

Similitude And Approximations In Engineering,
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Week - 06
Lecture - 20

Welcome back. We had mentioned earlier that while approximating there are sometimes we run into problems. In this lecture we will learn how to handle those problems.

Consider a flow between two parallel plates, long infinite parallel plates. The upper plate is stationary, and the lower plate a distance b below the stationary plate is oscillating with the velocity that is given as $V_o \sin \omega t$. In such a flow, the forces of interest are the unsteady inertial forces and the viscous forces.

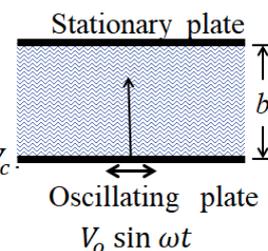
Oscillating boundary problem

The unsteady forces are estimated as

$$F_u \sim (\text{mass}) \times (\text{acceleration}) \sim (\rho L^3)_c \times (V/t)_c \sim \rho L_c^3 V_c / t_c$$

and the viscous forces are estimated as

$$F_\mu \sim \mu(\text{area}) \times (\text{velocity gradient}) \sim \mu(L^2)_c \times (V/L)_c \sim \mu L_c V_c$$



The natural or the characteristic values of the various quantities in these estimates can be taken as: $L_c = b$, $V_c = V_o$, and $t_c = 1/\omega$.

With these, $F_u \sim \rho b^3 V_o \omega$ and $F_\mu \sim \mu b V_o$. Thus, $F_u/F_\mu \sim \rho b^2 \omega / \mu$.

Let us estimate those forces. The unsteady forces are estimated as mass times acceleration and the mass is $(\rho L^3)_c$ times the acceleration $(\frac{V}{t})_c$ and so this gives $\rho L_c^3 V_c / t_c$, where t_c is a characteristic time, V_c is a characteristic velocity and L_c is a characteristic length. Similarly, the viscous forces are estimated as μ times area times the velocity gradient, and that, as before, results in $\mu L_c V_c$. The natural or the characteristic values of the various quantities in these estimates can be taken as $L_c = b$, the spacing between the plates.

We see the characteristic velocity as V_o the amplitude of the velocity of the oscillating plate, and a reasonable characteristic time is $1/\omega$, the circular frequency. With these the unsteady forces become $F_u \sim \rho b^3 V_o \omega$, and the estimate of the viscous forces is $F_\mu \sim \mu b V_o$. Thus, the ratio of the unsteady to viscous forces is given by $F_u/F_\mu \sim \rho b^2 \omega / \mu$. Now at equilibrium the

unsteady forces must balance the viscous forces. The ratio $\rho b^2 \omega / \mu$, then, determines the relative importance of two types of forces.

If this value is small, that is, if ω is much less than $\mu / \rho b^2$, the unsteady effects are small and may be neglected in comparison with the viscous forces. Once the unsteady effects are neglected the governing equation would require that the net viscous forces be 0 on any element of the fluid. Now what's going on? I think there's something funny. We claim that unsteady force term must be equal to the viscous force term. When we convert it this into normalized variable we get $\rho \left(\frac{\rho b^2 \omega}{\mu} \right) \frac{\partial V^*}{\partial t^*} = \frac{\partial^2 V^*}{\partial y^{*2}}$.

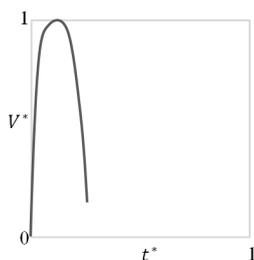
The left hand side represents the normalized acceleration and this is the normalized second gradient of velocity that occurs in the viscous forces. If the unsteady force term is small we said viscous term is equal to 0, and if viscous term is small we said unsteady term is 0. What's going on? It seems funny. If the unsteady term is small we claim that if this is small then the unsteady term is small and we set the viscous forces to be small. Now isn't this rather strange? If unsteady force is small we should set unsteady forces to be 0 and not viscous forces to be 0.

So what is going on? Let's look at this equation that we derived once again. Note that even when $\rho b^2 \omega / \mu$ is small, the left hand side is not smaller than RHS. The two sides are always equal. This whole is equal to this whole. We expect this to be order 1 we expect this to be order 1 and if this is small so hasn't something gone wrong? Yes, something has gone wrong.

When $\rho b^2 \omega / \mu$ is small it only means that at least one of the two derivatives which we were expected to be normalized are not normalized. At least one of the two derivatives is not of order 1. So either $\frac{\partial^2 V^*}{\partial y^{*2}} \ll 1$, or $\frac{\partial V^*}{\partial t^*} \gg 1$. So, if this is order 1 then this should be of order ϵ , where ϵ is this parameter: $\rho b^2 \omega / \mu$. And if this is of order 1 then this should be of order 1 over epsilon.

So we expect when a situation like this arises that is, when we have two terms in an equation that are equal, and after normalization one of the terms has a coefficient which is very small, then the only way out is that we should suspect that our normalization is not proper. So we said that either $\frac{\partial V^*}{\partial t^*} \gg 1$ much greater than 1 or $\frac{\partial^2 V^*}{\partial y^{*2}} \ll 1$ is much less than 1. Can $\frac{\partial V^*}{\partial t^*}$ be much greater than 1? Recall that t has been normalized by t_c , the characteristic time, which is 1 over ω . $\frac{\partial V^*}{\partial t^*}$ would be much greater than 1 if the actual t_c is much smaller than $1/\omega$. Is this possible?

This would be possible only if the variation of $\frac{\partial V^*}{\partial t^*}$ normalized by t_c is equal to $1/\omega$ is something like this:



Then it does not take the whole range of t^* for x^* to change from 0 to 1, but it changes from 0 to 1 in a much smaller interval. So the time it takes to rise from 0 to 1 is not like t_c but is much less than t_c that is much less than $1/\omega$. Is it possible? Can the fluid velocity inside or can the fluid velocity between the two plates oscillate faster than the fluid velocity at the bottom plate which is forcing the flow? That does not appear to be physically possible. It is not physically reasonable. Can $\frac{\partial^2 V^*}{\partial y^{*2}}$ be much less than 1, the other possibility? Sure, if $\frac{\partial V^*}{\partial t^*}$ is constant, the second derivative is 0.

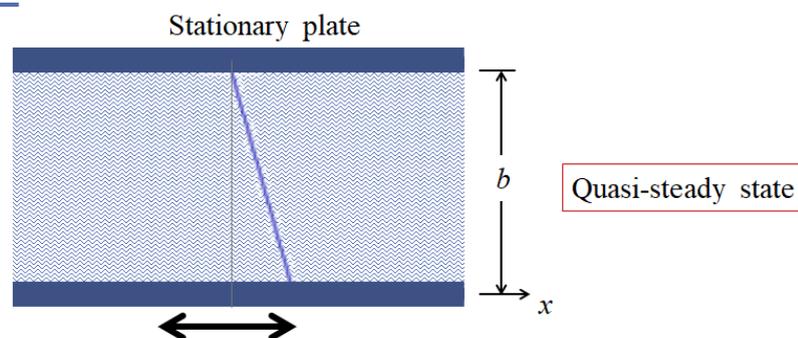
That is, if the variation of V^* by t^* is linear, clearly $\frac{\partial V^*}{\partial t^*}$ is constant and $\frac{\partial^2 V^*}{\partial y^{*2}}$ is 0. It does not need to be a straight line. Even if the variation $\frac{\partial V^*}{\partial t^*}$ is given by curve like the blue curve here, the second derivative $\frac{\partial^2 V^*}{\partial y^{*2}}$ would be much less than 1. And so there is a definite possibility that $\frac{\partial^2 V^*}{\partial y^{*2}}$ is much less than 1. If $\rho b^2 \omega / \mu$ is much less than 1, that is if ω is less than $\mu / \rho b^2$ divided by ρb^2 , the unsteady terms can then be dropped and we can set $\frac{\partial^2 V^*}{\partial y^{*2}} \ll 1$ equal to 0.

It is very small and we neglect it. So this is a valid approximation. This directly or through the momentum equation gives $\frac{d^2 V_x}{dy^2} = 0$. That is, there is no net viscous force acting on the fluid. This is to be solved with the boundary conditions.

The no slip boundary condition at the lower plate $V_x = V_o \sin \omega t$, and the velocity is 0 at the upper plate, again a no slip condition. So for $\omega < \frac{\mu}{\rho b^2}$ or in terms of kinematic viscosity, $\omega < \nu / b^2$: kinematic viscosity divided by the distance between the plates squared. We get a solution like this.

Oscillating boundary problem

$\frac{d^2 V_x}{dy^2} = 0$ to be solved with $V_x = V_o \sin \omega t$ at lower plate, and 0 at the upper plate for $\omega < \frac{\mu}{\rho b^2} = \nu / b^2$



This equation with this boundary condition will give you, what appears to be, a quasi steady solution. Depending upon the boundary condition for any given time t , we calculate the velocity at the lower plate and if this is the velocity at the lower plate then the velocity through the fluid is varies between 0 and this point linearly like a Couette flow.

But this boundary condition is changing with time and so the velocity profile is oscillating. At any given instant the velocity profile is exactly what is obtained in a steady flow, but as time changes the boundary condition at the lower plate changes the time and so the linear profile between the lower plate and the upper plate changes with time. This is what is termed as a quasi steady flow. Again for $\omega < \nu/b^2$, we can interpret b^2/ν as a penetration time τ . Notice that the dimension of b^2/ν are those of time.

So we consider the penetration time τ as the time it requires for the change of velocity at the lower plate to penetrate up to the upper plate. That is time required for the velocity fluctuation at the lower plate to penetrate right up to the upper plate. Then this condition $\omega < \nu/b^2$ translates into ω is much less than $1/\tau$. That means the time required for flow velocity to penetrate up to the upper plate is much less than the time of fluctuation of the velocity of the lower plate. The lower plate is fluctuating far too slowly such that at any given time the velocity appears to penetrate up to the upper plate almost instantaneously, and so we have a quasi steady flow.

In other words, if the time period $1/\omega$ of the oscillation of the lower plate is much more than this characteristic time the motion of the plate is so slow that the unsteady inertia of the fluid can be neglected. We can interpret this alternately also. We can consider $\sqrt{\nu/\omega}$ which is a dimension of length to be the penetration depth l_p , that is, in of the time of one cycle of the lower plate. How far does the effect of the lower plate motion penetrates upwards? The penetration depth within one cycle. How far does the effect penetrate and if this penetration depth is much larger than b , then the effect penetrates through the distance b almost instantaneously.

Same thing as before except here we interpreted in terms of penetration depth and earlier we interpreted in terms of penetration time. Penetration time is the time it takes for the effect to penetrate through distance b , and the penetration depth is the distance the effect penetrates in time $1/\omega$. For both these considerations we see that the motion of the plate is too slow. Therefore, for b less than $\sqrt{\nu/\omega}$ we can imagine that the effect of the motion of the plate penetrates throughout the thickness of the fluid layer before the motion of the lower plate changes direction, or that the oscillation appears too slow to the fluid and it can be assumed to respond instantaneously.

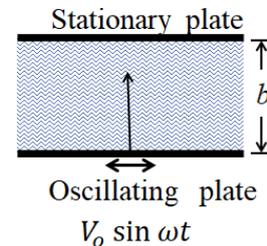
Oscillating boundary problem

The other limit with $\omega > \mu/\rho b^2$ leads to a very serious complication.

The principle of small causes producing small effects suggests that we may neglect viscous forces. This leads to a very strange result. The governing equation then becomes

$$\frac{dV_x}{dt} = 0$$

This is an ordinary differential equation simply stating that the velocity within the fluid does not change with time. It cannot now accommodate the lower boundary condition.



This was the easy part. Let us look at the other part. When ω is not less than v/b^2 , but the opposite. $\omega > \mu/\rho b^2$, and we will soon see that it leads to a very serious complication. The principle of small causes producing small effects suggests that now we can neglect viscous forces. This leads to a very strange result.

If we neglect the viscous forces that is the right hand side of our equation, the governing equation then becomes $\frac{dV_x}{dt} = 0$. This is the first order equation so it admits to only one boundary condition. But we have two boundary conditions the velocity V_x is 0 at y equal to b and the velocity V_x is $V_o \sin \omega t$ at y is equal to 0. It cannot admit both the conditions. So this cannot satisfy our boundary conditions.

So it does not appear to be a valid solution. This is an ordinary differential equation simply stating that the velocity within the fluid does not change with time. It cannot accommodate the lower boundary condition. So there is something that has gone wrong in a formulation. And what could go wrong? The only thing that could go wrong is that the quantities that we have taken to be the characteristic quantities may not be appropriate.

The resolution of this problem lies in recognizing that the distance l_p through which the disturbances penetrate in time $1/\omega$ is very small compared to b . This is the same approach that Prandtl took to explain the formation of boundary layer on a flat plate. The oscillating plate produces a viscous effect because of that viscous effect the velocity starts penetrating upwards. But before it can go through a significant distance the motion of the lower plate changes direction. And so the effect of a long time is now restricted to a thin layer about the lower plate.

The velocity changes from $V_o \sin \omega t$ to 0 over this small distance rather than over the whole distance b . This means that the velocity gradient normalized by $\frac{V_o}{b}$ is in error. When you obtain $\frac{dV_x}{dy}$ and normalize it then normalizing the distance y/b is in error. Only if it is

normalized by $\frac{V_o}{l_p}$ with l_p is equal to $\sqrt{\nu/\omega}$, μ is the kinematic viscosity μ/ρ should the velocity V^* be of order 1. So, there the characteristic distance the characteristic length in the y direction should not be taken as b, but as l_p the penetration depth which is related to the circular frequency ω of the oscillation of the lower plate.

So, how is the problem handled? We divide the flow field in two regions. One a thin layer of thickness l_p is equal to $\sqrt{\nu/\omega}$ near the lower plate and then the rest of the region. We term this region as the boundary layer and we term this region as the outer layer. Boundary layer is very thin region near the oscillating plate. Viscous stresses can then be neglected in the outer region in the outer layer, but not within the boundary layer when neither the viscous nor unsteady effects can be neglected.

So, that we now obtain two equations one valid for the boundary layer and the other for the outer region and the flow solution would be this. The region for the outer flow is simply $\frac{dV_x}{dt} = 0$ as we have obtained earlier and the upper boundary layer is applicable to this. So, the velocity is invariant through the outer region with time as well as with y and we get V_x is equal to constant, (equal to 0) throughout this region, but within the boundary layer both the terms must be present the unsteady force term as well as the viscous force term.

Oscillating boundary problem

For the boundary layer:

The unsteady forces are estimated as

$$F_u \sim (\text{mass}) \times (\text{acceleration}) \sim \rho L^2 \delta \times (V/t)_c \sim \rho L^2 \delta \omega V_o$$

and the viscous forces are estimated as

$$F_\mu \sim \mu (\text{area}) \times (\text{velocity gradient}) \sim \mu L^2 \times V_o / \delta \sim \mu L^2 V_o / \delta$$

The ratio of inertial to viscous forces then are $\frac{\rho \omega \delta^2}{\mu} = \omega \delta^2 / \nu$

Let us now talk about the boundary layer and derive the equation for that. The unsteady forces are estimated as mass times acceleration, and the mass within this region of thickness delta would be scaled like $\rho L^2 \delta$, L^2 in x and z direction and δ in the y direction. So, mass is rho times the volume and the volume would be like $L^2 \delta$ and the acceleration would be like $\left(\frac{V}{t}\right)_c$ and t_c is like $1/\omega$ and V_c by V_o . So, that this gives you $\rho L^2 \delta \omega V_o$. The viscous forces are estimated as μ times the area on which the viscous forces act times the velocity gradient. Now viscous forces act on an area which is like L^2 and now the velocity gradient is V_o / δ rather than V_o / b and this gives us the viscous forces of $\mu L^2 V_o / \delta$. δ is as yet unknown.

Now we assert that within this boundary layer both the unsteady forces and viscous forces should be of the same magnitude. So, this force should be about the same order as this. So, we

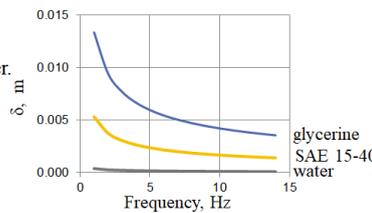
can set $\rho L^2 \delta \omega V_o \sim \mu L^2 V_o / \delta$. So, the ratio of the inertial to viscous forces is the inertial force divided by the viscous force is $\frac{\rho \omega \delta^2}{\mu}$, which in terms of ν , the kinematic viscosity, is $\omega \delta^2 / \nu$, and as I said, these two forces must be of the same order. So, this should be of order 1 and that gives us δ and this gives us δ should be order of $\sqrt{\nu / \omega}$. The same result that we got earlier.

Oscillating boundary problem

The ratio of inertial to viscous forces = $\omega \delta^2 / \nu$

Within the boundary layer where viscous effects and inertial effects are to play their parts, the ratio should be of order one.

Or, $\delta \sim \sqrt{\nu / \omega}$, the same result we got earlier.



This graph represents the results of the experiments done on three different fluids water, motor oil SAE 1540, and glycerin. Glycerin has the largest kinematic viscosity and we see that for each fluid, delta varies with frequency. As the frequency increases that is the value of omega increases, the boundary layer thickness decreases. This is in line with this result.

Penetration effects in thermal problem

$$\rho C \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$$

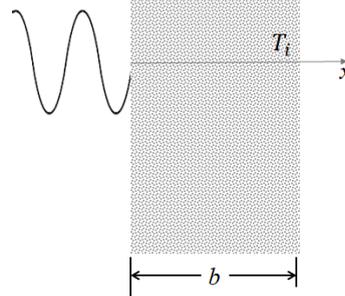
with $T = T_i$ for all x for $t \leq 0$,
and $T(x = 0) = T_o \sin \omega t$ for $t > 0$

Introduce $\theta^* = \frac{T - T_i}{T_o}$, $x^* = \frac{x}{b}$ and $t^* = \omega t$

$$\frac{\partial \theta^*}{\partial t^*} = \frac{k}{\rho C} \cdot \frac{1}{b^2 \omega} \frac{\partial^2 \theta^*}{\partial x^{*2}} \quad \frac{\alpha}{b^2 \omega}$$

with $\theta^* = 0$ for all x^* for $t^* \leq 0$,
and $\theta^*(x^* = 0) = \sin t^*$ for $t^* > 0$

$$T = T_i + T_o \sin \omega t$$



Let us do a similar problem, but in heat conduction. We have a slab of thickness b , one end of which is maintained at T_i , the initial temperature of the slab, and the other end is subjected to a fluctuating temperature $T = T_i + T_o \sin \omega t$. The amplitude T_o and the circular frequency ω . The applicable equation is $\rho C \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$. In this $\rho C \frac{\partial T}{\partial t}$ is the energy rate of storage of energy within this, and $k \frac{\partial^2 T}{\partial x^2}$ is the rate of conduction of energy. They are equal. To be solved

with the boundary condition $T = T_i$ for all x for $t \leq 0$, and $T(x = 0)$, that is this end, is $T_o \sin \sin \omega t$ for $t > 0$.

And then the temperature of this is always maintained at temperature T_i . Let us normalize this equation, and as we have done in other thermal problems, it is not the temperature but the temperature difference that matters. So we use a variable θ is equal to T minus T_i , and normalize it with the amplitude of the temperature fluctuation T_o . We normalize x by b , and time with $1/\omega$. And when we do this, these changes. This is the equation that results. Normalized equation is $\frac{\partial \theta^*}{\partial t^*} = \frac{k}{\rho C} \cdot \frac{1}{b^2 \omega} \frac{\partial^2 \theta^*}{\partial x^{*2}}$. With $\theta^* = 0$ for all x^* for $t^* \leq 0$ and $\theta^*(x^* = 0) = \sin \sin t^*$ for $t^* > 0$. Here again, if we expect these to be normalized then this parameter measures the relative importance of the conduction term and the heat storage term. This coefficient in terms of the thermal diffusivity, α , which is $\frac{k}{\rho C}$ becomes $\frac{\alpha}{b^2 \omega}$.

So this is the parameter that determines the relative importance of the two terms. Now for this parameter to be much less than 1 as before, we get $\frac{\partial \theta^*}{\partial t^*} = 0$. And since this is a first order equation, it can accommodate only one boundary condition, and so we run into problem. It cannot accommodate the boundary condition at $x=0$.

Penetration effects in thermal problem

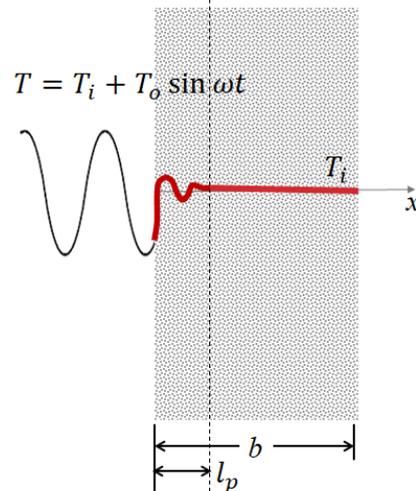
Near the left boundary, the characteristic length over which the temperature change takes place is like l_p , so that if we define a normalized variable $\tilde{y} = y/l_p$, we get

$$\frac{\partial \theta^*}{\partial t^*} = \frac{\alpha}{l_p^2 \omega} \frac{\partial^2 \theta^*}{\partial \tilde{y}^2}$$

$$\frac{\alpha}{l_p^2 \omega} \sim 1, \text{ so that } l_p \sim \sqrt{\frac{\alpha}{\omega}},$$

and we can write

$$\frac{\partial \theta^*}{\partial t^*} = \frac{\partial^2 \theta^*}{\partial \tilde{y}^2}$$



So similar to what we did before, there has to be an inner region, a boundary layer region, where the temperature fluctuates. So this area here is the area in which the temperature fluctuates. Near the left boundary the characteristic length over which the temperature change takes place is like l_p , a penetration distance. So near the left boundary, the characteristic length over which the temperature change takes place is like l_p . So that if we define a normalized variable $\tilde{y} = y/l_p$, we have chosen a different nomenclature \tilde{y} not y^* because with y^* you

may get confused by the variable when the characteristic length is b . So the resulting equation

$$\text{is } \frac{\partial \theta^*}{\partial t^*} = \frac{\alpha}{l_p^2 \omega} \frac{\partial^2 \theta^*}{\partial y^2}.$$

Now as before within this penetration region within this boundary layer, within this inner region, the two terms should be of the same order, and if the two terms should be of the same

order, then $\frac{\alpha}{l_p^2 \omega}$ should be order 1. So that the thickness of this layer is like $\sqrt{\frac{\alpha}{\omega}}$, and we can

write $\frac{\partial \theta^*}{\partial t^*} = \frac{\partial^2 \theta^*}{\partial y^2}$. And the temperature profile through this slab would be something like as

shown.

Of course this is at a given instant of time. At another instant in time this could change, but the change would be largely restricted to this region l_p , and the temperature in the outer region would be constant at T_i .

Thank you.