Similitude And Approximations In Engineering,

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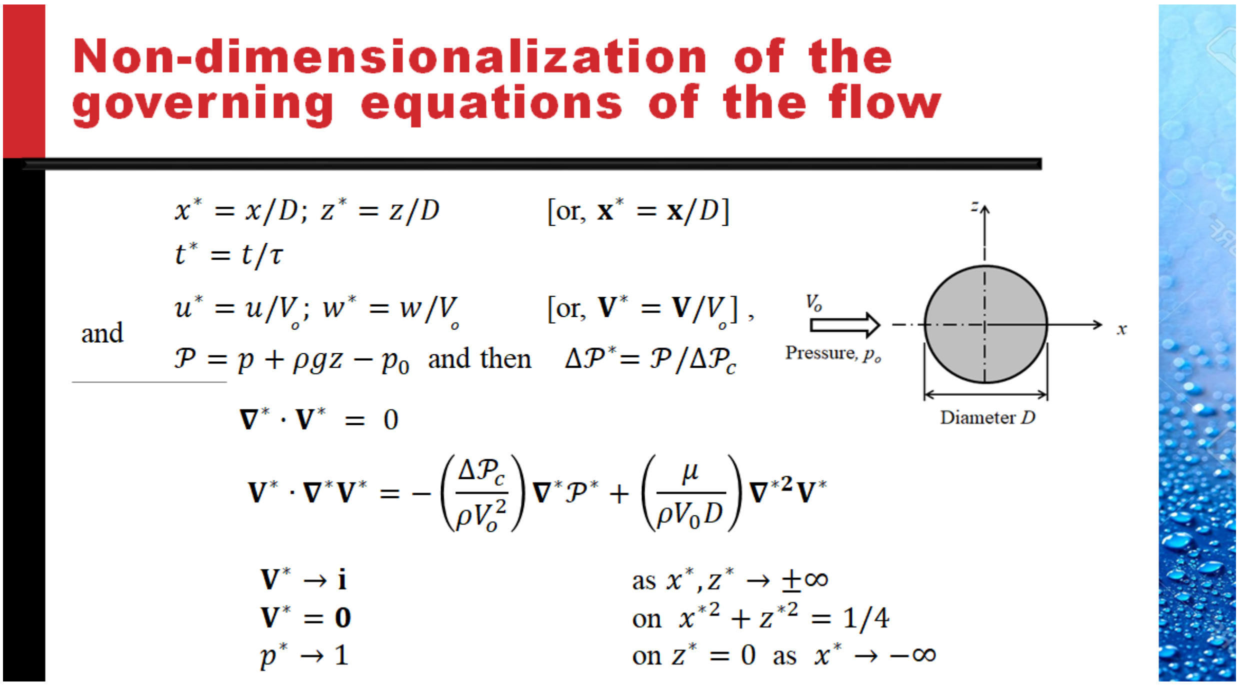
Week - 06
Lecture - 21

Welcome back once again. In this lecture, we will continue with handling approximations that lead to trouble. We saw that when the trouble arose, we could solve it by introducing a boundary layer in the two flows, the Stokes second problem, and the thermal conduction problem, transient thermal conduction problem. This approach has been taken from the approach that was introduced by Prandtl when we introduced the concept of boundary layers. So, we will see how the concept of boundary layer evolved.



We consider the problem of flow about a circular cylinder, the far upstream velocity is V_o with the pressure p_o, and we introduce the normalizing quantities: D for the lengths, τ for time, where τ is yet unknown, and we introduce V_o for velocity.

We introduce the non gravitational pressure term P which is defined as $P = p + \rho g z - p_0$. As before, this is termed non gravitational because under gravity in a stationary fluid, this parameter P remains constant. And we use Δp_c as the characterizing pressure difference. Δp_c is still unknown.

And if we do this then in steady flow, $\nabla^* \cdot V^* = 0$ is the normalized continuity equation and $V^* \bullet \nabla^* V^* = -\left(\frac{\Delta P_c}{\rho V_o^2}\right)\nabla^* P^* + \left(\frac{\mu}{\rho V_0 D}\right)\nabla^{*2} V^*$ is the normalized Navier Stokes equation. $-\left(\frac{\Delta P_c}{\rho V_o^2}\right)\nabla^* P^*$ represents the combined effect of pressure and gravity, $\left(\frac{\mu}{\rho V_0 D}\right)\nabla^{*2} V^*$ the viscosity, and $V^* \bullet \nabla^* V^*$ the convective acceleration. The boundary conditions are also normalized. As we said, the first term is inertial, the second term is non gravitational pressure, the third term is viscous force. The coefficient of the first term on the right $\left(\frac{\Delta P_c}{\rho V_o^2}\right)$ is like one over Euler number, and the coefficient $\left(\frac{\mu}{\rho V_0 D}\right)$ of the viscous force term is like one over Reynolds number.

So, this equation when we look for high Reynolds number flow. So, $\left(\frac{\mu}{\rho V_0 D}\right)$ is small, is inverse of Reynolds number. So, for high Reynolds number, this is small. So, the viscous term can be dropped, and what we get is what is termed as the Euler equation. Notice that this equation is a first order equation, not a second order equation which it originally was.

So, we cannot apply all the boundary conditions. In particular, the no slip condition on the body surface can no longer be applied. It results in what is known as the potential flow about the body, and as you studied in a course in fluid mechanics, this results in identically zero drag on any body within the fluid. This Euler equation is applied extensively in aerodynamics flow about airfoils about wing like bodies which are called streamlined bodies. The flow about these bodies can be calculated using Euler equation and it predicts lifts coefficient of the order 1 and we look at and when we look at the actual lift coefficients there is pretty good agreement.

So, this assumption of the neglecting viscosity when applied to problems in aerodynamics gives good results in predicting lift coefficients. As I said it matches quite well with the experimental results and it predicts drag coefficients exactly as a zero. The actual drag coefficients are of order 1 over under root and large number and for large Reynolds number this is purely within what we expect from approximations. We have neglected the viscous force terms of the order of 1 over Reynolds number. If that predicts zero drag, but the actual drag is of order $\frac{1}{\sqrt{Re}}$ there does not appear to be anything serious.

High Re flow past immersed bodies

Application to bluff bodies

→ predicts drag coefficients $C_D = \frac{Drag}{\frac{1}{2}\rho V_o{}^2 A_c}$ to be exactly 0

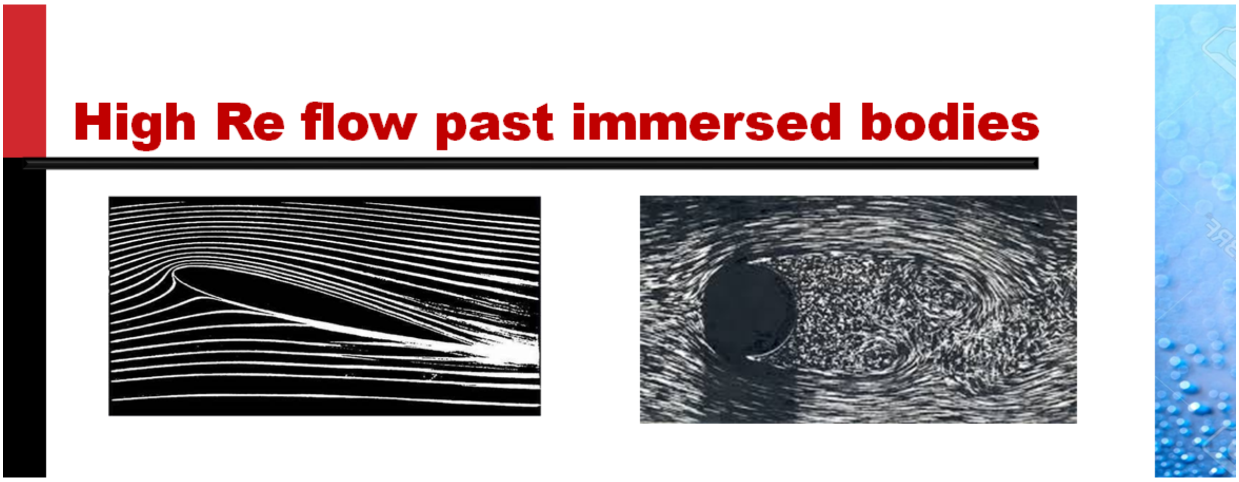
→ experimental result $C_D = O(1)$

→ D' Alembert Paradox

This appears to be well within the expectations from approximations and this is quite acceptable. However, when we apply this approximations to flow about bluff bodies we again predict the drag coefficients to be exactly zero, but the experimental results give us drag coefficients of order 1 not like 1 over Reynolds number for streamlined bodies. For bluff bodies like a cylinder or a sphere the drag coefficient is of order 1. So, our approximation appears to breaking down. This is what is known as the D' Alembert Paradox.

D' Alembert in 1749 concluded in one of his paper. "It seems to me that the theory developed in all possible rigor, this is the theory when the viscous term was neglected, gives at least in several cases a strictly vanishing resistance drag a singular paradox which I leave to the future geometers to elucidate."

The D' Alembert Paradox is not in the fact that the drag exists. The drag on airfoils which are order $\frac{1}{\sqrt{Re}}$ is not a paradox. This is what we expect. But with viscous forces of the order 1 over Reynolds number to get a drag of the order 1, when the theory predicts zero drag, is what is paradoxical. Prandtl worked on this.



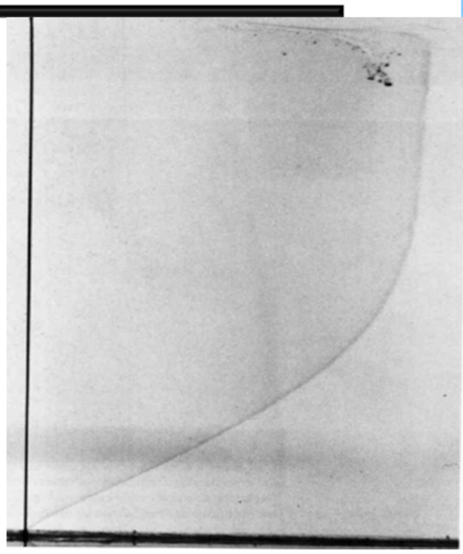
Let us look at the picture of two flows. And now we know, those of you who studied fluid mechanics, there is no separation in this picture, and there is massive separations in this picture. And it is this separation that causes large drag. Prandtl postulated the existence of a boundary layer in both bodies, but that boundary layer is well behaved in streamlined bodies like those on the left. But he could show that for bluff bodies like that on the right, there is a catastrophic separation from the body, which changes the flow field drastically from the inviscid flow picture, which would have been symmetric, fore and aft.

So, let us now see what Prandtl did. He showed that viscous flows of Reynolds number as high as one can imagine, that is, with viscosity as low as one wishes, are entirely different from the flows with zero viscosity. If a flat plate and the flow parallel to this. If the viscosity is zero, we cannot satisfy the no slip condition on the plate and we have uniform flow throughout. But that is not the case if there is viscosity present, howsoever small. He put forward the idea that at high Reynolds numbers, the no slip boundary condition causes a strong variation of the flow speeds over a thin layer near the plate. So, in the actual flow, the no slip boundary condition of the plate is satisfied, but there is a sharp variation of the velocity in a very thin layer near the plate. So, that while finding out the velocity gradients near the plate one does not have, or one cannot take the length of the plate as a characteristic length, but this thickness has to be taken as the characteristic length over which the normalized velocity varies from 0 to 1. He postulated the existence of this boundary layer within which large scale velocity variations are taking place. This boundary layer has been confirmed visually in this flow visualization with the air bubbles in water about a flat plate.

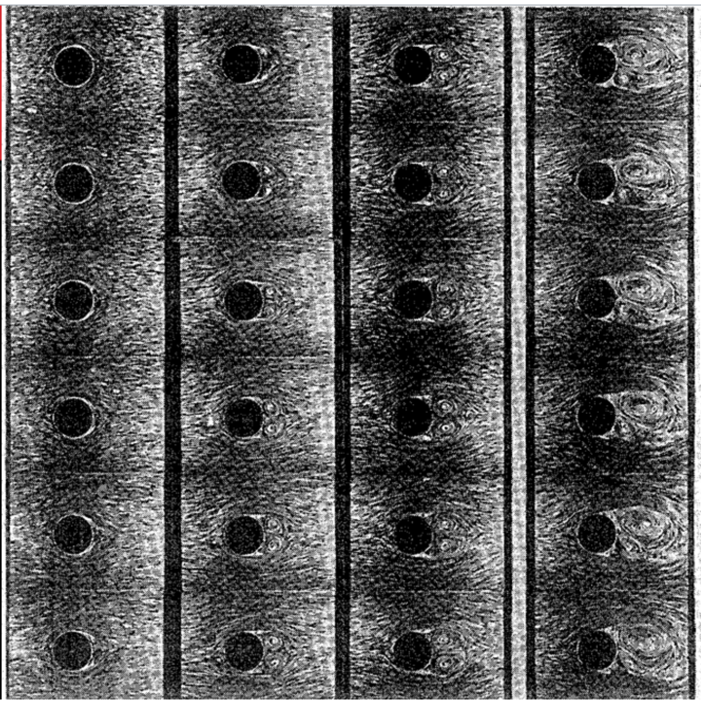
You can see this region where the bubbles are not passing through showing a boundary layer. In this experiment of 1970s Wortmann introduced a tellurium electrode. When a current passes through a tellurium electrode, tellurium ions leave the wire and are swept down. These tellurium ions produce a cloud like appearance and since they are traveling with the velocity of the fluid, this cloud region mimics the velocity profile on a flat plate. This is an act picture of an actual flow visualization.

High Re flow past immersed bodies

Visualization by a colloidal cloud produced by short pulse through a tellurium electrode. (Wortmann, 1977)



So, you can see a boundary layer like phenomena. In this picture we had a wire which emitted hydrogen bubbles and they formed a white strip and that this was upstream of the plate and it passed over the plate. You could see that this would appear like the velocity profile. A boundary layer velocity profile you can see and this becomes something like a boundary layer thickness and you can see that this is growing. There is an ample evidence of the phenomena of the development of boundary layer.



Sequential photos of flow starting from rest

From Prandtl and Tietjens

These are pictures from a book by Prandtl and Tijens. Here they show the when we have flow is started about a cylinder the flow is quite symmetric and as you go down this is increasing time. You can see here the development of separation. The separation multiplies you can see again going down and ultimately results in a typical flow behind a cylinder. Let us now look at the analysis of the boundary layer.

Within the boundary layer let us define the normalized x variable as $x^* = \frac{x}{L}$, where L is the length of the plate. But in the y direction the normalized variable ; $\eta = y/\delta_c$. The velocity is normalized with V_o and if we do this the continuity equation becomes $\frac{L}{V_o}\frac{\partial u^*}{\partial x^*} + \frac{\delta}{V_o}\frac{\partial v^*}{\partial \eta}$ in the new coordinates. Since both the terms are to be significant the characteristic velocity in the y direction should be like ; $u^* = u/V_o$ rather than V_o. Since L and δ are different, these two coordinates would be different if this had been normalized properly.

Since these coefficients are different that means v^* has not been normalized properly and v^* should have been normalized as $v^o = v/\left(\frac{V_o\delta_c}{L}\right)$. Since L is different from δ, these two terms cannot be equal unless we have made mistake in normalizing v^* and that is what has truly happened. v^* should be normalized by $\left(\frac{V_o\delta_c}{L}\right)$ so that we can write a new variable normalized variable $v^o = v/\left(\frac{V_o\delta_c}{L}\right)$. And then the continuity equation would have both terms in there. So, the characteristic velocity in the y direction is not V_o, but $\left(\frac{V_o\delta_c}{L}\right)$, much smaller.

Let me remind you that δ is as yet unknown. And we normalize the momentum equation by same definition of normalized variables, then the equation that we get is $\left(\frac{\rho V_o^2}{L_c}\right)u^*\frac{\partial u^*}{\partial x^*} + \left(\frac{\rho V_o^2}{L_c}\right)v^o\frac{\partial u^*}{\partial \eta} = \left(\frac{\mu V_o}{L_c^2}\right)\frac{\partial^2 u^*}{\partial x^{*2}} + \left(\frac{\mu V_o}{\delta_c^2}\right)\frac{\partial^2 u^*}{\partial \eta^2}$. And if we make the coefficient of the first term as 1, then the resulting equation is $u^*\frac{\partial u^*}{\partial x^*} + v^o\frac{\partial u^*}{\partial y^o} = \frac{1}{Re}\frac{\partial^2 u^*}{\partial x^{*2}} + \left(\frac{1}{Re}\frac{L_c^2}{\delta_c^2}\right)\frac{\partial^2 u^*}{\partial \eta^2}$. Recall that $\frac{\mu}{\rho V_o L_c}$ is like 1 over Reynolds number based on length L_c. So, this is how we can write this equation: $u^*\frac{\partial u^*}{\partial x^*} + v^o\frac{\partial u^*}{\partial y^o} = \frac{1}{Re}\frac{\partial^2 u^*}{\partial x^{*2}} + \left(\frac{1}{Re}\frac{L_c^2}{\delta_c^2}\right)\frac{\partial^2 u^*}{\partial \eta^2}$.

Now in this equation Reynolds number is very large. So, definitely $\frac{1}{Re}\frac{\partial^2 u^*}{\partial x^{*2}}$ drops out, but the second viscous term must remain in the equation. Otherwise, the viscosity would completely disappear from the equation, and we will get back a first order equation which definitely cannot satisfy the boundary conditions of no slip at the plate. So, this term should be retained in the equation and so this coefficient should be of order 1. That is possible only if $\frac{\delta_c}{L_c} = \frac{1}{\sqrt{Re}}$.

That is, the boundary layer thickness is like $\frac{L_c}{\sqrt{Re}}$. With very minimal analysis we have been able to obtain this estimate. We can do elaborate solving of these equations, and then we find out the boundary layer thickness is of order about 4.5 divided by under root Reynolds number. Same order.

Skin friction coefficient

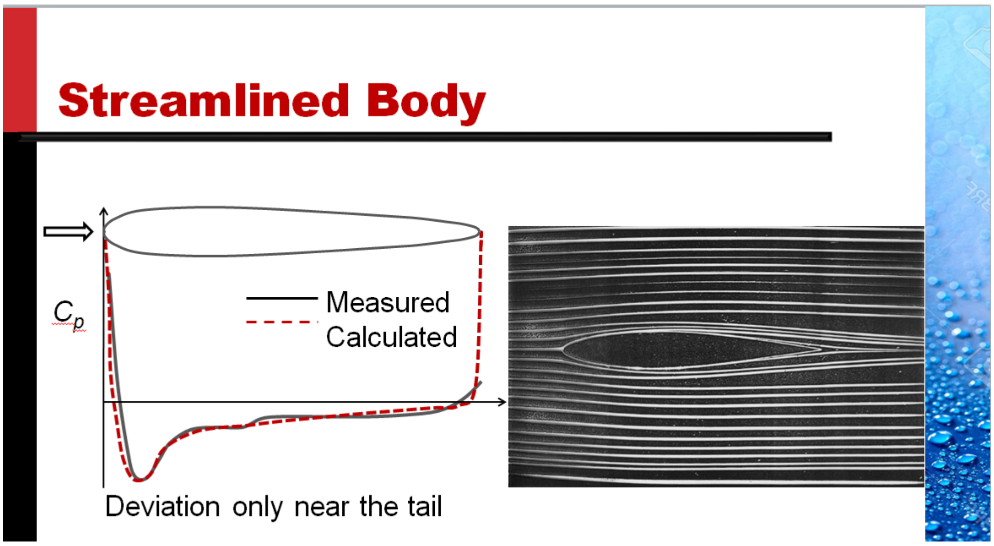
Estimate of the skin friction is obtained from the scale factor of viscous stresses:

$k_\tau = k_\mu \frac{k_V}{k_\delta}$, so that the estimates of shear stress is $\tau \sim \frac{\mu V_o}{\delta} \sim \frac{\mu V_o}{L_c/\sqrt{\text{Re}}}$

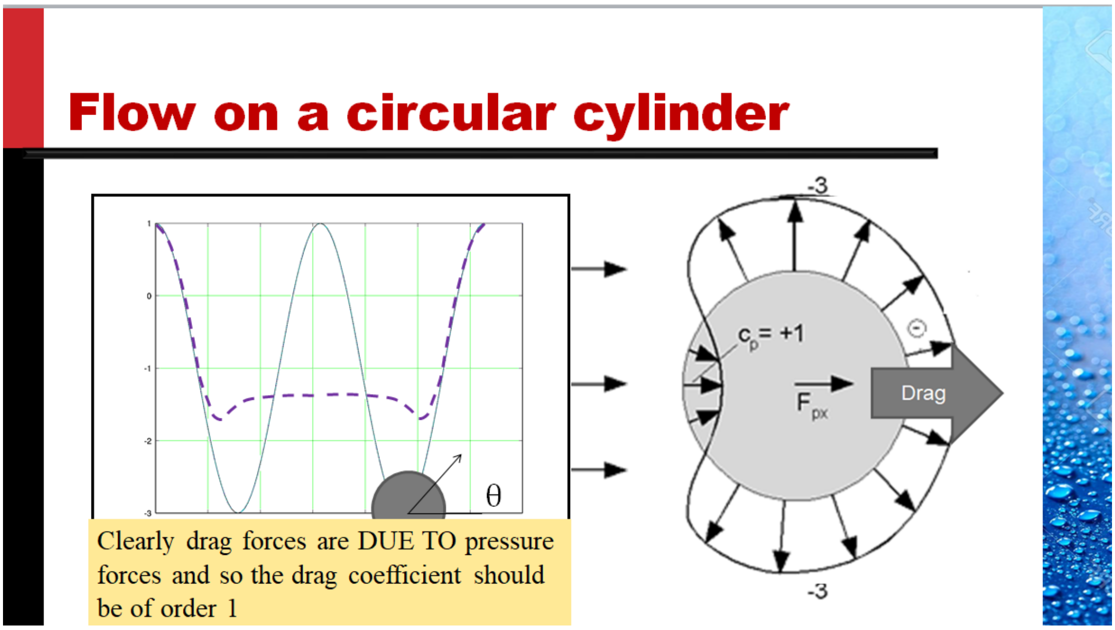
The resulting skin friction coefficient c_f defined as $\tau / \frac{1}{2}\rho V_o^2 \sim \frac{1}{\sqrt{\text{Re}}}$

Let us estimate the skin friction coefficient from the scale factors of various stresses. The scale factor for skin friction, that is k_τ became $k_\mu \frac{k_V}{k_\delta}$,, , the velocity gradient with length taken as δ. So, that an estimate of shear stress is $\tau \sim \frac{\mu V_o}{\delta}$. By plugging in the value of delta that we obtained. The resulting skin friction coefficient C f defined as $\frac{\tau}{\frac{1}{2}\rho V_o^2}$ is $\frac{1}{\sqrt{Re}}$on a flat plate.

The result is a very good result. The measured value $\frac{0.66}{\sqrt{Re}}$. So, it is a very good estimate that we obtain with very little work. This shows you the agreement on the measured coefficient of pressure and the calculated coefficient of pressure using Euler equation on a streamline body. Very close agreement except at the very tail.

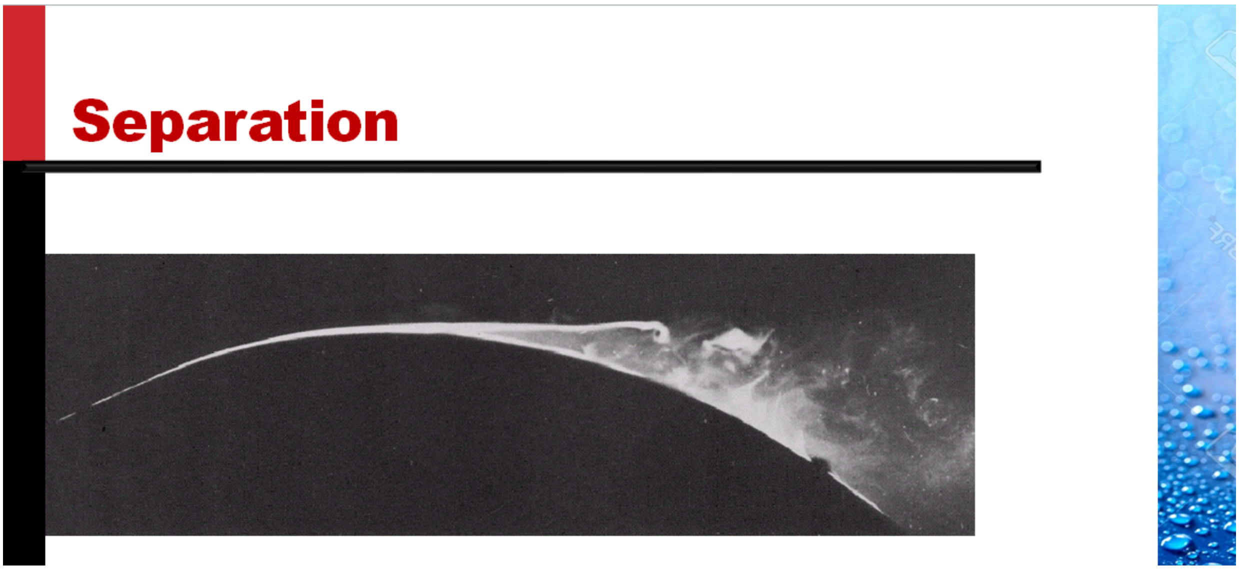


The flow as you see is only at the tail do we see some semblance of separation. On the other hand if we use Euler equation to calculate the coefficient of pressure on a circular cylinder we get this variation. This is 0. So, at the nose this positive pressure coefficient, same at the tail, and over the bulk of these surfaces we have negative pressure coefficients. This is what the pressure coefficient appears to be, if we calculate using Euler equation, but the actual pressure distribution is something like this.



Quite different from what we calculate using the Euler equation. We plot this coefficient on this chart. This is what is the measured pressure coefficient on a laminar flow about a circular cylinder. And this is Euler's prediction. You see there is very good agreement at the nose, but very bad agreement over this region.

Over the nose region, there is a good agreement, but over the rest of the region there is very bad agreement. Clearly the drag forces are due to pressure forces, and so the drag coefficient should be of order 1. What is happening? Separation.



This separation changes the flow picture drastically and the separation occurs because of the flow reversing near the wall because of viscous action. And so, the flow has separated from here, and this is the region of recirculating flow.

This makes the Euler equation break down completely, and that approximation is no longer valid. This is what the final flow looks like.

Thank you.