Similitude And Approximations In Engineering,
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Week - 01
Lecture – 05

Welcome back. In this lecture, we will discuss the procedures for obtaining modeling and prediction rules from the governing equations and the boundary conditions. We have stated earlier that the business of similitude is to determine the conditions under which we can predict the values of the dependent quantities, variables or parameters, for one set of independent parameters from those obtained from an experiment, physical or computational, with different though related values of the independent parameters. To introduce the first technique of obtaining these rules from the governing equations, let us consider the vibration of a spring mass dashpot system. There is a mass which is connected to the surroundings through a spring and a dashpot and is subjected to a periodic force sin $F_o sin\ \omega t$. The governing equation for this system can readily be obtained as $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = $ sin $F_o sin\ \omega t$. m is the mass, $m\frac{d^2x}{dt^2}$ is mass times acceleration, plus c is the damping constant, $c\frac{dx}{dt}$ is the damping force, k is the spring constant. So, kx is the spring force, and these forces must sum out to the force which are forcing the mass to vibrate. The solution of this equation is of the form $x = x_o$ sin $sin\ (\omega t + \varphi)$ where x_o is the amplitude of vibrations of x and φ is the phase difference from the wave frequency of the forcing force. The independent variable is t, the only one, independent parameters, those on which we have control and which define the problem, the uniquity parameters are m, the mass, c, the damping coefficient, k, the spring constant, F_o the amplitude of the force, and ω the frequency of the oscillating forcing force.

Obtaining modelling rules – First technique

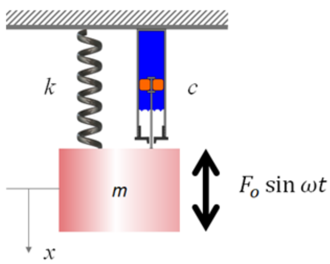
Consider the vibration of mass-spring-dashpot system.

The governing equation for this system can readily be obtained as

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F_o sin\ \omega t$$

The solution of this equation is in the form

$$x = x_o \sin(\omega t + \varphi)$$



Independent variables: t

Independent parameters: m, c, k, F_o, and ω

Dependent variables: x

Dependent parameters: x_o, φ

Independent variable is only one t, and the dependent parameters are x_o and φ, x_o the amplitude of vibrations and φ the phase angle. So, that the variable x the dependent variable x can be written as a function of t, the independent variables, semicolon and the least of unique parameters m c k F_o and ω. We could also write the dependent parameters x_o as a function of the unique parameters and φ also as a function of uniquity parameters. Note then in both these cases the independent variable would not be involved.

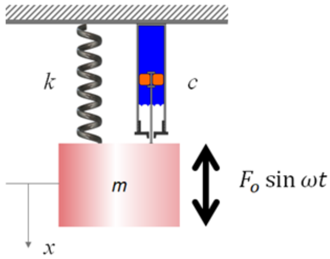
The question that arises is that if we are not able to do the experiments on the prototype system, on the actual system, can we do experiments on a model system, and from the results obtained there on predict the results for the prototype system that is the question that we need to answer?

If so, how should the unique parameters of the model system be related to those of the prototype and then what rules will be used for predicting the results for the prototype from the results from the model. The first technique starts with normalizing the variables the two variables x and t dependent and independent. Normalize in that we make them of the order unity, make them of order 1. How do you do that? We introduce the characteristic values of these variables the typical values of these variables and I define these variables with the typical values called the characteristic values. Then we should get variables non-dimensional of order 1.

So, the normalized independent variables can be written as $t^* = \frac{t}{t_c}$ where t_c is some characteristic time we will talk about this a little later. And the normalized dependent variable $x^* = \frac{x}{x_c}$, the characteristic value of the dependent variable. A good choice for t_c the characteristic value of time is 1/ω, 1 over the frequency of the forcing function, and a choice for x_c the characteristic value of x is the amplitude. It is a good choice. With this the equation would become this: $\frac{m\omega^2}{k}\frac{d^2x^*}{dt^{*2}} + \frac{c\omega}{k}\frac{dx^*}{dt^*} + x^* = \frac{F_o}{kx_o}\sin sin\, t^*$. That in this equation we have made the coefficient of springt force term as 1.

Non-dimensionalizing the Governing Equations

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F_o \sin \omega t$$



A good choice for $t_c = {}^1/_\omega$, and for $x_c = x_o$

With this, the equation becomes:

$$\frac{m\omega^2}{k}\frac{d^2x^*}{dt^{*2}} + \frac{c\omega}{k}\frac{dx^*}{dt^*} + x^* = \frac{F_o}{kx_o}\sin t^*$$

$$\frac{x_o}{F_o/k} = \mathcal{X}\left(\frac{m\omega^2}{k} \text{ and } \frac{c\omega}{k}\right)$$

$$\text{and } \varphi = \Phi\left(\frac{m\omega^2}{k} \text{ and } \frac{c\omega}{k}\right)$$

We could make the coefficient of any term as 1. It really does not matter. And if we do this in this equation there are now 3 groups of parameters. These two are made up of the values of the independent parameters and this the last one has x_o, the amplitude of vibrations as the dependent parameter. And from this it is clear that the value of $\frac{x_o}{\frac{F_o}{k}}$, the dependent parameter should be a function of only $\frac{m\omega^2}{k}$ and $\frac{c\omega}{k}$ parameters, which is written here like this: $\frac{x_o}{\frac{F_o}{k}} = X\left(\frac{m\omega^2}{k}\ and\ \frac{c\omega}{k}\right)$. If in two systems the values of $\frac{m\omega^2}{k}$ and $\frac{c\omega}{k}$ are equal, that is, the value of $\frac{m\omega^2}{k}$ in one system is equal to the value of this in the other system, and the value of $\frac{c\omega}{k}$ in one system is equal to the value of this in the second system, then the value of $\frac{x_o}{\frac{F_o}{k}}$ the left hand side should be identical. Similarly, for the phase angle φ. This equation is easy to solve analytically and the solutions are these. You do not need to worry about how they are obtained, but we will see that $\frac{x_o}{\frac{F_o}{k}}$ is a function only of m$\frac{m\omega^2}{k}$ and $\frac{c\omega}{k}$ occurring as they do. Note that each of these parameters is non-dimensional with no dimensions at all; no units.

It is conventional to use ω_n for $\sqrt{k/m}$. ω_n is the natural frequency or the resonant frequency of a spring mass system with no damping, and that in high school was obtained as $\sqrt{k/m}$. So, that $\frac{m\omega^2}{k}$ can be written as $\left(\frac{\omega}{\omega_n}\right)^2$. ω_n is the resonant frequency, ω is the actual forcing frequency. And we use ζ for $c/2m\omega_n$. So, that the parameter $\frac{c\omega}{k} = 2\zeta\omega/\omega_n$ and we get this expression for the non dimensional dependent parameters denoting the amplitude of oscillations, and this for the phase of oscillation.

Solution of the forced vibration problem

$$\frac{x_o}{F_o/k} = \frac{1}{\sqrt{\left(1-\frac{m\omega^2}{k}\right)^2 + \left(\frac{c\omega}{k}\right)^2}} \quad \text{and} \quad \varphi = \tan^{-1}\left(\frac{\frac{c\omega}{k}}{1-\frac{m\omega^2}{k}}\right)$$

It is conventional to use ω_n for $\sqrt{k/m}$ so that $\frac{m\omega^2}{k} = \left(\frac{\omega}{\omega_n}\right)^2$ and ζ for $c/2m\omega_n$ so that $\frac{c\omega}{k} = 2\zeta\omega/\omega_n$, and we get

$$\frac{x_o}{F_o/k} = \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2}} \quad \text{and} \quad \varphi = \tan^{-1}\left(\frac{\frac{2\zeta\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right)$$

Same equations. You notice that $\frac{m\omega^2}{k}$ or $\frac{\omega}{\omega_n}$ is non dimensional. We give it a name Π_1. The first Pi number, and $\zeta = c/2m\omega_n$ is the second Pi number. These can be used at modeling rules.

Two systems would have the same non dimensional results if the values of Π_1 and Π_2 of one system is the same as the values of Π_1 and Π_2 for the second system. The dependent parameter $\frac{x_o}{F_o/k}$ is a function only of Π_1 and Π_2. So, if two systems have the same values of Π_1 and Π_2 then the value of $\frac{x_o}{F_o/k}$ should be identical in the two cases and that can be used as the prediction rule. So, we can choose the value of m, k, c and omega of the model system in such a manner that the values of the two pi's, Π_1 and Π_2, are identical and then do the experiment on the model system. Measure the value of $\frac{x_o}{F_o/k}$ in the model system and this exactly would be the value of $\frac{x_o}{F_o/k}$ on the prototype system.

Solution of the forced vibration problem

$$\frac{x_o}{F_o/k} = \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2}} \quad \text{and} \quad \varphi = \tan^{-1}\left(\frac{\frac{2\zeta\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right)$$

$$\Pi_1 = \frac{m\omega^2}{k} = \frac{\omega}{\omega_n} \quad \text{with } \omega_n = \sqrt{k/m}$$

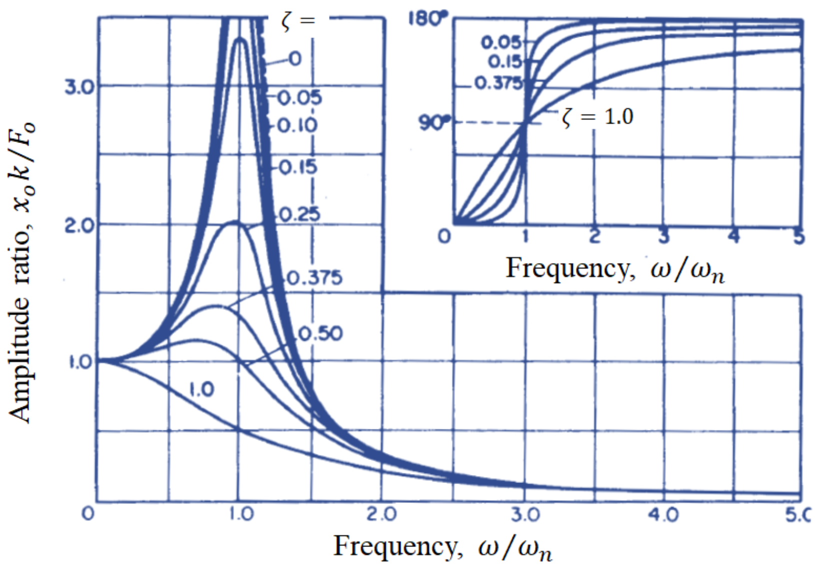
$$\Pi_2 = \zeta = c/2m\omega_n$$

Modelling rules

$$\frac{x_o}{F_o/k} = \mathcal{F}\left(\frac{\omega}{\omega_n}, \zeta\right) \text{ and } \varphi = \Phi\left(\frac{\omega}{\omega_n}, \zeta\right)$$

Prediction rules

In sense we know the value of $\frac{F_o}{k}$ for the prototype system, we can predict the value of x_o similarly of φ of the prototype system. This in essence is the way the modeling works. We establish the modeling rules. The number of pi's would be lesser or fewer than the numbers of independent parameters, the uniquity parameters. So, it would be possible to match the pi's without matching all the values of independent parameters.



The results have been plotted in this picture. This first graph on the left shows how the amplitude changes with the frequency omega by omega n and the inset graph shows how the phase angle varies with the frequency ratio omega by omega n for different values of zeta the damping parameter. These two graphs should be sufficient to predict the performance of any system of mass, dashboard, spring vibrating system. Given any values of the parameters omega k m and c, we can calculate the frequency ratio $\frac{\omega}{\omega_n}$ and zeta and read from this graph whatever the values of the non dimensional amplitude should be and what should be the phase angle. This is only possible through the concept of similitude.

Even if we are not doing experiments and we are solving this problem numerically, we could solve it for different values of these uniquity parameters in non dimensional forms. The 2 pi numbers obtain the results, plot this graphs and use them for predicting the performance of any vibrating system.

Another example: Transient conduction in an infinite slab

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

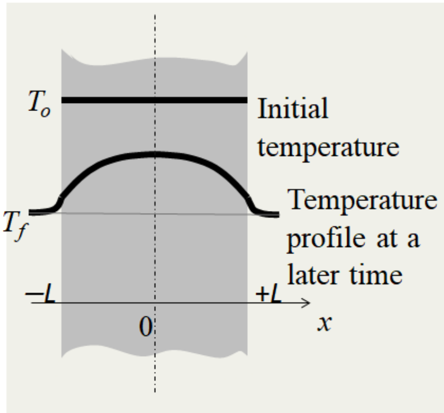
The equation is to be solved subject to the following initial and boundary conditions:

$T(x,t) = T_o$ for $t \le 0$

$-k\frac{\partial T}{\partial x} = q_x = h\left(T_f - T\right)$, at $x = -L$ for $t \ge 0$

and

$-k\frac{\partial T}{\partial x} = q_x = h\left(T - T_f\right)$, at $x = +L$ for $t \ge 0$

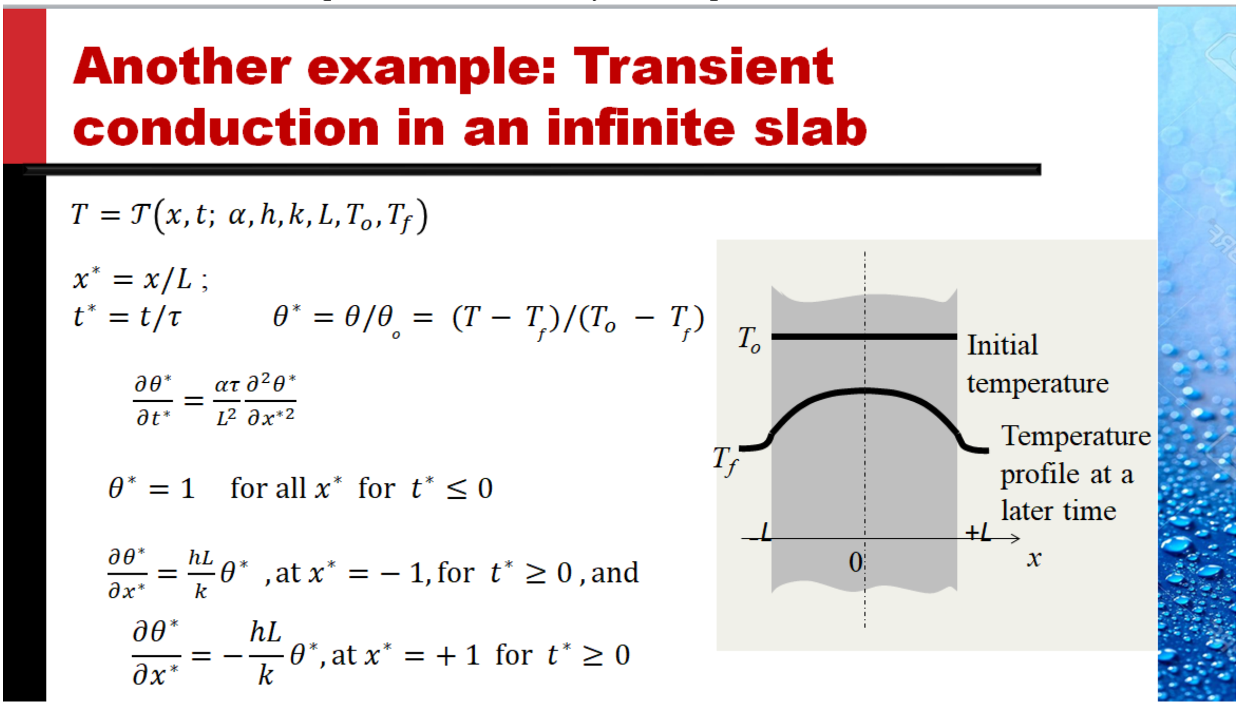


Let us do another example, and this example is from heat transfer. Let us discuss the problem or the transient conduction in an infinite slab. Let us have an infinite slab of thickness 2L and set up the coordinate system so that the left face of the slab is at $x = - L$ and the right phase is at $x = L$.

Let $\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2}$ be the initial temperature of the slab and the time $t = 0$, this is immersed in a fluid with a temperature T_f. There is a heat transfer coefficient h for the convection at the two faces of the slab, this conductivity k of the slab material, the density ρ and the specific heat c_p of the slab material. This is an elementary problem of heat transfer and after we submerged it, the temperature variations at any time t would look something like this. This profile will change with the time and after an infinite time, the temperature throughout the slab would again be constant at value T_f. To predict the value of temperature at any x at any time t, we write the governing equation.

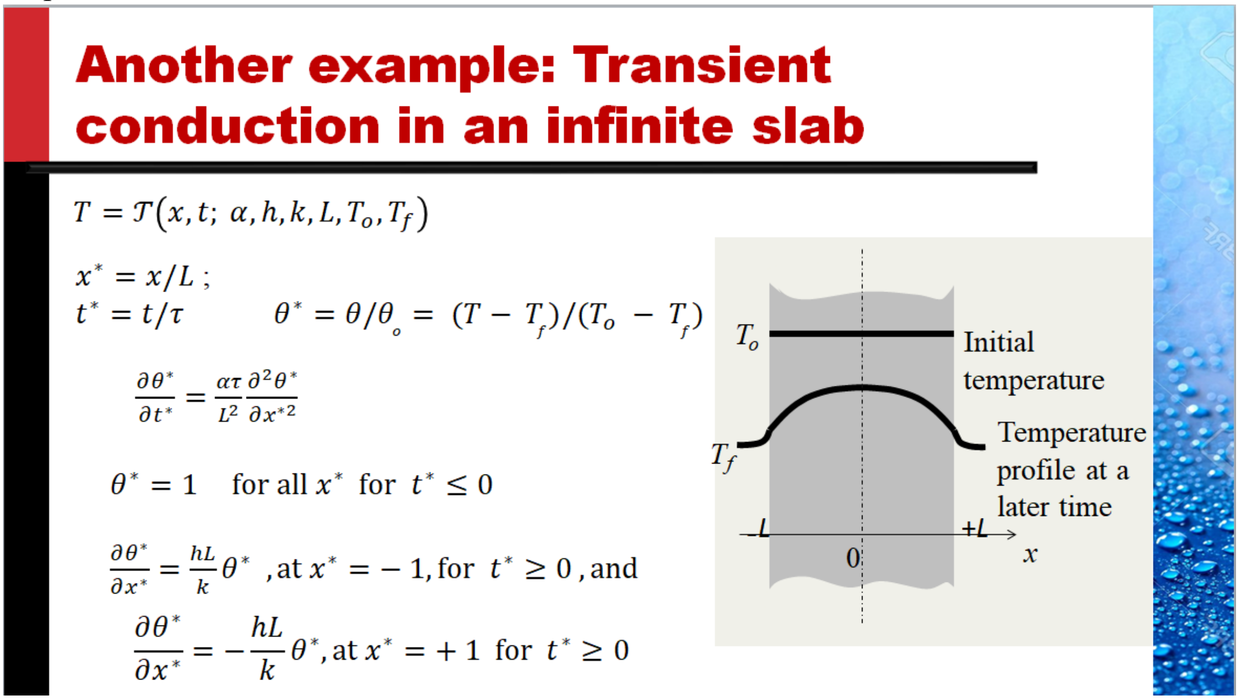
The governing equation is rather simple, $\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2}$, where α is the thermal diffusivity, $\frac{k}{\rho c_p}$, a material property of the slab material. This equation is to be solved subject to the following initial and boundary conditions. The initial condition T temperature at any point x in the slab is equal to T_o for time less than 0, for any time less than equal to 0, the temperature T_o. Then the time t when the heat transfer to the ambient fluid is permitted, we have the heat flux conditions at the two ends: $-k\frac{\partial T}{\partial x}$ is the heat transfer within the slab material at x is equal to -L and this should be equal to $\left(T_f - T\right)$. And similarly for the right face of the slab.

One initial condition and two boundary conditions. So, temperature T would be a function of x and t, the independent variables and alpha h K L T_o and T_f, the list of independent parameters, the uniquity parameters. Let us normalize the variables. Independent variables x and t: $x^* = x/L$, where L is the characteristic length dimension of the slab, half the thickness of the slab, and $t^* = t/\tau$, where τ is some characteristic time. We do not know what it is. We will talk about this a little later, but as of now let us take tau as yet undetermined. We will select a proper value towards the end. In non dimensional time, non dimensional temperature when we talk of temperature, there is a simplifying feature of this problem. Of any heat transfer problem, that we must exploit here. And that feature is that it is not the actual temperature that matters in a heat transfer problem, but it is only the temperature differences that matter.



The heat flux depends upon the temperature differences rather than temperature. So, the original temperature could be anything in Fahrenheit scale, we use the ice pot as minus 32, as 32 degrees Fahrenheit, and we use the same in centigrade scale as 0 degree Celsius. The starting point the 0 of the scale does not matter, it is only the temperature difference. So, let us not talk of temperature, let us introduce theta, the temperature difference from a reference. We could choose any reference, T_f the temperature of the surrounding bath surrounding fluid is a convenient reference.

So, we define a temperature difference as $(T - T_f)$and we normalize it by dividing it by characteristic temperature difference. Similarly in this problem $(T_o - T_f)$does characterize the temperature difference for this problem. So, we define a non dimensional temperature difference θ^*, a normalized temperature difference theta star, as $(T - T_f)/(T_o - T_f)$. And if we do this then the governing equations is simplified to $\frac{\partial\theta^*}{\partial t^*} = \frac{\alpha\tau}{L^2}\frac{\partial^2\theta^*}{\partial x^{*2}}$. Note the tau is as yet unspecified.



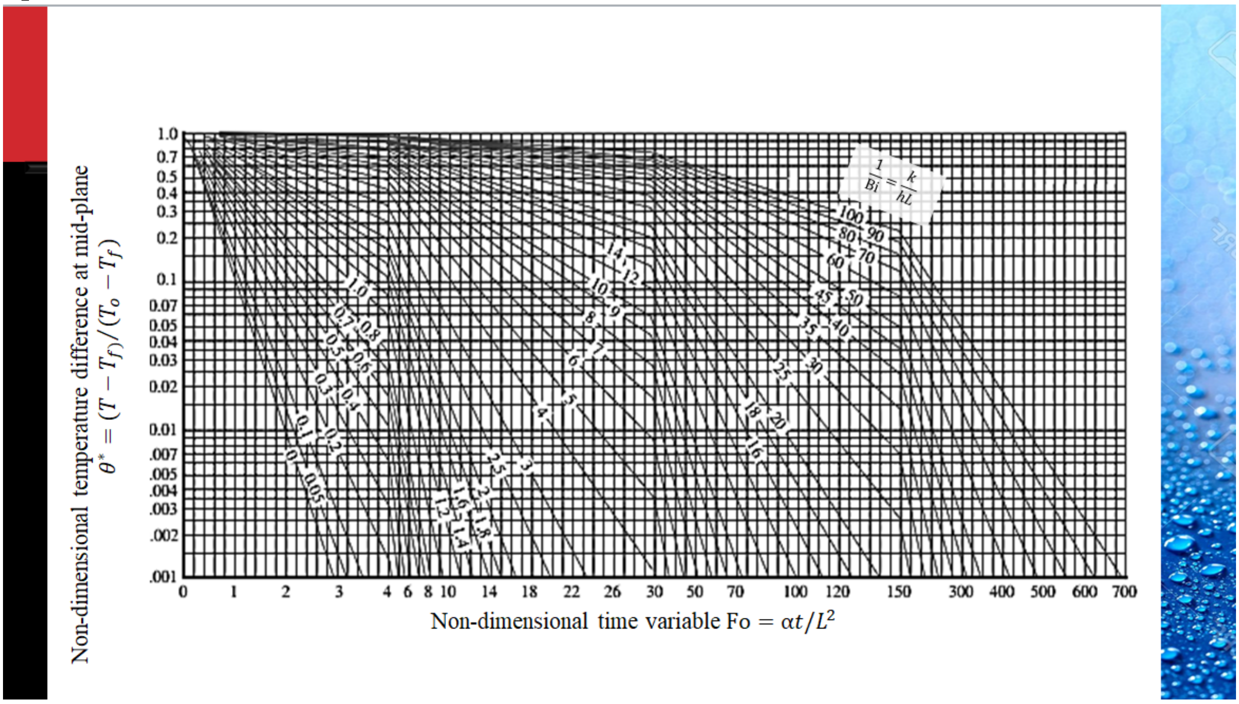
The boundary conditions change to $\theta^* = 1$ $for\ all\ x^*\ for\ t^* \le 0$ and the conditions at the interface boundary conditions $\frac{\partial\theta^*}{\partial x^*} = \frac{hL}{k}\theta^*$, $at\ x^* = -1$, $for\ t^* \ge 0$ and similarly for the right boundary condition at $x^* = +1$. Note that this definition the problem now has only only 2 groups of parameters. So, there are only 2 groups of parameters. If there are 2 geometrically similar situations and the values of these 2 parameters $\frac{\alpha\tau}{L^2}$ and $\frac{hL}{k}$ match in these 2 problems, then the non dimensional θ^* as a function of x^* and t^* would be identical in the 2 situations. And therefore, the problem now reduces to $\theta^* = F\left(x^*, t^*; \frac{hL}{k}, \frac{\alpha\tau}{L^2}\right)$.

From the original list of 5 parameters now we need to match only 2 parameters to obtain similarity. $\frac{hL}{k}$ has been given a name Biot number after the scientist Jean Biot a French scientist who did work related to conduction, unsteady conduction, and this is a number that has been named after him to honor him much later. Further, let us talk about tau as yet undefined. In this problem of unsteady conduction both sides of this equation, the governing equation in the normalized form, should be significant. Since we have normalized our variables, we expect the derivatives $\frac{\partial\theta^*}{\partial t^*}$ as well as $\frac{\partial^2\theta^*}{\partial x^{*2}}$ of order 1.

They are normalized. So, both terms would matter only if the coefficient $\frac{\alpha\tau}{L^2}$ is of order 1. And since τ is so far arbitrary in this problem we can set it equal to L squared by alpha without any loss of generality. $\frac{\alpha\tau}{L^2}$ is arbitrary. If I give it a value of L squared by alpha then $\frac{\alpha\tau}{L^2}$

becomes 1. And since it becomes 1, it drops out of the equation and we get theta star as a function of x^*, t^* and only 1 pi number $\frac{hL}{k}$ with t^* is equal to t/τ now becomes $\alpha t/L^2$.

This is really a variable, but is one of the few non dimensional group that is given a name. This is name named as Fourier number and abbreviated as Fo capital F and o. After the famous scientist Joseph Fourier would derived the equation that governs conduction. And the result is so powerful that we can solve all problems in unsteady heat conduction in this lab by a set of two graphs. You see in this problem theta star is a function of x star and t star with h L by k as a parameter.



The $\frac{hL}{k}$ is Biot number. With $\frac{hL}{k}$ as a parameter. There are two variables. A scientist by the name Heisler devised a scheme in which he said we will solve the first problem at x^* is equal to 0, that is, at the center line of the slab, and will produce the result of θ^* as a function of t^*, that is, the Fourier number with $\frac{hL}{k}$ as a parameter. This graph: you enter at a non dimensional time let us say 22 and go up to the value of bio number, 1 over bio number, say 5, and from this we read the normalized temperature difference here. This is for temperature at the center line of the slab.

I will not explain it further, but this graph is supplemented with what is called a position correction chart. From this we can, once we know the temperature at the center line, we can predict the temperature at any other location. You can refer to any textbook on heat transfer to understand how this graph is used.

Thank you.