

Similitude And Approximations In Engineering,
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Week - 01
Lecture – 03

Welcome back. In today's lecture, I will introduce the nature of the derived quantities and we will try to develop an understanding of the nature of dimensions of physical quantities. In describing physical things and events quantitatively, we refer to numerical values of base quantities and also introduce numbers derived by inserting these values into certain mathematical formulae. For example, if L is the distance travelled in time t, then we can define a quantity V as L divided by t. This is akin to velocities. If we introduce the numerical value of L and a numerical value of t in there, then it will give you a value of V in some units.

Derived quantities, dimensions, and dimensionless quantities

In describing physical things and events quantitatively, we refer to *numerical values* of base quantities, and also introduce numbers derived by inserting these values into certain mathematical formulas.

$$V = Lt^{-1}$$

Similarly, once we have established V from there, we can introduce the value of V in the second equation along with that of mass m to get a measure of the kinetic energy. These are called the derived quantities of the first kind. We will deal with the derived quantities of the second kind a little later. Now, there is a famous rule or famous principle due to Bridgman which says a number Q obtained by inserting the numerical values of the base quantities into a formula is a physical quantity if and only if the ratio of any two samples of it remains constant when the base unit sizes are changed.

Power law

Bridgman's principle of absolute significance of relative magnitude

A number Q obtained by inserting the numerical values of the base quantities into a formula is a physical quantity if the ratio of any two samples of it remains constant when base unit sizes are changed.

Bridgman went on to show that a monomial formula satisfies the principle of absolute significance of relative magnitude only if it has the power-law form

$$Q = \alpha A^a B^b C^c \dots$$

The principle of absolute significance of the relative magnitude. If we change the unit size of the base quantities, the value Q obtained would be different. But if two values of Q 's are obtained, first by one set of units and then by a second set of units and if $\frac{Q_1}{Q_2}$ remains unchanged when the units change. That means there is an absolute significance of the relative magnitude, and only in such a case would Q be a physical quantity. Bridgman went on to show that a monomial formula satisfies the principle absolute significance of relative magnitude, only if it is a power law form.

So, if Q is a function of quantities A , B and C , then Q must be a form $Q = \alpha A^a B^b C^c \dots$ etcetera. Now, a derived quantity does not necessarily represent something tangible in the same sense as a base quantity though it may. The velocity and the kinetic energy that we defined in the last slides are such quantities which represent something tangible. But the square of time for example, is a derived quantity because it has a required power law form, but we cannot point to any physical thing that is a square of time. Square of time is not physical.

Each type of base quantity has by definition its own dimension. If A is a numerical value of length we say it is a dimension of length and we write a in square brackets equal to L and read it like dimensions of A is L . By this we mean simply that if the length size is increased by a factor n the numerical value A will increase by a factor $1/n$. Consider a quantity defined by a

formula $Q = L_1^{l_1} L_2^{l_2} T_1^{t_1} T_2^{t_2} M_1^{m_1} M_2^{m_2} \dots$ etcetera. Let us change the units of the base quantities by a factor of n_L , n_T , and n_M . L , T and M are the base quantities, and we are changing the units by a factors n_L , n_T , and n_M . Then the measure of Q changes to $Q' = n^{-1} Q$ where

$n = (n_L)^{\sum l_i} (n_T)^{\sum t_i} (n_M)^{\sum m_i}$. This implies that q transforms like the numerical values of the base

quantities with a unit whose size is proportional to $(n_L)^{\sum l_i}$, $(n_T)^{\sum t_i}$ and $(n_M)^{\sum m_i}$. Where L , m

and T represent the sizes of length, mass and time units respectively. We therefore, say that the derived quantity Q has the dimension L raised to power summation $\sum l_i$, T raised to power $\sum t_i$ and m raised to power $\sum m_i$.

Dimensions

Each type of base quantity has, by definition, its own dimension. If A is the numerical value of a length, we say it *has the dimension of length*, and write this as $[A] = L$

By this we mean simply that if the length unit size is increased by a factor n , the numerical value A will increase by a factor $1/n$.

Consider a quantity defined by the formula $Q = L_1^{l_1} L_2^{l_2} T_1^{t_1} T_2^{t_2} M_1^{m_1} M_2^{m_2} \dots$

Let us change the units of base quantities by factors n_L , n_T , and n_M ,

Then the measure of Q changes to $Q' = n^{-1}Q$ with

$$n = (n_L)^{\sum l_i} (n_T)^{\sum t_i} (n_M)^{\sum m_i}$$

This follows directly from requiring that the relative values have absolute significance. Note that the dimensional quantity is simply a formalic indication of how the quantity's numerical value transforms when the size of the base units are changed. There is nothing fundamental to dimensions. Repeat, dimensions are not fundamental to physical quantities. It is how we define the base units.

We will elaborate on this again a little later. A derived quantity's dimension follows from its defining equation in the manner in which they are defined. In terms of the base quantities that are used to defining it. We shall see that the quantity's dimension depend upon the choice of the system or units and is, therefore, under the control of the observer rather than inherent attribute of that quantity. Let us summarize what we have learnt about derived quantities so far.

The dimension of any derived physical quantity is a product of the powers of the base quantity dimensions. Groups of the derived quantities with the same dimensions are derived quantities of the same dimension. Products and ratio of derived quantities are also derived quantities with dimension which are usually different from the original quantities. Third, all derived quantities with the same dimension change their values by the same factor when the sizes of the base units are changed. A derived quantity is dimensionless if the numerical values remains invariant when the base units are changed.

And special functions such as logarithmic, exponential, trigonometric, etcetera, of dimensional derived quantities are in general not derived quantities. In fact, we will learn that we can take logarithmic function and exponential function and trigonometric function only of non-dimensional quantities. Let us now discuss the various systems of dimensions that are used in mechanics. One of the system that you are all familiar with is the SI system units which is

rather international and followed quite universally except for a few countries like US and UK which is switching back to their old system recently. It is based on meter, kilogram and second.

There is a bar of metal sitting in a Paris museum which is defined as a length meter. Similarly, a slug of metal which is defined to have a mass of 1 kilogram. The second is defined in a little more complicated fashion, we will not go into it. This forms an MLT system that is the base quantities are mass, length and time. In this system, the unit of force is a derived unit and it follows from the Newton's second law.

Force is proportional to mass times acceleration which results in $F = kma$ where k is a constant. And in the SI system, k is taken to be non-dimensional of value 1. So, the $F = ma$. The unit of mass is kilogram as a dimension of mass. Acceleration has a dimension of velocity divided by time which is length divided by 2 dimensions of time.

So, that the dimension of force are MLT^{-2} . The unit of force is given the name Newton and this is the force that produces an acceleration of 1 meter per second square or the mass of 1 kilogram. But this is not the only system, the old British imperial system where the unit of force is the gravitational force of earth that acts on a unit mass. Thus, we define a pound force as the force by which the earth attracts a pound mass. So, we use pound mass abbreviated lbm and the pound force abbreviated as lbf, foot and second system of units.

The dimension system was FMLT force, mass, length and time system. This does cause a bit of bother. The Newton law in this system must be written F is equal to $K m a$, where K is dimensional. It is a constant and it is a universal constant in the sense that it is a same value in two different experiments. In any experiment it has the same value.

In the British system its value is $\frac{1}{32.2} \text{ lbf/lbm}\cdot\text{ft}\cdot\text{sec}^{-2}$. K is routinely represented by $1/g_c$, where $g_c = 32.2 \text{ lbm}\cdot\text{ft}\cdot\text{sec}^{-2}/\text{lbf}$. And the Newton second law is written as $F = \frac{1}{g_c} ma$.

Not a very clean representation. To reduce the inconvenience of using this dimensional constant, many trades have introduced a new unit of mass as a derived units from Newton law. So that the base units are treated as force, length and time and the mass is now a derived unit. A new unit slug was defined and slug was the quantity of mass that accelerates with 1 ft/sec^2 , where a unit force, pound force in this case is applied to it. Since a pound mass accelerates with 32.2 ft/sec^2 , when a pound force is applied to it, a mass that accelerates only 1 ft/sec^2 must be 32.2 times pound mass. Clearly then one slug is 32.2 pound mass. I will not elaborate on this any further, but this is a system known as F L T system.

The three common systems of units in mechanics

	Type 1	Type 2	Type 3
	M L T	F L T	F M L T
Velocity	LT^{-1}	LT^{-1}	LT^{-1}
Acceleration	LT^{-2}	LT^{-2}	LT^{-2}
Mass	M	$FL^{-1}T^2$	M
Area	L^2	L^2	L^2
Force	MLT^{-2}	F	F
Work	ML^2T^{-2}	FL	FL
Viscosity	$ML^{-1}T^{-1}$	$FL^{-2}T$	$FL^{-2}T$
Stress	$ML^{-1}T^{-2}$	FL^{-2}	FL^{-2}

This table shows you the various quantities that we use in mechanics and the units or their dimension in the three types of base systems that we defined. M L T, SI system is one example of this. F L T, where the mass is a derived quantity with a dimension $F L^{-1} T^2$ and the third is the F M L T system. They are old British imperial systems. All these dimensions you can work out very easily.

For example, work is force times the distance moved. So the dimensions of work would be the dimension of force time length. So $M L T^{-2}$ multiplied by L in type 1 system, and F times L in type 2 system, and also the same in type 3 system.

Let us now elaborate a bit on the dimensions of thermal quantities. This again presents us various options.

Dimensions of Thermal Quantities

There are various ways of looking at heat:

As a calorimetric quantity: $Q = mC\Delta\theta$

We can define a dimensional system as MLTH Θ , all independent dimensions. We recognize heat (thermal energy) and temperature as independent dimensions

Then	M	L	T	H	Θ
Specific heat, C: $Q = mC\Delta\theta$	-1			1	-1
Conductivity, k: $q = -k dT/dx$		-1	-1	1	-1
Joule's constant, J	1	2	-2	-1	
Gas constant, R: $pV = nRT$	1	2	-2		-1

First is the option with the mechanics and thermal quantities are quite separate. There is no conversion of work into heat energy. Heat is defined purely as a calorimetric quantity and the relation is Q is equal to $m C \Delta\theta$. Mass times the specific heat times the change in temperature is the quantity of heat that gets transferred to a body, where its temperature rises by $\Delta\theta$. We can define a system in which heat is given as a separate base quantity as well as the temperature or rather temperature difference.

We elaborate on this a little later: why temperature difference and not just the temperature. Actually it is quite simple. In all heat transfer problems the actual value of temperature is quite meaningless. It is only the temperature difference that matters, the change in temperature that matters. Heat, the energy that flows because of temperature difference, is directly related to the temperature difference.

So we use $\Delta\theta$, and $\Delta\theta$ is the temperature which we give a dimension capital theta. So now in a system we have 5 dimensions. So we have $m L T$ for the mechanics part and H and Θ for the thermal heat transfer part. They are all independent dimensions. We recognize heat as the thermal energy and the temperature as independent dimensions.

Then the specific heat would have a dimension of heat energy per unit mass per unit temperature difference. So $M^{-1} H^1$ and Θ^{-1} . Conductivity: the dimension of conductivity can be obtained from its defining equation. And the defining equation of conductivity is q , that is, the heat flux, heat transfer per unit area per unit time.

H divided by L^2 divided by T . Heat per unit area per unit time is equal to $-k \frac{dT}{dx}$, the temperature gradient. From this you can work out quite easily the dimension of q must be $L^{-1} T^{-1} H$ and Θ^{-1} . In this the Joules constant, if we have to equate the work done to the heat transfer dimensionally, then we must introduce a joules constant: Q times J is equal to W . W is work, work is $M L^2 T^{-2}$ divided by H . So the dimension of J is equal to be $M L^2 T^{-2} H^{-1}$. Similarly, the dimension of gas constant R divided by $p v$ is equal to $N R T$, N is the number of moles, is obtained as this.

Now we can complicate this a little bit. If we invoke additionally the first law of thermodynamics that is the work is like J times q . So dimensionally the energy in both forms, W and Q , have the same dimensions: $M L^2 T^{-2}$. So H is no longer treated as a base quantity and J is treated as a pure number, dimensionless. Then we have four dimensions $m L T$ and temperature theta. Now the dimensions of specific heat are obtained from the relation $Q = mC\Delta\theta$ as $L^2 T^{-2} \theta^{-1}$.

Conductivity would change as $M L T^{-3} \theta^{-1}$. The joules constant is now dimensionless by definition, and the gas constant would have dimension of 1, 2, minus 2, minus 1. I would recommend strongly that you work out the dimensions of conductivity and gas constant on your own.

Now we can further modify this and invoke in addition the gas kinetic theory and recognize temperature as related to the average kinetic energy of gas molecules. So temperature is now a derived quantity with the dimensions of energy kinetic energy of gas molecules and the dimension of energy in the $m L T$ system are $m L^2 T^{-2}$.

Dimensions of Thermal Quantities

We can in, addition, invoke gas kinetic theory and recognize temperature as related to the average kinetic energy of the gas molecules:

$$\text{Then, } [\theta] = \text{ML}^2\text{T}^{-2}$$

	M	L	T
Specific heat, C: $Q = mC\Delta\theta$	1		
Conductivity, k: $q = -k dT/dx$		-1	-1
Joule's constant, J			
Gas constant, R: $pV = nRT$			

So with this dimension of temperature the base quantities are reduced to 3: M L and T, and the dimensions of various thermal quantities are given by this temperature. Here both the joules constant and the gas constant are treated as non dimensional, and the specific heat has the units of only mass conductivity has L minus 1 T minus 1. We have spent time on this thermal quantities or the dimensions of this thermal quantities in preparation for the dimensional analysis we will carry out on thermal quantities a later in the course. So in three different systems the dimensions of various thermal quantities that we have obtained so far are consolidated in this table. Note that we can have as many as 5 or 4 or 3 dimensions to the thermal quantities.

Let us next talk about the nature of physical equations. In quantitative analysis of sciences one seeks mathematical relationships between the numerical values of the physical quantities that describe the event. What is the velocity when a car travels 30 kilometers in 1 hour? What is the kinetic energy of a body 1 kilogram moving at 1 meters per second? So, you want a relationship between the numerical values of the various physical quantities. We are not however interested in just any relationship that applies between the values of the physical quantities. Size is considered only with expressing a physical relationship between one quantity and a set of others that are relevant.

The resulting relation is termed as a physical equation. Nature is indifferent to the arbitrary choice of the base units. We are interested therefore only in numerical relationship that remain truly independent of the choice of base units, as well as, of the sizes of the base units. So, the relationship that we obtain between acceleration mass and force should be valid in any system of base units. Whether foot pound and second system or meter kilogram and second system.

Let in a specified physical event the numerical value Q_o of a physical quantity is determined by the numerical values of a set of four physical quantities $Q_1, Q_2, Q_3, \text{ and } Q_4$, that is, $Q_o = F(Q_1, Q_2, Q_3, \text{ and } Q_4)$. The principle of absolute significance of relative values tells us that the relationship implied by this equation can be physically meaningful or physically

relevant only if Q naught and F change by the same factor when the magnitude of any of the basic unit is changed. And that will happen only if the physical equation is dimensionally homogeneous because the value of any quantity changes as the powers of the change of units raise to power the dimensions of that unit. So, the dimension homogeneity imposes the following constraint on any mathematical representation of a relationship.

First, both sides of the equation must have the same dimensions Q must have the same dimension as $Q_1^{n_1} Q_2^{n_2} Q_3^{n_3} Q_4^{n_4}$, if that is the relation between Q_o and the various Q_1, Q_2, Q_3 and Q_4 . Second, wherever a sum of quantities appear in F, that is, F is a polynomial, there are more than one terms added or subtracted in F then each of this term must have the same dimensions. Also all arguments of any exponential logarithmic trigonometric or other special function that appear in F, the relationship, must be dimensionless. Taking a log of a dimensional quantity is meaningless because it will not be satisfying the property of the absolute significance of relative magnitude. Similarly, for trigonometric functions and logarithmic functions.

For example, let us say a physical equation is represented by $A = Be^{-C} + \frac{D_1 + D_2}{E} - F$. This can be a valid equation valid physical equation representing the results of a physical phenomena only if it is dimensionally homogeneous. And the dimensional homogeneity requires that C must be dimensionless because we take the exponential of C. We cannot take exponential of dimensional quantities and have a meaningful dimension. B and F must have the same dimension as A since C is dimensionless e raise to power minus C is also dimensionless.

So, A and B must have the same dimension for homogeneity and similarly F must have the same dimensions as A or B. D_1 and D_2 are added. So, they must have the same dimensions. In addition D_1 by E or D_2 by E must have the same dimensions and dimension as of A, B and F. Only if all these conditions are satisfied would this equation be dimensionally homogeneous and could be a candidate for being a physical equation.

Let us summarize what we have done today. A base quantity is a property that is defined in physical terms by two operations a comparison operation and an addition operation. Base quantities are properties for which the following concepts are defined in terms of physical operations. Equality, addition, subtraction, multiplication by a pure number and division by a pure number. Not defined in terms of physical operations are product, ratio, power, logarithmic, exponential, trigonometric and other functions of physical quantity.

These are not defined. Only things that define equality, addition, subtraction and multiplication and division by a pure number. Third, a base quantity can be measured in terms of an arbitrarily chosen units of its own kind. You can compare only with a similar quantity and a numerical value. We deal with these numerical values in mathematical equation in the physical equations. A derived quantity the first kind is a product of various powers or numerical values of base quantities.

A derived quantity is defined in terms of the numerical value which depends on the base size unit and does not necessarily have a tangible physical representation. It should have, but it is not necessary that always have. I mean does not have a tangible physical representation, it is not a physical quantity. The dimensions of a physical quantity whether base or derived is a formula that defines how the numerical value of the quantity changes when the base unit sizes

are changed. So, if the base unit sizes change by a factor n then the quantity changes as $1/n$ if the dimension is 1. If the dimension is 2 then it changes like $1/n^2$ and so on. This same quantity may have different dimensions in different systems of units and quantities that are clearly physically different may have the same units. Like the torque and the work done, they are two entirely different physical quantities, but have the same dimensions in MLT systems. So, they are two entirely different system of units like our SI system of units. Next, relationships between physical quantities may be represented by mathematical relationships between their numerical values leading to the concept of physical equations.

A mathematical equation that correctly describes a physical relationship between quantities is dimensionally homogeneous. Such equations remain valid when base unit sizes are changed arbitrarily. The categorization of physical quantities as either base or derived is to some extent arbitrary. If a particular base quantity turns out to be uniquely related to some other base quantity via some universal law, then we can, if we so desire, use the law to redefine that quantity as a derived quantity of the second kind. Like we have derived mass as a derived quantity from the Newton's law in the British system using the slug as a mass unit, and the magnitude of that depends upon the units chosen for the other quantities.

All base quantities that are transformed into derived quantities in this way retain their original physical identities. But the numerical values are measured in terms of the remaining base quantities either directly via defining equation or indirectly by using a unit that is derivable from the remaining base units. A system of unit is defined by the general the base quantities, the units and the derived quantities each with either a defining equation or the form of the physical law that has been used to cast the quantity into the derived category. Both the type and number of base quantities are open to choice. This choice is what we are going to exploit in the remaining part of this course. Thank you very much.