Similitude And Approximations In Engineering,
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Week - 05
Lecture - 19

Welcome back. We continue in this lecture with the basis of approximations. We learn how to make and justify approximations in engineering sciences. We in the last lecture had analyzed the problem of damped oscillators starting from the governing equations and the initial and boundary conditions. We showed that the two non dimensional pi numbers $\frac{ct_c}{m}$ and $\frac{kt_c^2}{m}$ measured the relative magnitudes of the damping force and the spring force terms respectively. Here t_c stands for a characteristic time.

Damped oscillator revisited

	Law	Scale factor	Pi-number
Inertial	$F_i = ma$	$k_{F,i} = k_m k_L k_t^{-2}$	
Damping	$F_\mu = CV$	$k_{F,\mu} = k_C k_L k_t^{-1}$	$\frac{ct_c}{m}$
Spring	$F_s = kx$	$k_{F,s} = k_k k_L$	$\frac{kt_c^2}{m}$

We could arrive at these estimates of relative magnitudes directly from the scale factors as well without invoking the governing equations and that is what we will do now. In this example there are three forces the initial forces, damping forces and the spring forces. We write the law for each force the initial force is ma, damping force is cv, where c is the coefficient of damping, and spring force is kx, where k is the spring constant. We can convert these laws into scale factors relations like shown.

And then by equating the scale factors of inertial and damping forces we can get $\frac{ct_c}{m}$ as a pi number, and by equating inertial and spring forces we could get $\frac{kt_c^2}{m}$ as the pi numbers. The first pi number $\frac{ct_c}{m}$ is the ratio of the damping force to the inertial force while this last one is the ratio of the spring force to the inertial force.

We could, similarly, obtain the relative magnitude of terms in the baked-potato problem using the scale factors for the energy currents:

	Applicable relations	Scale factor relations	Pi-numbers
Conduction in solid	$\dot{Q} = Aq = Ak_s d\theta/dx$	$k_{c,s} = k_{k_s} k_L k_{\Delta\theta}$	$\frac{\dot{Q}_{c,s}}{k_s L \Delta\theta}$
Convection	$\dot{Q} = Aq = Ah\Delta\theta$	$k_{cov} = k_h k_L{}^2 k_{\Delta\theta}$	$\frac{\dot{Q}_{cov}}{hL^2 \Delta\theta}$
Heat storage in solids	$\dot{Q} = (\rho V)c_v (d\theta/dt)$	$k_u = k_\rho k_L{}^3 k_{c_v} k_{\Delta\theta} k_t{}^{-1}$	$\frac{\dot{Q}_u t}{\rho L^3 c_v \Delta\theta}$

$\dfrac{hL}{k_s}$ Bi number

$\dfrac{L^2 \rho c_v}{k_s \tau} = \dfrac{L^2}{\alpha\tau}$ Fo number

We could similarly obtain the relative magnitude of the terms in baked potato problem using the scale factors for the energy currents. The three energy currents are important: conduction in solid, convection, and heat storage in solids. The applicable relations $\dot{Q}$ is area times the conductive heat flux. So, it is $Ak_s d\theta/dx$ and so the scale factor relation is the scale factor of conduction solids $k_{c,s} = k_{k_s} k_L k_{\Delta\theta}$. From this we obtain this as the primary pi number $\dfrac{\dot{Q}_{c,s}}{k_s L \Delta\theta}$ where the characteristic nature of the quantities involved is understood. Similarly, the applicable relation for convective flux or the convective current is $\dot{Q}$ equal to A times the convective flux given by the Newton's law of cooling. So, this is $Ah\Delta\theta$, and so $k_{cov} = k_h k_L{}^2 k_{\Delta\theta}$. The heat storage in solid is $\dot{Q} = (\rho V)c_v\left(\dfrac{d\theta}{dt}\right)$.

And from this we get the scale factor relation as $k_u = k_\rho k_L{}^3 k_{c_v} k_{\Delta\theta} k_t{}^{-1}$, and so $\dfrac{\dot{Q}_u t}{\rho L^3 c_v \Delta\theta}$ is the resulting pi number. From the first two we get $\dfrac{hL}{k_s}$, which can be seen as the ratio of the convective flux divided by the conductive flux in solid. And from the first and third basic pi numbers we get $\dfrac{L^2 \rho c_v}{k_s \tau}$. Further, $\dfrac{k_s}{\rho c_v}$ is replaced by α, the thermal diffusivity of the solid material. So, that we get $\dfrac{L^2}{\alpha\tau}$ as the Fourier number and this is the ratio of heat storage in solid divided by the convective flux at the boundary. So, thus we see in both these examples we obtain the same ratios of the forces or the same ratios of the fluxes as we obtained from the original governing equations and boundary conditions.

Spreading of oil slick on oceans

Spreading of an oil slick on sea-water is controlled primarily by

- the viscous forces that the sea-water applies on the slick,
- the (unsteady) inertial forces,
- the buoyant weight of the slick, and
- the surface tension forces

It is asserted that the surface tension forces are insignificant to begin with, and come into play only after the slick is very thin, which may be days after the spill.

How long before surface tension forces come into play?

There is no need to start from those equations we could start from this consider the ratios of scale factors relation. Let us do a new example the spreading of all slick on oceans. When an oil is dumped in an ocean the oil spreads out on the sea water. This spreading of oil on sea water is controlled primarily by the viscous forces that the sea water applies on the slick. The unsteady inertial forces in the oil the buoyant weight of the slick that is its own weight minus the buoyancy and the surface tension forces.

It is asserted that the surface tension forces are insignificant to begin with and come into play only after the slick is very thin which may be days after the slick. How long before the surface tension forces come into play we have to determine this. So, we will take the law approach since we are worried about the surface tension forces and when they come into play. Let us begin with estimating the surface tension forces and then compare it with one of the other forces say the inertial forces. The surface tension forces are estimated as $F_{\sigma,c}$ which is approximately σL_c, where σ is the surface tension between the spilled oil and the sea water.

Spreading of oil slick

We first obtain an estimate of the surface tension forces and then compare it with one of the other forces, say, the inertial forces.

The surface tension forces are estimated as $F_{\sigma,c} \sim \sigma L_c$, where σ is the surface tension between the spilled oil and the sea-water.

The unsteady inertial forces in the slick are estimated as $F_{i,c} \sim (mass)_c \times (\partial V/\partial t)_c \sim (\rho_s L_c^3) \times (V_c/t_c) \sim \rho_s L_c^3 V_c/t_c$, where ρ_s is the density of the slick.

The surface tension forces are negligible as long as $F_{\sigma,c}/F_{i,c} \sim \sigma t_c/\rho_s L_c^2 V_c$ is small compared to unity.
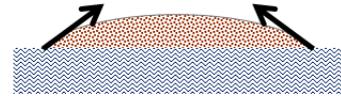
This red drop represents a mass of oil on sea water. The surface tensions acting on this with the angle of contact act in these directions, and the component of these horizontally will try to oppose the expansion of the slick. The unsteady inertial forces on the slick $F_{i,c}$ are estimated as mass times the acceleration, dV/dt. So, that is like $(mass)_c \times \left(\frac{\partial V}{\partial t}\right)_c$ or $\left(\rho_s L_c^3\right) \times \left(V_c/t_c\right)$, where ρ_s is the density of the slick. The surface tension forces are negligible as long as $\frac{F_{\sigma,c}}{F_{i,c}} \sim \frac{\sigma t_c}{\rho_s L_c^2 V_c}$ is small compared to unity.

That is, this is the ratio. As long as this is less than 1, the surface tension forces are small compared to the inertial forces. The characteristic length L_c of the slick can be taken as the cube root of the volume of the oil that has been spilt on the ocean. The velocity V_c can be estimated as L_c/t_c, and so the actual time t can be used for t_c, and then the relevant criteria for neglecting the surface tension forces is that $\frac{\sigma t^2}{\rho_s V}$ be less than 1, or t be less than $\sqrt{\frac{\rho_s V}{\sigma}}$, where sigma is the surface tension. So, surface tension does not come into play for times less than under the $\sqrt{\frac{\rho_s V}{\sigma}}$.

The typical values of ρ_s and V, this may yield a period of several days during which surface tension forces are not significant. One estimate of sea water and crude surface tension force is sigma is equal to 0.02 N/m. A 100,000 ton oil spill is considered a medium size spill. For this spill the above equation suggests the surface tension forces do not show up for times of the order of 23 days.

It is only after 23 days would the surface tension force would need to be considered. This picture shows an oil slick in the Gulf of Mexico that has been spreading since 2004. You see the rainbow colors. The colors are visible only for film thicknesses between 150 nanometers and 1 micrometers. So, this suggests that in last 18 years the film has spread so much that its thickness is now of the order of 1 micrometer or less.

Let us do another example. An interesting example of flow past a pulsating sphere. Consider the flow of a uniform stream of fluid at a velocity V_o about a pulsating sphere whose diameter can be represented as $D_o(1 + \varepsilon \sin \sin \omega t)$, where ε is small. Under what conditions can the flow be treated as a quasi steady flow? What would a quasi steady flow mean in this context? A quasi steady flow would mean that at any given instant the flow would behave like a steady flow. That is, we could neglect the inertia forces.

Flow past a pulsating sphere

Since the diameter of the sphere varies like
$D = D_o(1 + \varepsilon \sin \omega t)$, the normal velocity at the
surface is $\frac{dD}{dt} = \frac{1}{2}\varepsilon\omega D_o \cos \omega t$, and the acceleration
of the surface is like $\frac{dD}{dt} = \frac{1}{2}\varepsilon\omega D_o \cos \omega t$

$V_o \rightarrow$
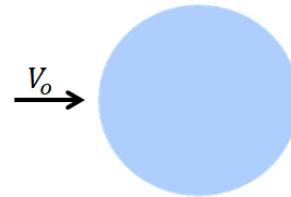
If F_u is the force of unsteady inertia in the fluid, then
$k_{F_u} = k_\rho k_L{}^3\left[\varepsilon k_\omega{}^2 k_L\right]$

This gives $F_u \sim \varepsilon\rho\omega^2 D_o{}^4$

If F_i is the force of convective inertia in the fluid, then
$k_{F_i} = k_\rho k_L{}^2 k_V{}^2$

This gives $F_i \sim \rho D_o{}^2 V_o{}^2$

But the flow is not really steady It is changing with time. Because at any given time, it must accommodate the boundary conditions that is, the normal flow at the surface of the sphere is equal to the velocity of the surface or the diameter is as it is. The unsteadiness in the flow enters through the normal velocity boundary condition that the normal velocity of fluid that the surface must be equal to the normal velocity of the surface itself at every instant of time. Since the diameter of the sphere varies like D is equal to $D_o(1 + \varepsilon \sin \sin \omega t)$, the normal velocity of the surface is the first derivative of D which gives $\frac{1}{2}\varepsilon\omega D_o \cos \cos \omega t$, and the acceleration on the surface is like another derivative of this and this would be like $\varepsilon\omega^2 D_o \sin \sin \omega t$. So, we know what is the acceleration at the surface and if we know this we can estimate the unsteady force, the unsteady inertia in the fluid.

The scale factor for that k_{F_u} is k of mass times k of acceleration, which is $\varepsilon\omega^2 D_o \sin \sin \omega t$. So, $k_{F_u} = k_\rho k_L{}^3\left[\varepsilon k_\omega{}^2 k_L\right]$. This gives the unsteady force varies like $F_u \sim \varepsilon\rho\omega^2 D_o{}^4$. This is the unsteady force. If Fi is the force of convective inertia in the force then $k_{F_i} = k_\rho k_L{}^2 k_V{}^2$. This we have obtained a number of times earlier. So, this gives you $F_i \sim \rho D_o{}^2 V_o{}^2$. So, the unsteady force is $\varepsilon\rho\omega^2 D_o{}^4$, and the inertia force of convective acceleration is $\rho D_o{}^2 V_o{}^2$. So, the unsteady force can be neglected if $\varepsilon\rho\omega^2 D_o{}^4$ is much less than $\rho D_o{}^2 V_o{}^2$, which gives $\frac{\varepsilon\omega^2 D_o{}^2}{V_o{}^2}$ is much less

than 1. That is the unsteady forces can be neglected if εSt^2 is much less than 1, with Strouhal number as $\omega D_o / V_o$. If this condition is obtained, then the flow can be taken as a quasi steady flow about a sphere having the instantaneous value of the diameter.

Another example the decay of a vortex. When an aircraft takes off from a runway, a swirling flow is set up. This picture which is from original flow visualization experiments of Ludwig Prandtl from the turn of the last century shows an aerofoil moving in that direction and when the aerofoil moves in that direction, there is a vortex that is formed in the flow.

And as the aerofoil moves away to the left this vortex stays there. In an actual aircraft, this vortex, which is termed as a starting vortex, is formed at the airport, and as the aircraft takes off this vortex is left on the runway and decays with time. This swirling flow as I said is termed as a starting vortex and the flow within it is modeled by a free vortex, in which the velocity components are given by $V_\theta = k/r$, and V_r is equal to 0. No radial velocity and the tangential velocity varies inversely with radius. Small far away, and increasing as you move towards the center of the vortex.

This is the kind of the tangential velocity profile that we get. As r decreases, the velocity increases. The large velocity gradients near the origin set up significant viscous forces changing the flow pattern with time. We show a similar flow here. Flow in a tornado is a swirling flow that goes around like this.
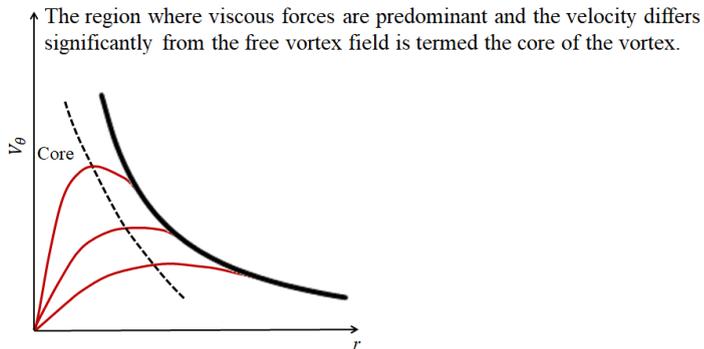
Decay of Vortex

The large velocity gradients near the origin set up significant viscous forces, changing the flow pattern with time, as shown and dissipating the energy of this vortex.

$$V_\theta = k/r$$
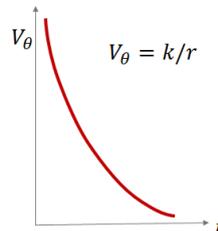
The velocity is large at the center and small far away at any given cross section. The large velocities near the center give rise to very low pressure that picks up the debris from the ground and so that makes the core visible. The core is also visible because of condensed moisture in there at low pressures. This velocity profile V_θ versus r. Because of the large gradient near core, the flow near the center is viscous and since it is viscous it is more like a forced vortex, where the velocity varies directly with r; increases with r, and after achieving a maxima the free vortex takes over and the flow velocity varies like this.

Decay of Vortex

The region where viscous forces are predominant and the velocity differs significantly from the free vortex field is termed the core of the vortex.

The viscous action in this core results in dissipation of energy and with time the velocity profile changes. The location of the maximum velocity marks the edge of what we can consider as a core. So, as time increases the radius of the core increases. The region where viscous forces are predominant and the velocity differs significantly from the free vortex field is termed as the core of the vortex. This graph shows the actual radius versus velocity of an actual tornadoes which is known as a Spencer tornado in literature, and a predicted model by Hao, which has been published recently and this is based on considering a viscous cores superimposed on a free vortex.

We have to deduce how the radius of this core varies with time. So, we obtain the estimates of the various forces: the viscous forces and the unsteady forces are the two relevant forces in this decay. The viscous forces can be estimated, as always, as F_μ is equal to $\mu V_c L_c$, exactly the same as we obtained a number of times earlier wherever we estimated the viscous forces. And the unsteady forces are like mass times the acceleration, the unsteady acceleration with time, and that is V_c/t_c. And so, the viscous forces are negligible whenever $F_\mu \ll F_u$, which gives us that the characteristic length is like $\sqrt{v t_c}$. If we take the characteristic time as the actual time, then the radius of the core would be like the characteristic length, and that varies like $\sqrt{v t_c}$, as a square root of time.

Another example let us calculate the time required to acquire the terminal velocity in a low Reynolds number flow. When a ball falls in water or a helium balloon rises in air, a steady velocity, termed as the terminal velocity, is attained when the drag force on the ball or the balloon becomes equal to the net weight. For what order of time the accelerating motion cannot be neglected. This is of the same order as the time required to acquire the terminal velocity.

Also determine the time after which the unsteady inertial terms for the fluids are unimportant too. The earlier case was when the unsteady forces of the solid were important. Now, we ask the time after which the unsteady inertial forces for the fluid are also unimportant. Then as a numerical we will do, for a steel sphere of diameter 0.1 millimeter, a very tiny steel sphere, settling in water, we estimate these two characteristic times and then try to explain the physical significance of these two times.

Time to acquire the terminal velocity in low Re flows

1	Inertial forces on sphere	$\sim \rho_s D^3 V_c / t_c$
2	Gravity forces on sphere	$\sim (\rho_s - \rho_f) g D^3$
3	Viscous forces	$\sim \mu V_c D$
4	Inertial, fluid convection	$\sim \rho_f D^2 V_c^2$
5	Inertial, fluid unsteady	$\sim \rho_f D^3 V_c / t_c$

Ratio to the viscous forces is $\rho_s D^2 / \mu t_c$.

Thus the unsteady forces on the solid can be neglected if $\frac{\rho_s D^2}{\mu t} < 1$, or $t > \rho_s D^2 / \mu$

For a steel sphere of diameter 0.1 mm settling in water ($\rho_s = 7{,}480 \text{ kg/m}^3$) this is 0.028 s.

   There are five kinds of forces in this problem. The inertial force of the sphere itself, the mass of the sphere times the acceleration of the sphere, unsteady forces. For sphere there will be only unsteady forces. There is no convective acceleration for a solid sphere. The gravity forces of the sphere are the net of the weight and the buoyancy, and they would be like the density differences of the solid and the fluid, the volume, and the acceleration due to gravity.

   Viscous forces, as before, would be like $\mu V_c D$. C everywhere stands for the characteristic value of the relevant quantity. The inertial force of fluid convection, and that also we obtained many times as $\rho_f D^2 V_c^2$, which is exactly the mass, $\rho_f D^3$ times the convective acceleration which is like VdV/dx. VdV/dx would be like V_c^2 / D. So, the net is this rho $\rho_f D^2 V_c^2$.

   And the unsteady fluid inertia from the mass of fluid times the acceleration which is V_c / t_c. We take two forces, viscous and inertial, and from this we construct the ratio of the inertial force of the sphere to the viscous force as $\rho_s D^2 / \mu t_c$. Thus, the unsteady forces on the solid can be neglected if $\rho_s D^2 / \mu t_c$ is less than 1, or for times greater than $\rho_s D^2 / \mu$. For a steel sphere of diameter 0.1 millimeter settling in water, the density of steel is 7480 kilogram per meter cubed. And so this time is obtained as 0.028 seconds, a very small time, less than three hundreds of a second after which that steel sphere would have acquired a constant velocity, the terminal velocity. This is because of a very low diameter of the sphere, which results in a very small Reynolds number. Now, let us look at when the unsteady inertial forces in the liquid can be neglected.

Time to acquire the terminal velocity in low Re flows

1	Inertial forces on sphere	$\sim \rho_s D^3 V_c / t_c$
2	Gravity forces on sphere	$\sim (\rho_s - \rho_f) g D^3$
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Following the same logic, the unsteady or inertial effects in *fluid* are negligible for $t > \rho_f D^2 / \mu$. This time for a sphere of diameter 1 mm is 0.011 s.

This time for a sphere of diameter 0.1 mm is 0.011 s.

Following the same logic the unsteady or inertial effects in liquids are negligible for $t > \rho_f D^2 / \mu$, from this and this. So, we get $t > \rho_f D^2 / \mu$ we can neglect the inertial effects of the fluids. This time for a sphere of diameter 1 millimeter is 0.011 second for water. ρ_f, we have to choose the fluid. So, we choose this as water, and this comes out to be 0.011 a second. The earlier time was 0.028 seconds. So, for time like 0.028 second the inertia of the solid cannot be neglected and we can neglect it only after this time. In this region up to 0.011 second, the inertia of the fluid cannot be neglected. So, over this time we can neglect the inertia of the fluid, but cannot neglect the inertia of the solid. That is this sphere is still accelerating during this time, but the motion of the fluid can be considered to be quasi steady. At any instant during this interval, the fluid would behave like in a steady flow, except that the velocity of the fluid is is changing. That seems like a contradiction, but it is not. The whole picture at time, say 0.015 seconds, whatever be the velocity of the solid sphere at that time, the fluid will behave like it is moving past the solid at that velocity, at that constant velocity. At a later time the velocity of the solid sphere would be larger.

So, the velocity field in the fluid would be like a steady velocity field with the ball motion at a larger speed. This is what is meant by quasi-steady fluid flow. A human sperm with a head of 50 micrometer diameter is traveling at 200 micrometer per second by moving its flagella in seminal plasma of kinematic viscosity about 10^{-4} m²/s which is about a 100 times that of water. It develops a Reynolds number of order of 10^{-2}, a very small Reynolds number. Let us calculate the length over which the sperm will coast if it stops moving flagella and just coast.

So, that at time t is equal to 0, if it stops moving its flagella, how long would it continue moving? What distance would it cover before it comes to rest, if it is traveling at 200 micrometers per second when it stops moving its flagella. The deceleration in experience is purely because of the viscous forces which have been estimated earlier as $\mu D V_o$. The mass of the sperm is like $\rho_s D^3$. So, that the deceleration is of the order of $\mu V_o / \rho_s D^2$, force divided by mass. And the coasting distance can then be estimated from V_o^2, the initial velocity squared

divided by acceleration, a formula that you did in your high school. Or that or $\frac{L}{D} \sim \frac{\rho D V_o}{\mu}$, which is like $\sim \frac{\rho_s}{\rho} Re$, where Re is the Reynolds number. Since $\frac{\rho_s}{\rho}$ is like 1, the density of the sperm and the density of the plasma is about the same, the Reynolds number itself becomes the non-dimensional cruising distance. The value of Reynolds number is 10^{-2}. So, that L/D is 10^{-2} or L is like 0.5 micrometers. This means that the sperm will stop moving almost immediately after it stops moving its flagella.

We see from this that the response of fluid at low Reynolds number to the motion of the boundaries is practical instantaneous. As soon as the flagella stops moving, the response is immediate and the sperm stops moving immediately. One more example: flow through channels of slowly varying area. We have done in a fluid mechanics course the motion of a fluid in a straight pipe of a constant diameter pipe.

Flow through channels of slowly varying area

Flow through a constant area duct is fully-developed: inertia forces are neglected all together.

For small dR/dx, we can neglect, area variation and consider the channel to be '*locally* fully-developed'
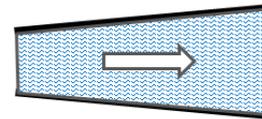
VR^2 is constant. So, $\frac{dV}{dx} = -\frac{2V}{R}\frac{dR}{dx}$

Estimate of convective acceleration $\sim V\frac{\partial V}{\partial x} \sim \frac{V_c^2}{D}\alpha$, or $\frac{V_c^2}{D}\alpha$

And, the estimate of convective forces is $\rho L D V_c^2 \alpha$

Estimate of viscous term is $\sim \mu \frac{V_c}{D} DL$

Consider next the flow in a pipe. Now if the pipe walls are diverging slowly, the radius is changing with x, very slowly. Then, can we approximate this as a locally fully developed flow? If dR/dx is small, we can definitely neglect area variations and consider the channel flow to be locally fully developed. But how long should that small dR/dx be for this to be considered small so that we can make this approximation. Let us find that out.

Now from continuity equation we get VA, velocity times the area as constant. So, that VR^2, if R is the radius of the tube as a function of x. VR^2 is constant. So, $\frac{dV}{dx} = -\frac{2V}{R}\frac{dR}{dx}$. This is the change of velocity with time. The estimate of convective acceleration would be like VdV/dx, and so this would be $\frac{V_c^2}{D}\alpha$. dR/dx is replaced by α, the angle of the slope of the wall. And the estimate of the convective force are obtained by multiplying it by mass, which is $\rho L D^2$. And so we get the estimated convective force $\rho L D V_c^2 \alpha$. Notice that here we consider the mass as ρ times the volume, and the volume is written as D^2L. Estimate of the viscous force $\mu \frac{V_c}{D} DL$, the

surface area around the walls times the velocity gradient in the radial direction, which would be $\frac{V_c}{D}DL$.

This means that the inertia forces are negligible if if $\rho LDV_c^2\alpha$, is much less than μLV_c, or $\alpha \ll \frac{\mu}{\rho DV_c}$, which is 1 over a Reynolds number. Thus, if the angle of pipe is less than 1 over Reynolds number then the flow in the channel can be considered to be locally fully developed. This shows you the variation of alpha permissible for making this assumption versus the Reynolds number. For very low Reynolds number much larger alphas are permissible.

At the critical Reynolds number 2300, it turns out to be about 0.025 degrees or a very small angle. These examples should illustrate for you how to make approximations in various areas. We will continue with this in the next lecture. Thank you.

Thank you.