Similitude And Approximations In Engineering,
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Week - 05
Lecture - 18

Welcome back once again. We will continue discussing the basis of approximations in engineering. Here again we will do more examples to learn to make and justify approximations in engineering sciences. In the last lecture we have said that in order to establish which terms in the governing equations are negligible, their magnitudes need to be estimated. And one very powerful method of making such estimates is to normalize each variable by scaling it with its characteristic value. If the characterizing values of the quantities are chosen properly, the normalized variables are then expected to be order unity, that is, except at certain isolated points in the flow field The values of the transformed variables are expected to be neither very large nor very small compared to unity.

Further, it is hoped that the dimensionless derivatives of the various physical quantities appearing in the equations are also of order 1. When we saw that in well-behaved function this is possible, but sometimes this breaks down and the approximations or the estimates obtained are erroneous. We will learn to deal with the situation in a later lecture. If the proper estimates are obtained, then the coefficient of the various terms reflect the estimate of the corresponding term in the equation.

Based on these estimates we can then decide what terms can be neglected. We have been doing this for two examples that we discussed in the last class. But life is never easy, there are pitfalls on the way. Be on the lookout. We had seen one pitfall when we discussed the large Reynolds number flows.

A damped oscillator

$$m\ddot{x} + c\dot{x} + kx = 0$$

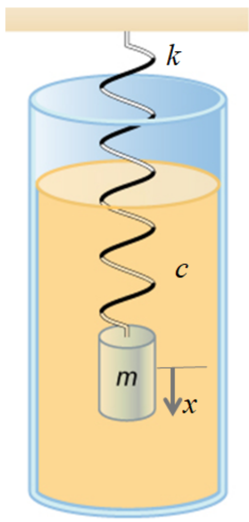
BCs: $x(t=0) = x_o$; $\dot{x}(t=0) = 0$

Normalize:

$$\frac{mx_c}{t_c^{\,2}}\ddot{x}^* + \frac{cx_c}{t_c}\dot{x}^* + kx_c x^* = 0$$

$$\ddot{x}^* + \frac{ct_c}{m}\dot{x}^* + \frac{kt_c^{\,2}}{m}x^* = 0$$

with $x^*(t^*=0) = 1$ and $\dot{x}^*(t^*=0) = 0$



We start with a simple example of a damped oscillator. The equation for a damped oscillator is $\ddot{x} + c\dot{x} + kx = 0$. The first term represents the inertial terms mass times acceleration. The second is the damping term, the damping coefficient c times the velocity is the damping force and kx is the spring force. So, these three forces sum out to 0 as there is no external force applied to the system.

We start with a very simple boundary condition that $x(t = 0) = x_o$; $\dot{x}(t = 0) = 0$. We normalize the dependent variable x with its characteristic value x_c and we normalize time with a characteristic time t_c and this is what we obtain: $\frac{mx_c}{t_c^2}\ddot{x}^* + \frac{cx_c}{t_c}\dot{x}^* + kx_c x^* = 0$. And if we make the coefficient of the first term that is the inertial term 1, then this is the form that we get: $\ddot{x}^* + \frac{ct_c}{m}\dot{x}^* + \frac{kt_c^2}{m}x^* = 0$. $\frac{ct_c}{m}$ is an estimate of the viscous force and $\frac{kt_c^2}{m}$ is the estimate of the spring force provided that we have taken x_c and t_c appropriately, that is they are indeed characteristic of the system. The boundary conditions become $x^*(t^* = 0) = 1$ and $\dot{x}^*(t^* = 0) = 0$.

If we solve this equation and it is easy to solve, this is the kind of variation that we see of x^* star with t^*. Recall that we are not chosen t_c yet we said only the t_c is the characteristic time for what is it value. Now if this mass is to oscillate, then the spring force must be present. The coefficient of inertia term is 1. So, this term to be present means that this coefficient should be of order 1. And if this is order 1, then this suggest that t_c may be chosen as $\sqrt{\frac{m}{k}}$ without any loss of generality. This is a characteristic time in this problem. We will go further than this. We introduce a symbol $\omega_n = \frac{1}{t_c = \sqrt{\frac{k}{m}}}$. This at times is called the resonant frequency of an undamped oscillator of mass m with a spring constant k. From a high school physics you must recognize the resonant frequency. So, in terms of ω, the equation now reduces to $\ddot{x}^* + \frac{c}{m\omega_n}\dot{x}^* + x^* = 0$.

We could have also used $\frac{kt_c^2}{m}$ for defining t_c, the characteristic time. And if we did this, we will get characteristic time of order of $\frac{m}{c}$ rather than $\sqrt{\frac{m}{k}}$. We will get it as $\frac{m}{c}$. So, there really are two time scales in this problem. One time scale, $\sqrt{\frac{m}{k}}$, and the other time scale $\frac{m}{c}$. What is the meaning of these two time scales?

A damped oscillator

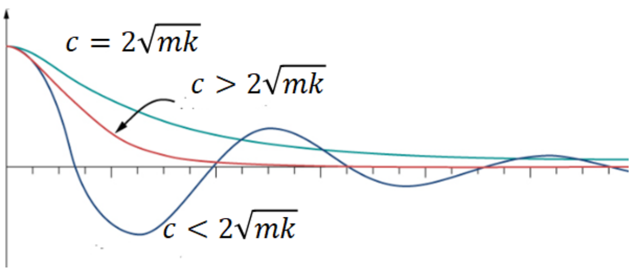
$$\ddot{x}^* + \frac{ct_c}{m}\dot{x}^* + \frac{kt_c^2}{m}x^* = 0$$

We could also have defined t_c from the damping term to get $t_c \sim O\left(\frac{m}{c}\right)$.

Two time scales:

$$t_c = \sqrt{\frac{m}{k}}$$

$$t_c = \frac{m}{c}$$



This dark blue curve is the oscillations when is when c is less than $2\sqrt{mk}$. This the blue line is the variation of x star with the t star when c is less than $2\sqrt{mk}$. That is, when $\sqrt{\frac{m}{k}}$ is much more than time $\frac{m}{c}$. This is a shorter time, and we get these fluctuations of x^*, the vibrations of the oscillator and the frequency that is suggested by $\sqrt{\frac{m}{k}}$. But when $\sqrt{\frac{m}{k}}$time is less, that is, when c is greater than $2\sqrt{mk}$, then the red line represents the solution. It is no longer oscillating. So, this is where this characteristic time dominates. It is a lower value. So, damping takes over before the vibrations take place. And this line is what is called the critical damping. This separates the two domains.

A damped oscillator

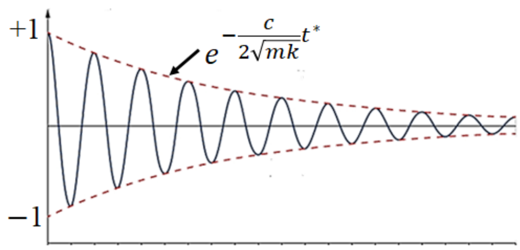
$$\ddot{x}^* + \frac{ct_c}{m}\dot{x}^* + \frac{kt_c^2}{m}x^* = 0$$

Sticking with the first time scale

$t_c = \sqrt{\frac{m}{k}}$, we get the solution

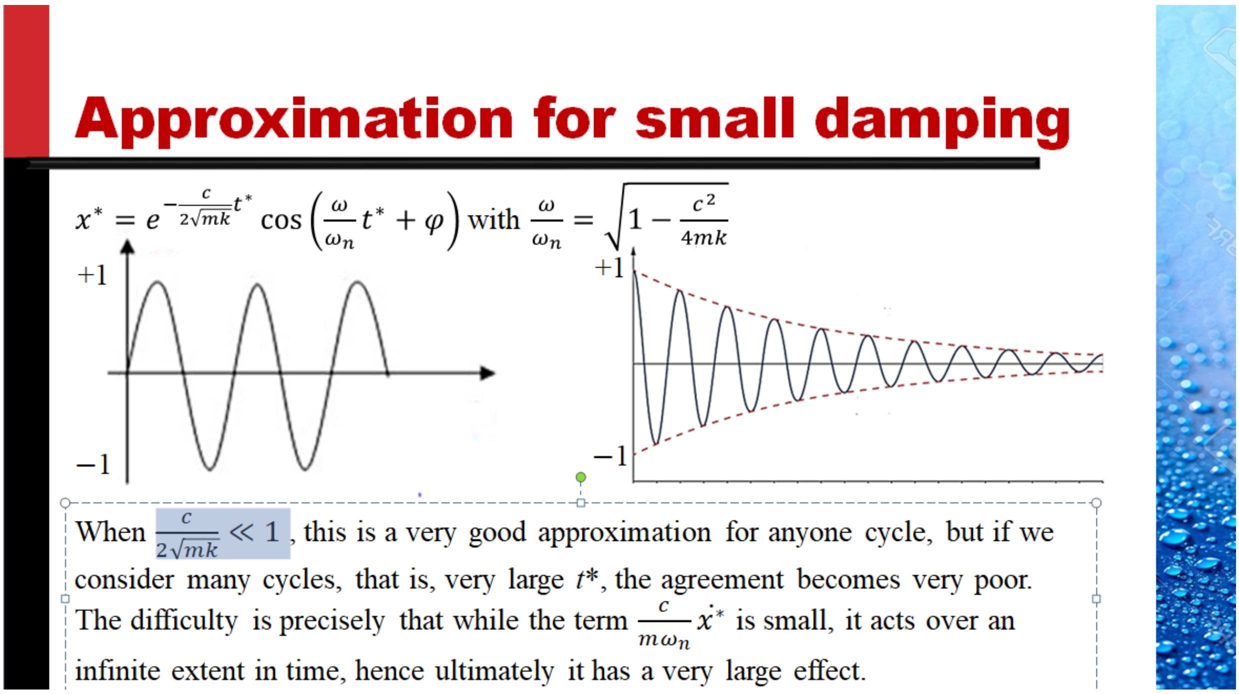
$$x^* = e^{-\frac{c}{2\sqrt{mk}}t^*}\cos\left(\frac{\omega}{\omega_n}t^* + \varphi\right)$$

with $\frac{\omega}{\omega_n} = \sqrt{1 - \frac{c^2}{4mk}}$



The original equation again. If we look at the first case where the time scale is $\sqrt{\frac{m}{k}}$, then we get these solutions. The exact solution of this equation $x^* = e^{-\frac{c}{2\sqrt{mk}}t^*} \cos cos \left(\frac{\omega}{\omega_n}t^* + \varphi\right)$. Notice that in this solution both the time scales are present. This dotted curve, that marks the envelope of the amplitudes, has the time scale of this $\sqrt{mk}/c$. So, the time scale $\frac{m}{c}$ is for the damping of the amplitude, and this time scale, $\sqrt{\frac{m}{k}}$, is the time scale for the vibrations per se.

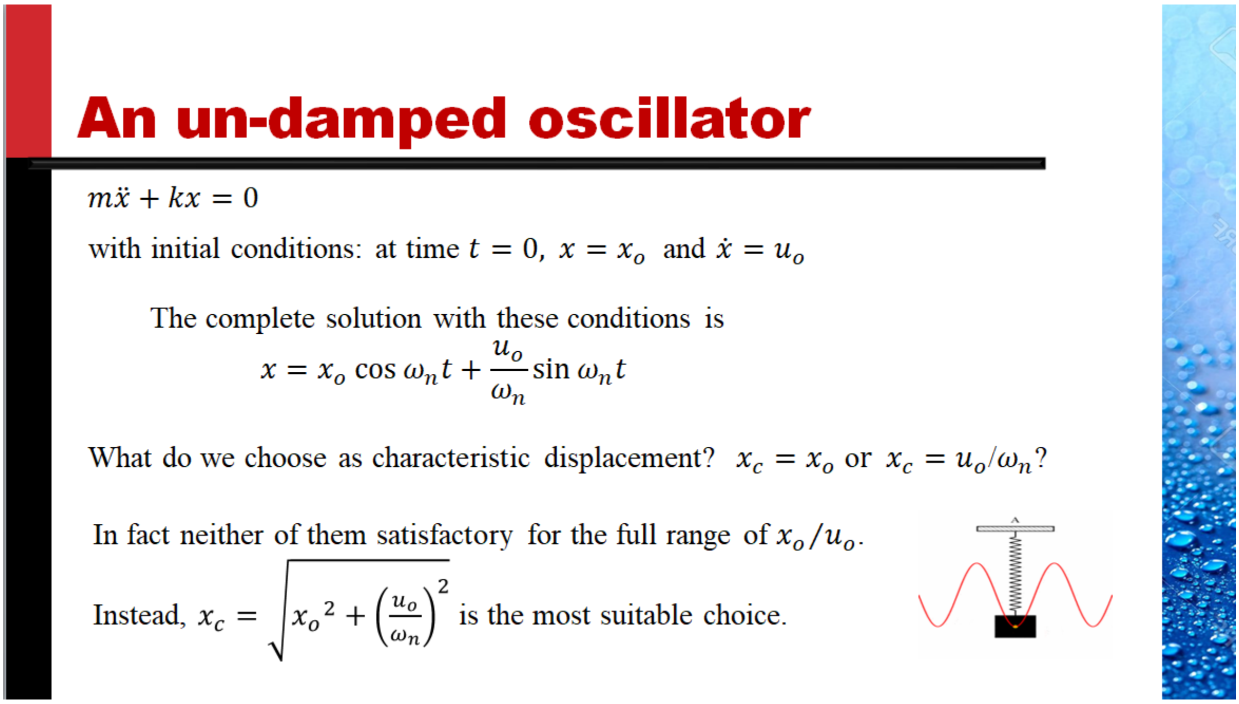
Now, let us consider $\frac{c}{\sqrt{mk}}$ much less than 1. This term $\frac{c}{m\omega_n}\dot{x}^*$ can be dropped from the equation, and the equation reduces to $\ddot{x}^* + x^* = 0$. The equation for simple harmonic motion whose solution is $x^* = \cos cos\, t^*$ with the boundary conditions that were given that with $x^*(t^* = 0) = 1$ and $\dot{x}^*(t^* = 0) = 0$. When $\frac{c}{2\sqrt{mk}} \ll 1$, this approximation, the simple harmonic motion is a very good approximation for any one cycle, but if we consider many cycles, that is, for very large t^*, the agreements become very poor. Initially this picture, and this picture for very small time may be same, but for large times the second picture is entirely different from the first picture. The difficulty is that though this is small the viscous term is small, but now it is acting over a very large time.



This is small, but t star is becoming very large, and when t star becomes very large compared to 1 such that this no longer is small then the approximation breaks down. So, the effect of this viscosity integrated over a large time gives a large effect. So, we need to worry about this while making approximations. A problem like this, that behaves like this, is said to be singular in its limiting behavior. For example, here for time t star very large, that is, the behavior of the solution when the parameter is very small is altogether different from when the parameter is identically 0.

When we dropped this term, the viscous damping was assumed to be 0, but actually it is not 0 it is small, the coefficient is small. So, the behavior of the solution when the parameter is very small is altogether different from when the parameter is identically 0. On the left is the behavior when the parameter is identically 0 and on right is the behavior when the parameter is very small. They are entirely different. This is called singular behavior.

When singular behavior in the limit occurs, we usually try to resolve the difficulty by expanding in the powers of that parameter. So, that this equation is written like this with this term is replaced by ε, ε small. Now, we obtain a series solution of this equation. We assume that $x^* = x_0^* + \varepsilon x_1^* + \varepsilon^2 x_2^* + ...$ We plug this x^* into this equation and then set it equal to 0, and then get a series of equation by equating the coefficients of ε and ε^3, separately. So, the equation when we drop $\frac{c}{m\omega_n}\dot{x}^*$ term results when we take only this, not this. This solution perturbed by this or and this to a higher order. We will do an example of this a little later.



Let us look at undamped oscillator in a little more details to understand what we do when we make approximations. This is the undamped vibrator with the initial conditions that at time $t = 0$, $x = x_o$ and $\dot{x} = u_o$.

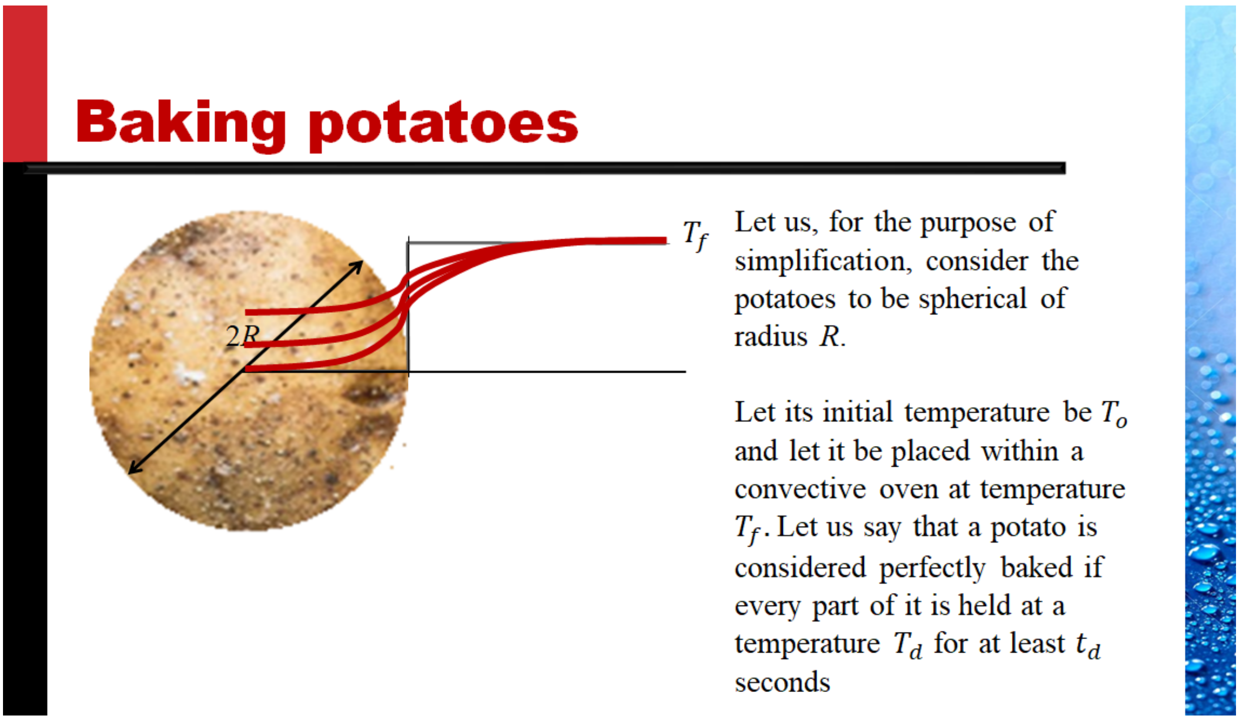
Earlier we had taken $\dot{x}$ to be 0 at time t is equal to 0. Now, we assume there is a initial velocity u as well. The complete solution with these cases is this. This equation is not difficult to solve and we can solve this. This is easy to solve. Now, if we normalize this equation, what is the characteristic value of x that we choose? Do we choose this x_o or do we choose this? In fact, neither of them is satisfactory for the full range of x_o and u_o. Instead $x_c = \sqrt{x_o^2 + \left(\frac{u_o}{\omega_n}\right)^2}$ is the most suitable choice, but it is not easy to see. It is only after we work with the solution get stuck that we realize that if we choose this then we get the best results. Let us choose a characteristic time t_c arbitrarily as τ. The equation and the boundary conditions become $\ddot{x}^* + \frac{k\tau^2}{m}x^* = 0$ with $x^* = \frac{x_o}{x_c}$ and $\dot{x}^* = \frac{u_o \tau}{x_c}$.

Now, let us consider the case when $\frac{k\tau^2}{m}$ is much less than 1. If this is less than 1 this term drops out and we get $\ddot{x}^* = 0$. A trivial equation. Double integration and use of the two boundary conditions give you $x^* = \frac{x_o}{x_c} + \frac{u_o\tau}{x_c}t^*$ as the solution, which matches with the exact solution $x^* = \cos cos\, \omega_n \tau t^* + \frac{u_o}{x_c\omega_n}\sin sin\, \omega_n \tau t^*$. For small values of $\omega_n \tau t^*$, but deviates more and more as t^* increases the same problem as we begin.

For very small t^* it is valid. To increase the range, we will have to use a series solution $\frac{k\tau^2}{m}$ is equal to ε as a small parameter. However, when $\frac{k\tau^2}{m}$is much greater than 1, the first term, the inertia term, drops and the only term that we get left with is x^* and so x^* is 0. And now we cannot even match the boundary conditions. We have no solution at all unless the boundary condition is trivial. It is exactly like what we did earlier for small value of $\frac{k\tau^2}{m}$ we have a solution which is valid for small values of τ.

But when $\frac{k\tau^2}{m}$is much greater than 1 we have no solution at all. This is much like the boundary layer of Prandtl. This much like the high Reynolds number flow where we cannot satisfy the boundaries and a similar solution would be in order. This type of difficulty often occurs whenever we try to drop the highest order term. In such cases we can no longer satisfy all the boundary conditions and we can therefore, obtain solutions only for a certain very limited values of boundary data.

This type of difficulty can sometimes be resolved by various kinds of expansion in powers of the parameter or by a technique which we call boundary layer theory. We will come to that later in the course. Now after this discussion let us do a very simple example.

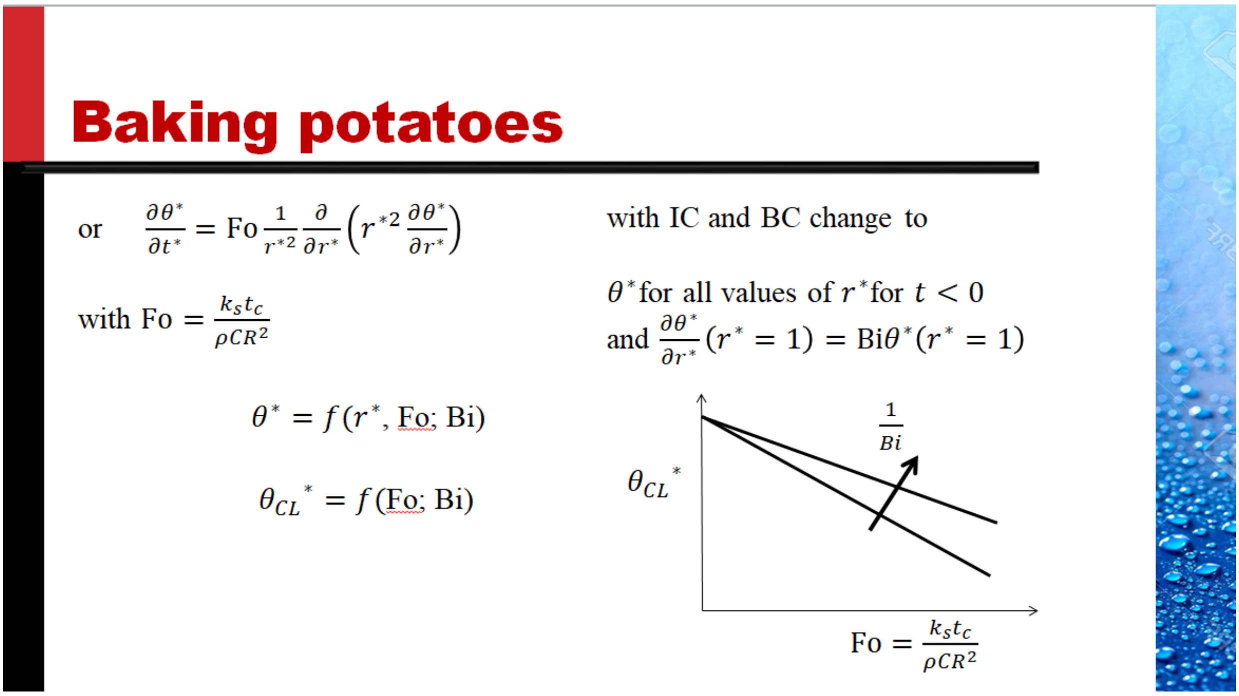


Let us consider determining the time required to prepare perfect baked potatoes in a convective oven. Let us for the purpose of simplification consider the potatoes to be spherical of radius R. Initially this black line represents the temperature. This is the initial temperature T_o throughout the potato, temperature T_f is the temperature in the oven, the air in the oven is temperature T_f and is heating the potato by convection. Then after a little time the temperature profile may look like this. The temperature at the surface would increase, there will be variation of temperature within the potato. The at a very small time the center of the potato may still be at temperature T_o and the temperature in the boundary layer around the potato would look like this.

At a little larger time, the temperature at the center would also start rising, temperature everywhere in the potato was rising and this might be a profile and still a later time the profile may be this. Let the initial temperature of potato be T_o, and let it be placed within a convective oven at a temperature T_f. Let us say that the potato is considered perfectly baked if every part of it is held at a temperature T_d the final temperature for at least a time small t_d seconds. Then the time to bake the potato would be time to bring the center point which is the coolest to the temperature T_d plus this t_d. The sum of two times would be the time required for baking the potatoes.

The unsteady heat transfer equation is spherical coordinates that you would have studied in a course in heat transfer is this. This term represents the heat that increases or the quantitative energy the thermal energy the increases within the potato and $\rho C \frac{\partial T}{\partial t} = k_s \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right)$ represents the heat that is being conducted at the surface of the potato into the potato mass. This equation is to be solved subject to the initial boundary conditions that $T = T_o$ for all values of r for $t < 0$ and $- k_f \frac{\partial T}{\partial r}(r = R) = h\left(T - T_f\right)|_{r=R}$. We normalize the variables by defining a normalized temperature difference $\theta^* = \frac{(T - T_f)}{(T_o - T_f)}$, and $r^* = \frac{r}{R}$ and $t^* = \frac{t}{t_c}$, where t_c is a characteristic time.

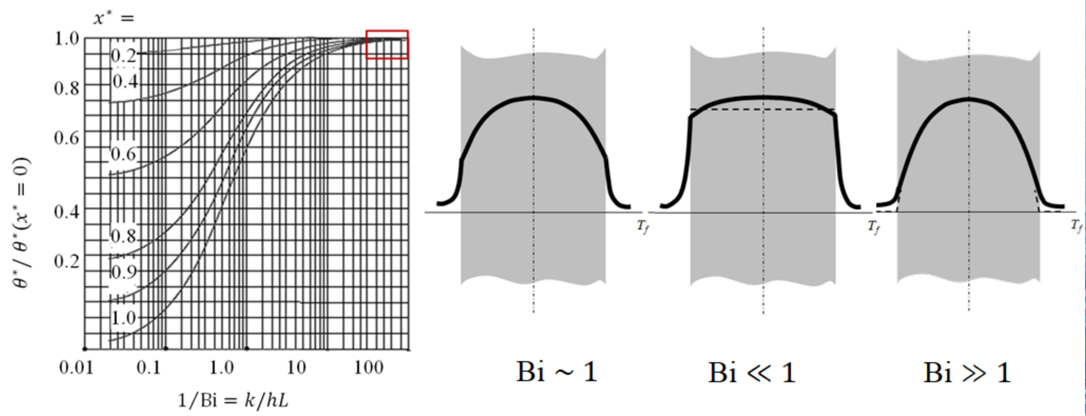
We as yet unknown will try to determine what the t_c should be from the equations and then we normalize the equation this is what we get and by making the coefficient of the first term as 1 this is the coefficient that we get here or $\frac{\partial \theta^*}{\partial t^*} = \frac{k_s t_c}{\rho C R^2} \frac{1}{r^{*2}} \frac{\partial}{\partial r^*}\left(r^{*2} \frac{\partial \theta^*}{\partial r^*}\right)$. We presume that $\frac{\partial \theta^*}{\partial t^*}$ is a order 1 and $\frac{1}{r^{*2}} \frac{\partial}{\partial r^*}\left(r^{*2} \frac{\partial \theta^*}{\partial r^*}\right)$ is a order 1 a regular behavior. So $\frac{k_s t_c}{\rho C R^2}$ measures the importance of the convective term. The initial and the boundary condition change to θ^* for all values of r^* for $t < 0$, and $\frac{\partial \theta^*}{\partial r^*}\left(r^* = 1\right) = \frac{hR}{k_f} \theta^*\left(r^* = 1\right)$. Two parameters, one is $\frac{k_s t_c}{\rho C R^2}$ and using α, the thermal diffusivity for $\frac{k}{\rho C}$, the properties of potato. So, α is the thermal diffusivity of potato $\frac{\alpha t_c}{R^2}$ is a term that we came across earlier, and we had named it the Fourier number Fo and we get this other parameter $\frac{hR}{k_f}$ which it term as Biot number. h is the heat transfer coefficient in the fluid, R is the radius of the potato and k_f is the thermal conductivity of the fluid. Notice that in Fourier number it is thermal conductivity the solid and the biot number is the thermal conductivity of the fluid. So, $\frac{\partial \theta^*}{\partial t^*} = Fo \frac{1}{r^{*2}} \frac{\partial}{\partial r^*}\left(r^{*2} \frac{\partial \theta^*}{\partial r^*}\right)$ is what the equation look like after we inserted Fourier number and Biot number in the equations. So, this will give you $\theta^* = f(r^*, \mathrm{Fo; Bi})$.



We are interested in the center line temperature for center line $r^* = 0$. So, $\theta_{CL}{}^* = f(\mathrm{Fo; Bi})$, and we have seen before that it is given by Heisler chart. This is θ, the non dimensional normalized temperature the center line. This is the time variable Fourier number and there are different curves for different Biot numbers.

So, $\frac{1}{Bi}$ is used as a parameter. If we determine what is the θ that we need we go to the relevant Biot number curve and from this we read what is the Fourier number that we want, and this would then from this we can calculate what is the time that would be needed to get the temperature to the required temperature.

Baking potatoes



We had earlier discussed that we could find the temperature at any other point by using what are called the position correction chart which gives you the non dimensional temperature difference divided by the non dimensional temperature difference at the center line the ratio as a function of $\frac{1}{Bi}$ with the location as a parameter. So, this is for the center line the ratio is 1 and at x^* is equal to 0.2 it could be this x^* is equal to 0.4 equal to this and at the surface with x^* is 1 this curve would be this.

Interesting thing to notice that for $\frac{1}{Bi}$ large that is a Bi small there is very little temperature variation across the bulk of the potato. In fact, we could neglect all the variations. We could assume that the whole bulk of the potato is at the same temperature. What is happening? Let us in a different context in the case of a parallel plate one dimensional flow we notice that the biot number is 1 there is the temperature variation within the solid within the potato and within the fluid of the same order, but if the biot number is very small that is $\frac{1}{Bi}$ is large the variations within the potatoes are small and negligible compared to the variation of temperature in the fluid. So, we are not going to make serious mistakes if we assume the temperature throughout the bulk of the potato to be constant only one temperature.

This is the other case where Bi is very large all the variations are within the potato there hardly any variation in the fluid. So, this problem can be solved by changing the boundary conditions that the temperature T_f is not in the fluid far away, but it is the temperature at the surface of the potato itself. So, the convective boundary conditions can be dropped and can be assumed that the temperature at the surface of potato is always T_f Typical heat transfer coefficients of domestic ovens set at 180 degree centigrade is of the order of 25 W/m^2K .

The conductivity of potato is between 0.545 to 0.957 depending upon the water content within the potato, depending upon how old the potato is, and how long it has been stored. Let us take it as 0.8, a typical value, and let us take the diameter of potato to be 4 centimeters.

A rather small potato Then the Biot number is of order 1.2. Biot number is order 1, and so the Biot number would be important. We could neither make the assumption Biot number is very small and assume the temperature throughout the potato be constant, nor we can assume that the temperature variation within the bath are negligible, and we can transfer the temperature

T_f to the surface of potato. So, we will have to use the Heisler charts. We will continue with discussion on approximations in the next lecture as well. Thank you.

Thank you.