Similitude And Approximations In Engineering,
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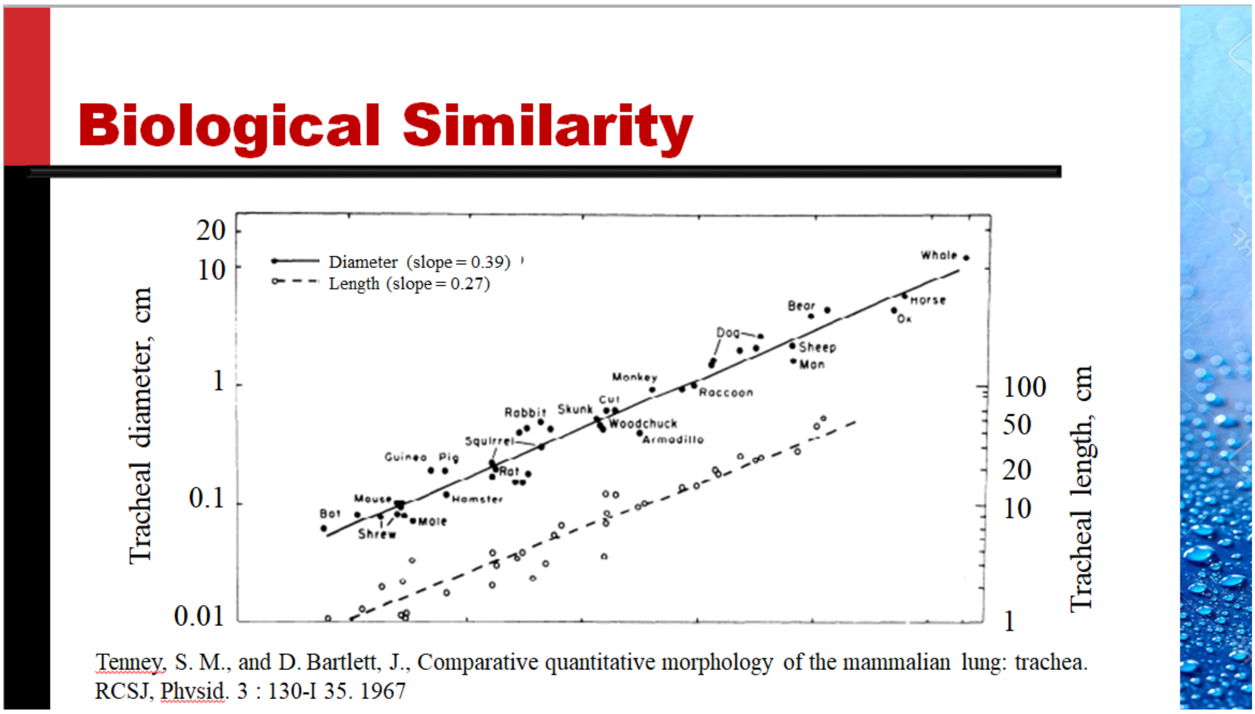
Week - 07
Lecture - 23

Welcome back once again. In today's lecture, we will discuss concepts of similarity applied to biology. It has been claimed that the various organisms and physiological mechanisms operate similarly and governed by the same laws in different species. Therefore, various species may be treated as scaled model of the same machine, and that the various biological functions must be scaled following the rules of similarity. In this lecture, we will explore this concept using some interesting examples. At its simplest, we obtain that k_L is one-third power of k_M, simply because k_M is like $k_\rho k_L^3$, and k_ρ is one, the tissue density is the same in almost all species, very close to 1000 kilograms per meter cubed. So let us see what results.



This is some published data about various animals. The diameter of the trachea, the windpipe, and the length of the windpipe. And this is plotted against the body weight of animals. And we see, very interestingly, that for almost all animals starting from bat to whale, the points lie almost on the same line. Solid line for the diameter of the trachea, and broken line for the length of the trachea. The scale for diameter in centimeters is on right and scale for tracheal length is on left.

The slope for diameter is 0.39, and for length is 0.27, compared to our prediction of 0.33 using strict similarity. So over this whole region of a bat weighing less than 100 grams to a whale over 2000 kilograms, the trachea is scaled by mass raised to power one-third.

A trivial but dramatic example. Determine the relation between scale factors for the weight of the animal and their surface area, the body surface area. From scale factor relation that should be relatively easy. If the length scale factor is k_L, the area scale factor is k_L^2, and the mass scale factor like k_L^3, assuming that k_ρ is one, that is, the density of all mammals is about the same.

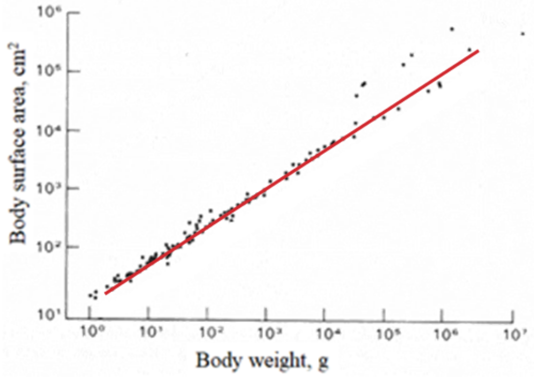
Thus, the surface areas of mammals should vary like mass raised to power two-thirds or S is like m raised to power two-thirds. And what is the data that we see?

A trivial (but dramatic) example

Determine the relation between the scale factors for the weight of the mammals and their surface areas

If the length scale factor is k_L, the area scale factor is $k_L{}^2$, and the mass-scale factor is like $k_L{}^3$, assuming that the density of all mammals is about the same.

Thus, the surface areas of mammals should vary like mass raised to power 2/3, or $S \sim M^{2/3}$



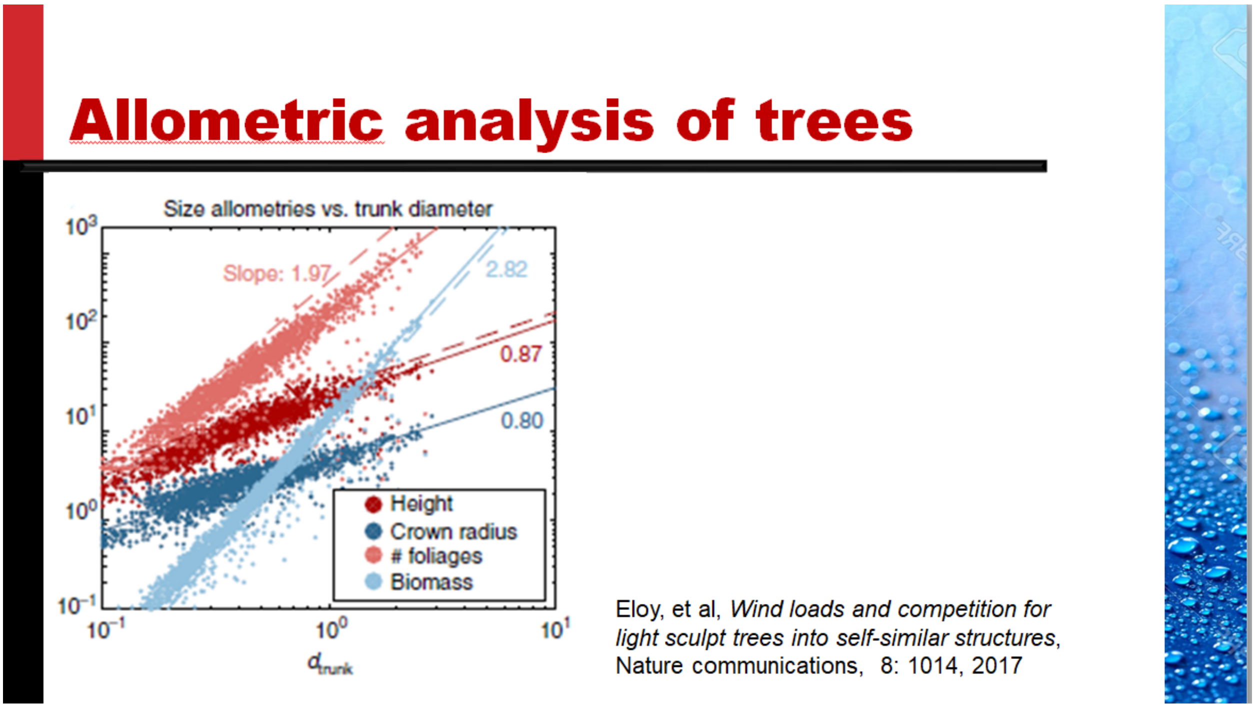
*Lightfoot, E. N., *Transport phenomena and living systems*, Wiley-Interscience, London,1974, p348

We see the body surface area and the body weight are following the red line which has a slope of mass raised to power two-thirds. Log, log plot. So that seemed to confirm our assertion.

$X = am^b$

Item and Measurement	Parameter (a)	Exponent (b)	Theoretical Exponent (b)
Gross somatic measurements, cm			
Vertex-heel height	46.0	0.30	
Trunk height	21.3	0.28	0.33
Chest circumference	17.1	0.37	
Thoracic width	5.11	0.32	
Skeletal lengths, cm			
Humerus	7.72	0.354	
Radius	7.08	0.31	
Ulna	8.22	0.37	
Femur	9.38	0.34	0.33
Fibula	8.15	0.24	
Skeletal weights, g			
Femur	3.78	1.08	
Tibia	2.82	1.02	
Fibula	0.60	1.04	
Humerus	2.41	1.07	1.00
Ulna	1.28	1.15	
Radius	1.00	1.17	
Organ weights, g			
Heart	4.77	0.84	
Right lung	3.30	1.02	
Empty stomach	5.80	0.96	
Liver with gallbladder	30.90	0.80	
Large intestine	13.70	1.00	1.00
Pancreas	1.11	0.94	
Spleen	1.12	0.99	

Stahl and Gummerson measured the dimension and masses of some skeleton and organs of five different species of adult primates and fitted them to the equation $X = am^b$, a and b are constants, and m is the mass of the primate. And this is what he obtained? He measured for a large number of individuals of five primate families, and measured something like vertex to heel height, trunk height, chest circumference, thoracic width, all linear dimensions. And to this equation if we try to fit our data the value of parameters a is given in third column, and the value of exponent b are given in fourth column. Since all these parameters are length, the theoretical exponent b should be the value 0.33, and you see these values are very close to 0.33. Similarly, the skeletal length, humerus, radius, ulma, femur and fibula, for five different species. And fitted the data to this equation and obtained these values of a and b. Again the exponents are very close to the theoretical exponent of 0.33. Skeletal weights in grams, these are the fitted value of the exponent and the expected value of the exponent if we assume complete similarity. Similarly, for organ weights. Pretty good agreement, I would say. Tends to confirm what we started out with.

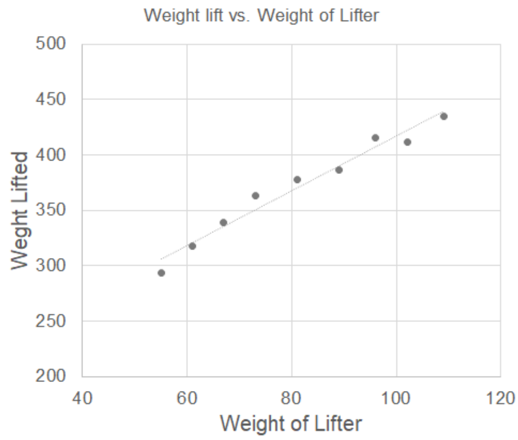


Now, we go to another example to different domain altogether. The allometric analysis of trees. Here in this graph is data of the diameter of the trunk of the tree, a length versus the height of the tree, the crowded edges of the tree, the number of leaves, and the biomass. We obtained pretty good agreement. The height and the crown radius, blur data, and this is a line. A length scale, so they should be linear with the diameter of the trunk. Here we see the power is 0.80 and 0.87, not a very good agreement, but quite there. The biomass is density times the volume and the volume should be three lengths and we get 2.82, quite close to 3. And therefore, this confirms that tree are also quite similar.

Let us extend this analysis further. The human weightlifting capability are taken data from the weightlifting championship, the world championship weightlifting last year. How much weight can a human being lift? Obviously, there must be the same kind of stress, the maximum stress in the muscles that a well-trained human beings should experience. And so the force that a muscle could apply would be like stress times the area. And the stress scale factor is one. The area scale factor is like length squared, and so the weightlifting capacity should be like the mass of the subject, mass of the human being raised to part two thirds. To repeat, if it is argued that the maximum stress level in the muscles of all persons is same irrespective of the size of the individual, then k_{mass} of the individual is like k_L^3, and the k_W, the weightlifted, is like k_{area}, k_{stress} being one, and so k_W is like k_L^2. So, the scale factor for the weightlifting k_W should be like the scale factor of mass raised to part two thirds. Let us look at the data.

World MEN weight-lifting records as of July 2020:

Men		
Weight class, kg	Total weight lifted, kg	$M^{2/3}/W$
55	294	0.0492
61	318	0.0487
67	239	0.0487
73	363	0.0481
81	378	0.0495
89	387	0.0515
96	416	0.0504
102	412	0.0530
109	435	0.0525
+109	484	

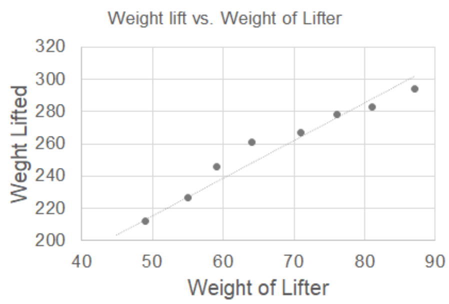


World men weightlifting records as of July 2020. Weightlifted versus weight of the lifter. Weight of the lifter here, and weightlifted here. These are the records. And you see there is a pretty good agreement with this line. These are the records in the weight class 55 kilogram. The total lifted weight in kilogram was 294. So, the weight of the individual raised to part two thirds divided by the weightlifted, it comes to 0.0492. For the 61 kilogram weight class, it is this. So, this ratio is increasing only very slightly over this range. The lowest value is 0.0481 and the highest value is about 0.0530. This line is a plot of m raised to part two thirds divided by weight.

This appears to justify our assertion that the weightlifting capacity can be predicted from mass raised to part two thirds of the individual weight. Similar data for women weightlifting records.

World WOMEN weight-lifting records as of July 2020:

Women		
Weight class, kg	Total weight lifted, kg	$M^{2/3}/W$
45	191	0.0662
49	212	0.0632
55	227	0.0637
59	246	0.0616
64	261	0.0613
71	267	0.0642
76	278	0.0645
81	283	0.0662
87	294	0.0668
+87	332	0.0662



Here, masses towards two thirds divided by weight has a higher value, but the values are almost the same across the weight classes. This higher value means that the maximum stress level in women is slightly less than the maximum stress level in men. But does not take away from a basic proposition.

Scale factor for breathing rates. Consider the metabolic rates in mammals. This rate should depend upon the rate of heat loss. The metabolic rate should be enough to supply the heat being lost. And the heat loss is in turn proportional to the surface area of the mammal. If all the organs of mammals are assumed geometrically similar and are made of similar material, what is the scale factor of breathing rates in terms of k_m, the scale factor of mass? Now to determine that, let us consider that the breathing rate would depend upon rate of metabolism. That is the amount of heat that is being produced in the body. The heat produced in the body must be equal to the heat lost by the body. And if the temperature differences are of the same order, then the heat lost would be proportional to the surface area of the body. And so the heat produced must also be proportional to the surface area of the body. That is like k_L^2. Now heat is produced by oxidizing food. So the scale factor of heat produced is the same as scale factor for food consumed, the mass of food consumed. The mass of food consumed would be proportional to the mass of oxygen that is used for oxidizing that food. So the mass of oxygen per unit time should be like k_L^2. This is the same as the scale factor for the surface area.

So scale factor for need of oxygen must be same as scale factor for area. Since the scale factor for the volumes of the lung would be like k_L^3, so the scale factor for the breathing rate will be kL2/k_L^3, or $1/k_L$. So, if k_ρ is 1, k_L is $k_m^{1/3}$, and then the scale factor of the breathing frequency would be like $1/k_m^{1/3}$, or $k_m^{-1/3}$. This is the scale factor for breathing rates. The heavier the mammal, the lower would be the breathing rate with the power of one third. If the mass is eight times, the breathing frequency would be one half. The Lilliputians and the Brobdingnagians of the Gulliver travel fame differed by height ratio of approximately 1 is to 144. The Lilliputians was 1 is to 12. Our foot was like an inch. The same scale between Brobdingnagians and humans.

Let us estimate using biological scaling, their relative masses, frequencies of breathing, velocity of walking and the maximum jumping height. Treat the Lilliputians and the models and the Brobdingnagians as prototypes, so k_L is 144, 12 is to power 2. Mass scale factor k_m would be k_ρ k_L^3 and that would be about 3 million. The mass of a Brobdingnagians was 3 million times the mass of a Lilliputian. We have already calculated the scale factor of frequency of breathing for warm-blooded animals and that is 1 divided by k_L. So it would be like 6.94×10^{-3}. Walking velocity, the scale factor of walking velocity will be same as scale factor of any other velocity. It would be k_L/k_t, and k_L/k_t would be $k_L k_f$. k_f is the frequency. k_t is $1/k_f$ and we have seen just now that k_f is like $1/k_L$.

So k of walking velocity is 1. So the speed at which the two species walk would be about the same. To calculate the scale factor of jumping height, assume that the maximum stress level in all species are the same, so that the maximum work done by muscle and jumping is scaled by k_{work} which is $k_{force}\times k_L$, and k_{force} is like $k_\rho\, k_V^2\, k_L^2$. So k_{work} become $k_\rho\, k_V^2\, k_L^3$. The work done is converted into potential energy in the jump, so that if H is the height jumped, $k_{potential\ energy}$ is like $k_{mass}\times k_g\times k_H$, and so it gives you $k_L^3 k_g k_H$. Equating this to k_{work}, we get with k_V is equal to 1, we get k_H is equal to 1, so that the two species can jump up the same height. Interesting.

Which species would have more difficulty with the surface tension of water, with the viscosity of water, and with the earth gravitational field? Consider the ratio of the inertial forces to the forces of surface tension. Inertial forces are like $k_\rho\, k_V^2\, k_L^2$. The surface tension forces scale factor k_F,σ is like $k_{sigma} k_L$. From this we get the ratio of the two forces. The surface tension forces to inertial forces is $1/k_L$ or 1/144.

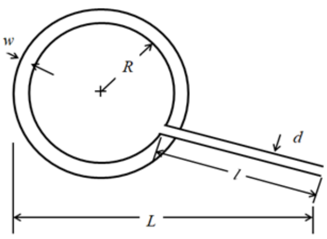
So the prototypes, the Brobdingnagians will have with less difficulty with the surface tension. The larger the L, the less significant is the surface tension. Ratio of the viscous to inertial forces is like 1/Reynolds number, and here this turns out to be 1, so that the two populations would have the same difficulty with the viscous forces.

Time required for emptying the bladder

David Hu and co-workers report* the discovery of the law of urination, according to which *all* animals ranging in size from mice to elephants empty their bladders in nearly constant time interval of 21 ± 13 s independent of their sizes.

We are not sure what are the dominant forces controlling urination

- Gravity and inertial forces,
- Gravity and viscous forces,
- Elasticity of bladder and inertial forces, or
- Elasticity of bladder and viscous forces



Determine the mechanism involved.

*Yang, P. J., J.C. Pham, J. Choo and D.L. Hu, *Law of urination: all mammals empty their bladders over the same duration*, arXiv:1310.3737v2.

Let's do this last example, a very interesting example. David Hu and his co-workers report the discovery of law of urination, according to which all animals, ranging in size from mice to elephant, empty their bladders in nearly constant time interval of 21 plus minus 13 seconds. This variation, but this is a random variation, independent of their sizes so that the time that they need to empty their bladders is about the same, independent of the size from a mouse to an elephant. We are not sure what are the dominant forces controlling the urination. Is it gravity and inertial forces? The gravity which accelerates the flow downwards the inertial forces that need to be overcome, gravity and viscous forces, gravity the accelerating force and viscous forces, the resistance that need to be overcome or the elasticity of bladder and the inertial forces, or the elasticity of bladder and the viscous force. Let's examine these four pairs. Let's first look at the pair gravity and inertial forces.

The scale factor of gravity force is like scale factor of mass times the scale factor for g, the acceleration due to gravity. So, that is like $k_{F_g} = k_\rho k_L^3 k_g$. For the inertial forces we have determined, very large number of times, is $k_{F_i} = k_\rho k_L^2 k_V^2$. So, if these are the two forces of concern, then $k_L k_g = k_V^2$. So this gives $\left(\frac{g\tau^2}{L}\right)_c = Constant$, or that the $\tau_c \sim \sqrt{\frac{L_c}{g}}$. So, clearly this is not the pair that determines urination. Because τ_c is independent of L_c, the observed fact. But if gravity and inertial forces are the controlling forces, then τ_c should vary like $\sqrt{L_c}$. So, incorrect.

Let's look at gravity and viscous forces. Scale factor for gravity forces again $k_{F_g} = k_\rho k_L^3 k_g$, and for viscous forces as determined before is $k_{F_\mu} = k_\mu k_L k_V$ which gives $k_\rho k_g k_L^2 = k_\mu k_V$ and from this we get $\left(\frac{\tau\rho g L}{\mu}\right)_c = Constant$ and which results in $\tau_c\left(= \frac{\mu}{\rho g L_c}\right)$ being inversely proportional to L_c. So, this again is not a valid pair of forces controlling urination.

Let's look at the third pair. The elasticity of the bladder and the inertial forces involved. Elasticity would be $k_E = k_\rho k_V^2$ and the inertial forces of course, $k_{F_i} = k_\rho k_L^2 k_V^2$ and so this yields $k_E = k_\rho k_V^2$ and which gives you rho $\left(\frac{\rho V^2}{E}\right)_c = Constant$ from this we get $\tau_c \sim L_c\sqrt{\frac{\rho}{E}}$. So, this also is incorrect.

Let's look at the last pair now. Elasticity of bladder and viscous forces. The scale factor elastic forces is $k_{F_E} = k_E k_L^2$ for the viscous forces $k_{F_\mu} = k_\mu k_L k_V$ and this gives $k_E k_L = k_\mu k_V$ which gives you $\left(\frac{\tau E}{\mu}\right)_c = Constant$ is constant. We are talking of the same urine, so k_μ is one. Elasticity of all fibers muscle fibers should be same. So, k_E is one. So, the only valid law which is independent of L_c is this. So, it is the elasticity of bladder and viscous forces that control urination. We can construct models for this and analyze further.

We have in this lecture covered a few examples of applications of scale factors to biological systems. I hope you enjoyed them.

Thank you.