Similitude And Approximations In Engineering,
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Week - 07
Lecture - 24

Welcome back. In today's lecture, we will start a new topic: Relaxation of the Modeling Rules. We have seen earlier that as we add laws to any problem, for each law a new pi number is added. Are there any situations where the number of laws from which the pi numbers are obtained are too many to be satisfied? In principle, as we add a law to be modeled, we add at least one uniquity parameter to be determined and that should not cause any problem. But in actual practice, the situation is a little more complicated. The uniquity parameters are usually material properties, and it is not easy to tailor material properties to our requirements.
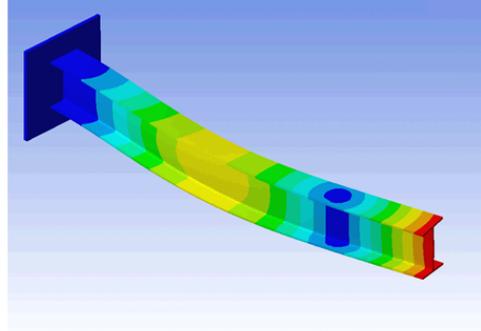
Too many pi numbers?

To illustrate the point, let us consider a beam vibrating under the action of an oscillating force. There are three forces (three laws) we should worry about:

Inertia: $F_i \sim \rho L^3 \cdot \frac{L}{T^2} \rightarrow \Pi_1 = \frac{F_i T^2}{\rho L^4}$;

Elasticity: $F_e \sim E\varepsilon L^2 \rightarrow \Pi_2 = \frac{F_e}{E\varepsilon L^2}$;

Internal damping: energy loss per unit volume $\sim c\sigma^3 \rightarrow FL/L^3 \sim cF^3/L^6 \rightarrow \Pi_3 = \frac{cF^2}{L^4}$

$\delta_{tip} = \mathcal{F}(t; F, L, \rho, E, c)$

To illustrate the point, let us consider a beam vibrating under the action of an oscillating force. There are three forces, basically three laws that we should worry about. The law of inertia, the initial force is like mass into acceleration and that gives you a $\Pi_1 = \frac{F_i T^2}{\rho L^4}$. Another is we have to worry about the elasticity of the material. Elastic force would be $F_e \sim E\varepsilon L^2$. So, that gives you a $\Pi_2 = \frac{F_e}{E\varepsilon L^2}$. The third force that we worry about is the force of internal damping and we write in terms of energy loss per unit volume and this is given a $c\sigma^3$ where c is a damping coefficient and this translates into a pi number $\Pi_3 = \frac{cF^2}{L^4}$.

The deflection of the tip or rather the amplitude of the reflection of the tip is a function of F, L, rho, E, and c, and the actual deflection is a function of time, the independent variables and the independent parameters F, L, rho, E and C. The three pi numbers that we obtained are these.
$\delta_{tip} = F(t; F, L, \rho, E, c)$

Using scale factors for F, L, and t, we can calculate the scale factors for density, elasticity and the damping coefficient. k_ρ, the scale factor for density turns out to be $\dfrac{k_F k_T^2}{k_L^4}$4 from the first one. For E, the elasticity, the scale factor is $k_E = k_F/k_L^2$, and for the damping $k_c = k_L^4/k_F^2$. If we choose the model material to be same as the prototype material, the three scale factors are one, all of them are one, and then $k_F = k_L^2$ and $k_T = k_L$.

Too many pi numbers?

But suppose the self-weight of the beam was also significant.

If gravity was added, $F_g \sim \rho L^3 g \rightarrow \Pi_4 = \dfrac{F_g}{\rho L^3 g}$ and for $k_g = 1$, we get $k_F = k_L^3$

All conditions can be satisfied only if $k_T = k_L = k_F = 1$!

But suppose the cell weight of the beam was also significant then the weight would come into picture and the gravity force F_g which is $F_g \sim \rho L^3 g$. This gives $\Pi_4 = \dfrac{F_g}{\rho L^3 g}$, and for $k_g = 1$, we get $k_F = k_L^3$.

Now these conditions can all be satisfied only if $k_T = k_L = k_F = 1$!. It does not give us any freedom to do the experiments. So, what do we do? These considerations lead us to an important point in scale modelling: the more physical laws are involved in the phenomena the more pi numbers are to be satisfied, and the less freedom exist in choosing the material and or the scale factors.

Fortunately conflicting claims on scale factors can frequently be resolved by what we call relaxation. It means the resolution of scaling conflict by using all prior knowledge of the phenomena to be scaled. Relaxations might consist for instance of neglecting less important laws or of infusing the experiment with analytical knowledge, or even of dividing the whole phenomena into smaller and manageable parts. We have tried all of these strategies earlier, but now we will do them in a systematic manner.

In an engineering problem we begin with a selected and simplified part of nature. Eliminating most of the marginal and non-incision laws to arrive at a useful approximation. Thus, even the full scale test is often a relaxed version of the true event. But even then we may find that we have too many laws and that scaling is difficult or impossible. When this happens further relaxations are necessary.

As a first step in relaxation it is helpful to determine whether the laws causing scaling conflicts are governing the given system with equal or unequal strengths. If they are of unequal strengths the weakest laws can be disregarded within segments of the investigation if not throughout its entire range.

Emori & Shuring give a summary of the various relaxation techniques in modeling requirements. It is divided into two parts. First, when the phenomenon governed is by law of unequal strength and on the right are the strategies used when the phenomena governed are by laws of equal strength. When the laws are of unequal strength, the strategy is the neglect of weak laws, and this neglect of weak law could be throughout the entire range of the investigation, or within the segments of the investigation. And when we neglected within segments of the investigation the segmentation could be temporal or could be spatial.
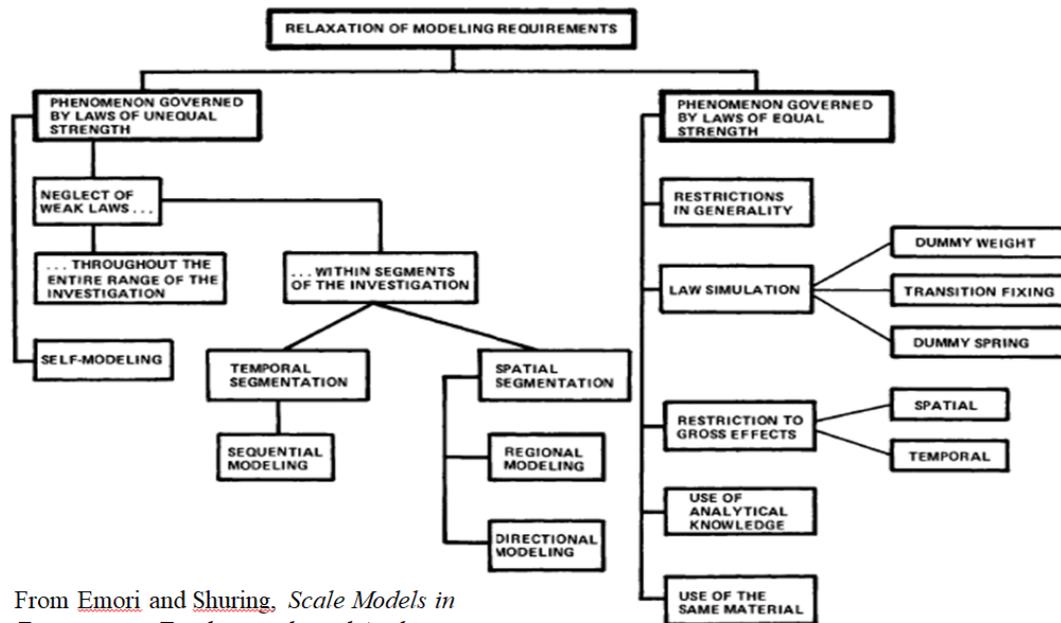

From Emori and Shuring, *Scale Models in Engineering, Fundamentals and Applications*

In temporal segmentation we do sequential modeling. In spatial segmentation we could adopt regional modeling or directional modeling. When the phenomena governed by laws of equal strength then we run into a little problem, and we have to be more innovative. One of the technique is when we restrict the generality. We look at some region where we could neglect the law if not throughout.

We could simulate the law by using dummy weights, by using dummy springs or by fixing transition. Some of the techniques we will discuss later. We could also adopt restriction of the gross effects, spatial or temporal, and of course, the always available technique of use of analytical knowledge, or we could work with the same material which may restrict the range of scale factors that we can use. First we will talk about what we mean by strength of laws. Let us suppose that the phenomenon is controlled by three pi numbers π_1, π_2 and π_3.
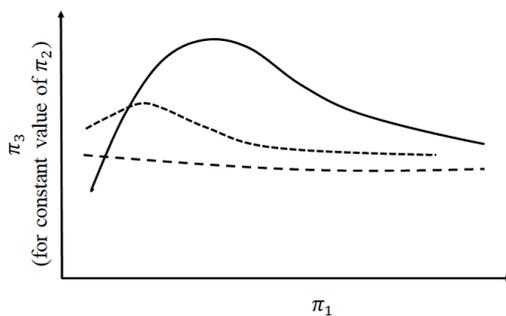
Let us study the variation of π_3 with π_1 for constant values of π_2. If the behavior of π_3 with respect to π_1 is as shown by these broken lines this means this is a weak law. We could neglect π_3 throughout this region. Or the behavior could be like this, the solid line. Here we see the effect of variation of π_1 on π_3 is large.

So, we cannot neglect the effect of π_1. π_1 is a strong law. Another possibility is the behavior is like this. In that case we see that over this range of π_1, the effect of π_1 is small, π_1 represents a weak law in this region of the values of π_1. But on this side the strength of the law is not negligible and we need to worry about π_1. So, if we are doing segmented modeling we could neglect π_1 over this range, but not over this range.

Disregarding Weak Laws

Flow of fluid in a pipe

Inertia: $F_i \sim \rho L^3 \cdot \frac{V^2}{L} \rightarrow \Pi_1 = \frac{F_i}{\rho L^2 V^2}$;

Viscous: $F_\mu \sim L^2 \mu \cdot \frac{V}{L} \rightarrow \Pi_2 = \frac{F_\mu}{\mu V L}$; $\Big]$ Re

pressure: $F_p \sim \Delta p L^2 \rightarrow \Pi_3 = \frac{F_p}{\Delta p L^2}$;

$\text{Eu} = \frac{\Delta p}{\rho V^2}$

Geometric similarity: $\frac{L}{D}, \frac{\varepsilon}{D}$

$\text{Eu} = \frac{\Delta p}{\rho V^2} = fcn\left(\text{Re}, \frac{L}{D}, \frac{\varepsilon}{D}\right) \implies \Delta p = \left(\frac{1}{2}\rho V^2\right)\frac{L}{D}f\left(\text{Re}, \frac{\varepsilon}{D}\right)$

Let us give some examples. We had considered the flow of fluid in a pipe, three force laws are important: inertia, viscous and pressure. As we have done a number of times we can write the pi numbers for these three forces. The pi number that results from the inertia forces is $\frac{F_i}{\rho L^2 V^2}$;. The pi number that results from viscous forces is like $\frac{F_\mu}{\mu V L}$, μ is the viscosity. And from

pressure we get the pi number $\frac{F_p}{\Delta p L^2}$. Of course, we have geometric similarity parameters : $\frac{L}{D}$ and $\frac{\varepsilon}{D}$. ε is the roughness parameter, : $\frac{\varepsilon}{D}$ is the relative roughness parameter, L is the length of the pipe and D is the diameter of the pipe. From the first two laws inertia and viscous we get Reynolds number $rho\,L\,V\,divided\,by\,\mu$ and from the first and third law we get the Euler number Eu= $\frac{\Delta p}{\rho V^2}$. We treat Δp as a dependent parameter.

So, the Euler number is a dependent parameter and we can write the functional relationship as Euler number $Eu = \frac{\Delta p}{\rho V^2} = fcn\left(Re, \frac{L}{D}, \frac{\varepsilon}{D}\right)$. And this can be rewritten as $\Delta p = \left(\frac{1}{2}\rho V^2\right)\frac{L}{D}f\left(Re, \frac{\varepsilon}{D}\right)$. f is called the friction factor for the pipes. We had obtained this equation earlier. Now, this function f as a function of Reynolds number and the relative roughness parameter $\frac{\varepsilon}{D}$ has been plotted by Moody in the Moody chart to obtain this relation.

On this the friction factor f is on vertical scale the Reynolds number is on horizontal scale and the various curves are for the various values of the relative roughness parameter $\frac{\varepsilon}{D}$. Now, notice in this case $\frac{\varepsilon}{D}$ does not occur this is true for all values of $\frac{\varepsilon}{D}$. Since f is a ratio of the representative pressure force to the representative inertial force, Reynolds number the ratio of the representative inertial force to the representative viscous force. The relationship of f proportional to 1/Re as is denoted by this straight line sloping down valid in the lower Reynolds number range means that the representative pressure force becomes proportional to the representative viscous forces and hence the influence of inertia is negligible. So, for modeling this laminar flow we could neglect the inertia force completely.

The result is obtained by balancing the pressure force with viscous force alone. In the other limit for larger Reynolds number, in this gray area, the friction factor seems to be independent of the Reynolds number, and just a function of $\frac{\varepsilon}{D}$, the relative roughness parameter. So, this is the region where viscous forces could be neglected, and the pressure forces are being balanced by the inertial forces alone. So, we have used two approximations: neglect of inertial forces in the lower Reynolds number region, and neglect of viscous forces in the higher Reynolds number region.

Let us revisit the wind tunnel testing of aircrafts and aircraft components. A wind or the air blows at large speeds in a tunnel. The aircraft or aircraft model is kept within the tunnel and forces that the air applied on this model are measured and from this prediction of the forces on the full scale aircraft are obtained. The forces of interest are the inertial, viscous and compressibility forces in the air. The scale factor for the inertial force is $k_{F,i} = k_\rho k_V^2 k_L^2$, obtained a number of times throughout this course. The scale factor for the viscous force is $k_{F,\mu} = k_\mu k_V k_L$, also obtained a number of times in this course. And the scale factor of the compressibility forces $k_{F,c} = k_{E_s} k_L^2$. And the elasticity of the material, compressible gas, can be written as $\left(\frac{k_\rho k_a^2}{k_\gamma}\right)$ where a is a speed of sound in that material times k_L^2. γ is the ratio of the specific heats, C_p divided by C_v. From the first two, of course, we obtain $k_V = k_\mu / k_\rho k_L$. The first two, when we equate them, result in Reynolds number and from the Reynolds number we

find that the scale factor of velocity must be like scale factor of μ divided by scale factor of ρ and scale factor for length.

And from the first and third, that is, inertial and compressibility forces we find $k_V = \dfrac{k_a}{\sqrt{k_\gamma}} = \sqrt{k_{R*}k_\theta}$, where R *is the gas constant times and θ is the absolute temperature of the gas. So these are the two laws that we obtained for modeling. $k_V = k_\mu/k_\rho k_L$, and $k_V = \sqrt{k_{R*}k_\theta}$. Now notice that if we use the same gas that is we use air at standard temperature, the material properties are same. So k_μ, k_ρ, k_{R*}, and k_θ are all one. So, the second of these relations gives you k_V is equal to 1, and if k_V is 1, then the first relation dictates that k_L must be 1.

So this rules out the possibility of testing scaled model in the wind tunnel using the normal atmospheric air. Is there a way out? Yes there is a way out. If we could modify k_μ divided by k_ρ. If we pressurize the gas, if we pressurize the atmospheric air so that the density increases, the viscosity mu does not increase appreciably or does not change appreciably even when we increase the density by a factor of 10 or so. So that as you pressurize the gas k_ρ increases and the kinematic viscosity decreases. So k_μ/k_ρ becomes much larger than 1, and so that permits us to obtain a k_L which is smaller. So one way to test models at the same value of Reynolds number and Mach number is to test them in what is called the high density tunnels.

Wind-tunnel testing

$k_V = k_\mu/k_\rho k_L$

$k_V = \dfrac{k_a}{\sqrt{k_\gamma}} = \sqrt{k_{R*}k_\theta}$
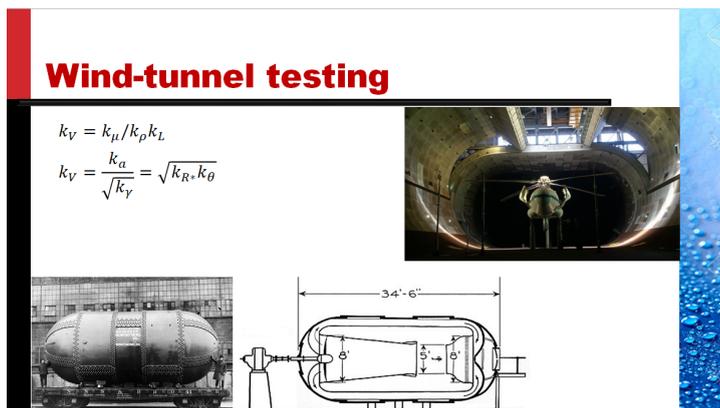
This is NASA's high density tunnel which has now been decommissioned. It is a large vessel inside which the pressurized air is kept with a pressure of about 10 atmosphere. So the density is 10 times and so we can work with the scale model of k_L is equal to 10 that is one tenth the full scale model. This was one facility that would be used for testing the components at the same value of Reynolds number as the actual proton.

Wind-tunnel testing
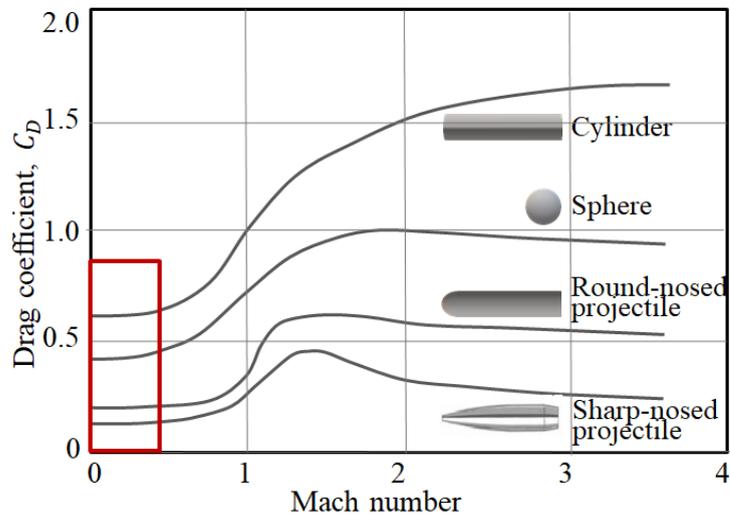
There is more to it if we study the drag coefficients of different bodies and as they change with Mach number. We see curves like this for a cylinder, for a sphere, for a round roads projectile and for a sharp nose projectile. And the important thing to note is this region where the drag coefficient is quite independent of the Mach number. This permits us to ignore the compressibility effects for Mach number below a value of 0.3 or 0.4. So if we remain below 0.3 or 0.4 we could ignore the effect of the Mach number. So we can work with only two forces inertial forces and viscous forces. Further if we study the drag coefficient of a body here a smooth sphere and we study the variation of drag coefficient with Reynolds number we can see this variation. Obviously this region where the drag coefficient varies inversely with Reynolds number is a region where inertial effects could be neglected lower Reynolds number region and we had covered this quite a bit of details earlier. But the interesting thing to note is this region where the drag coefficient is independent of Reynolds number, first over this range from 10 raised to the power 4 to about 3 into 10 raised to the power 5, C_D is constant.

Then, again at a much larger values. We had seen in a fluid dynamics course that the first region represents the region of laminar separation around the smooth cylinder and the second is when the boundary layer is turbulent so there is turbulent separation. But the Reynolds number is constant. Since most of the flows in aerodynamics are high Reynolds number flow in which we are interested, so postulating that the drag coefficient is independent of Reynolds number on the aerodynamic body is an often made assumption. So we use the relaxation that viscous effects could be neglected and the drag force is governed largely by pressure forces and inertia forces. It is interesting to see that the lift coefficient of NACA 4412 aerofoil for various Reynolds number established the truth of this.

Wind-tunnel testing
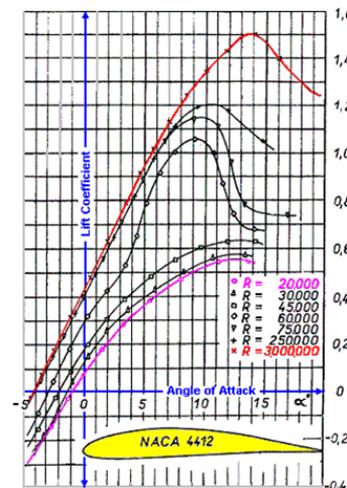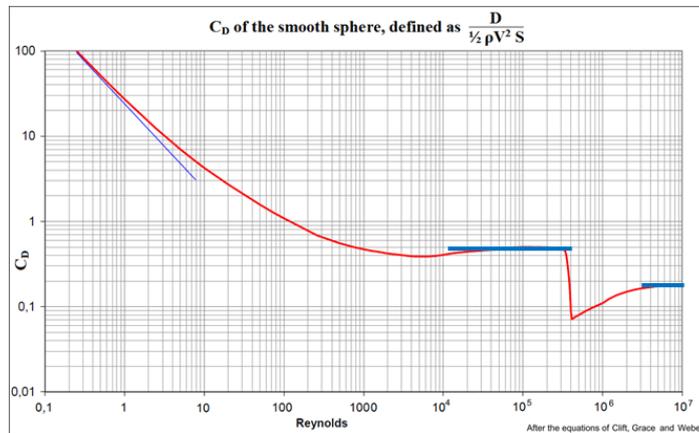
These 3 curves cover a Reynolds number range from 75,000 to 3 million, a very large range. Most flights are within this range and over this range of α, the 3 curves are almost coinciding. Again confirming the fact that the lift coefficient is quite independent of Reynolds number in this range of Reynolds number. So that simplifies the testing of models in wind tunnel. For large enough Reynolds number we do not worry about Reynolds number at all. We neglect the viscous law. We take a model of the aircraft or the aircraft component, put it in the wind tunnel, and run the test at a   large enough Reynolds number so that we are within this range of Reynolds number, and then   coefficients, drag coefficient, lift coefficient, moment coefficient that you obtain can be used to predict the forces or moments on the prototype. So all wind tunnel testing or rather major wind tunnel testing is quite independent of Reynolds number. Other applications of this neglect of heat loss are the neglect of radiation in low temperature heat transfer applications. There is radiation in almost every situation, but in low temperature heat transfer applications when I say low temperature I mean even up to a 300 ℃, we do not worry about the radiation effects. Similarly, neglect of gravity in the vibration of lightweight structure.

The self-weight of structure is almost never considered when structures are light. The weight of sub microscopic particles in colloidal materials is also neglected. Surface tension effects in opal channel floors are not considered. This is an important application and we will consider this later in more details, particularly because it exposes a new paradigm. There are situations when surface tension is negligible in the prototype but could have serious implications in the model, if the model is not made carefully. We will discuss this later.

How do we discover weak laws? Self-modeling is one of the techniques. In self-modeling the prototype itself becomes the model, so that the length scale factor as well as the scale factors for all the material property parameters are unity. Any quantity other than length and the material properties however can be changed. Self-modeling experiments are frequently used in fundamental heat and mass transfer experiments to clarify the influence of certain laws whose parameters can be varied without affecting the length and material properties. A great number of weak laws can be identified through self-modeling.

Discovering weak laws: Self- Modelling

Hydrodynamic torque converter

A hydraulic transmission is principally governed by inertial forces of both the working fluid and the moving mechanical parts.

Viscous forces appear to be negligible because the flow in the hydraulic circuit is thought to be turbulent.

The correctness of this assumption Is to be verified in a self-modelling test.
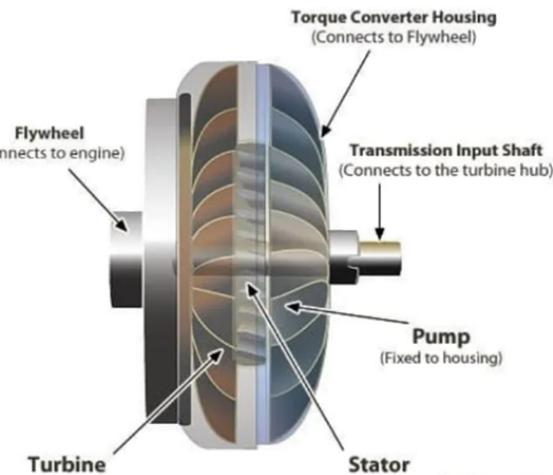
Self-modeling test often permits us to establish the governing laws of a given machinery. We will discuss this to a hydrodynamic torque converter. Hydrodynamic torque converter is used to transmit power from an IC engine to the wheels without the use of a conventional gearbox. Here the casing of the torque converter is connected to the flywheel of the engine.

So, this is where the input power goes. The whole thing rotates. This portion works like a pump. There is a hydraulic fluid inside and as the casing rotates with the vanes, it acts like a pump and pumps the fluid to the left into the turbine section, which makes the turbine part rotate and the power output is available on this shaft. Such a transmission is principally governed by inertial forces of both the working fluid and the moving mechanical parts. Viscous forces appear to be negligible because the flow in the hydraulic circuit is thought to be turbulent.

Is this assumption correct? The correctness of this assumption is to be verified in a self-modeling test. So, two forces of importance if the viscous forces are not negligible inertial forces like m a and which gives you $\rho L^4 \omega^2$, where omega is the angular velocity, and the viscous forces, which are like $\mu L^2/\omega$. It is understood that all quantities are the characteristic values. Converting these forces into torque, the inertial torque would be like $\rho L^5 \omega^2$. Multiplying the force by L. The viscous torque would be like $\mu L^3/\omega$. For self-modeling test, then, where the length scale is 1 and all the material properties are the same. So, that the inertial force requirement will give you $\frac{T}{\omega^2}$ is constant. And the viscous forces would give the requirement the $\frac{T}{\omega}$ is constant. Two quite different things.

Now, we will run the test on the torque converters for various values of omega and various values of torque. Input torque and the output torque. Input rotation ω_1 and output rotation ω_2. And if we plot the curves against $\frac{T_1}{\omega_1^2}$, the pump side value of torque and omega, the value of ω_2/ω_1, and the value of $\frac{T_1}{T_2}$ are obtained as these at various ω_1's and the various values of T_1.

What does this show? This shows that $\frac{T}{\omega^2}$ is constant is the correct modeling rule for the self-modeling test.

 That means that the torque converter is governed by inertial forces, and we can forget about the viscous forces. So, the assumption of negligible viscous forces, because the flow is expected to be turbulent within the casing of the torque converter, stands the test of self-similarity.

 Thank you.