 6 parameters over their entire ranges, but only manipulate these 3 pi numbers over their ranges of values.

The results obtained with one set of values or dimensional parameters could be used to predict the results for many more sets of parameters as long as the value of these 3 pi numbers match. The same holds for numerical solutions as well not just the experimental solutions, but numerical solutions as well. Since the variables and their various derivatives have all been normalized and are expected to be order 1, the non dimensional groups of parameter which are now rendered as coefficients of the various terms indicate the importance of the term in that equation. Thus if the coefficient of any one term is much less than 1, the term may be ignored as an approximation. Consideration of this point forms the bulk of the second part of this course: the theory of approximations.

The 3 pi numbers are named after Osborne Reynolds, a British scientist after Leonard Euler, and William Froude, another British scientist. The two flows, model and prototype, are similar flows if the values or non dimensional pi numbers formed with the uniquity parameters are identical in the two flows. In such situations the normalized dependent variables have the same values at all sets of homogeneous points. The statement can be broken down into two parts. A: the modeling rules, the requirement of similarity.



Two flows are similar if the values of pi numbers formed with the independent parameters in the two flows are the same. Thus, if the three groups of parameters have identical values in two flows, model and prototype, then we say the two flows are similar. And if that is obtained then the values of the non dimensional dependent parameters would have identical values and they form the prediction rules. If the two flows are similar, the values of the normalized dependent variables in one flow are the same as in the other flow at homologous points. What is homologous? Homologous are two points like this and this which bear geometrically similar relationship to the boundaries of the flow.

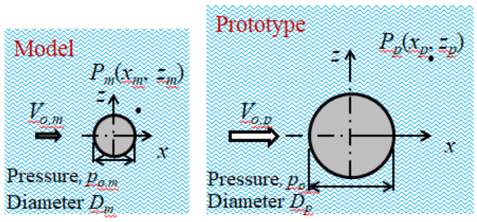
To illustrate this, let us consider the dependent variable shear stress τ at any location within the flow field. We know from Newton Stokes relation for stresses that in the flow the shear stress τ is measured by $\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$. u and w are the velocity component in x and z direction respectively. So $\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$ give you the rate of distortion of the fluid element. The rate of distortion multiplied by the viscosity is the shear stress.

On non dimensionalization or normalization this becomes $\frac{\mu V_o}{L}\left(\frac{\partial u^*}{\partial z^*} + \frac{\partial w^*}{\partial x^*}\right)$. So that the tau is now given by $\rho V_o^2 \cdot \frac{\mu}{\rho V_o L}\left(\frac{\partial u^*}{\partial z^*} + \frac{\partial w^*}{\partial x^*}\right)$. I take $\rho V_o/L$ and this is the shear stress square which has the dimensions of pressure or stress and the normalized τ by dividing by ρV_o^2 to get normalized value of τ as τ^*. And this τ^* is $\frac{1}{Re}\left(\frac{\partial u^*}{\partial z^*} + \frac{\partial w^*}{\partial x^*}\right)$. If the Reynolds number of two flows match and so also the Euler number and Froude number, then the values of these are identical in the two cases.

Concept of similitude

$$\tau^* = \frac{\tau}{\rho V_o^2} = \frac{1}{\mathrm{Re}}\left(\frac{\partial u^*}{\partial z^*} + \frac{\partial w^*}{\partial x^*}\right),$$

In similar flows, the non-dimensional RHS of the above would have same values at homologous points, and therefore, $\tau_p^* = \tau_m^*$.



And since Reynolds number also matching that means, tau star would be identical in the two cases. Now, if I measure the value of tau in the model and divide it by the corresponding value of rho v naught squared, then I get tau star for the model and this would have the same value for the prototype. So, once I know tau star for the model I can use it as tau star for the prototype, and multiply that tau star by the value of rho v naught squared for the prototype to obtain the tau for the prototype at the homologous point. This is the whole nature of similitude in fluid flows. So, again in similar flows the non dimensional RHS of the above would have the same values in homologous point and therefore, tau star p is equal to tau star m.

The tau star value in prototype is the same as tau star value in the model. Thank you.

Thank you.