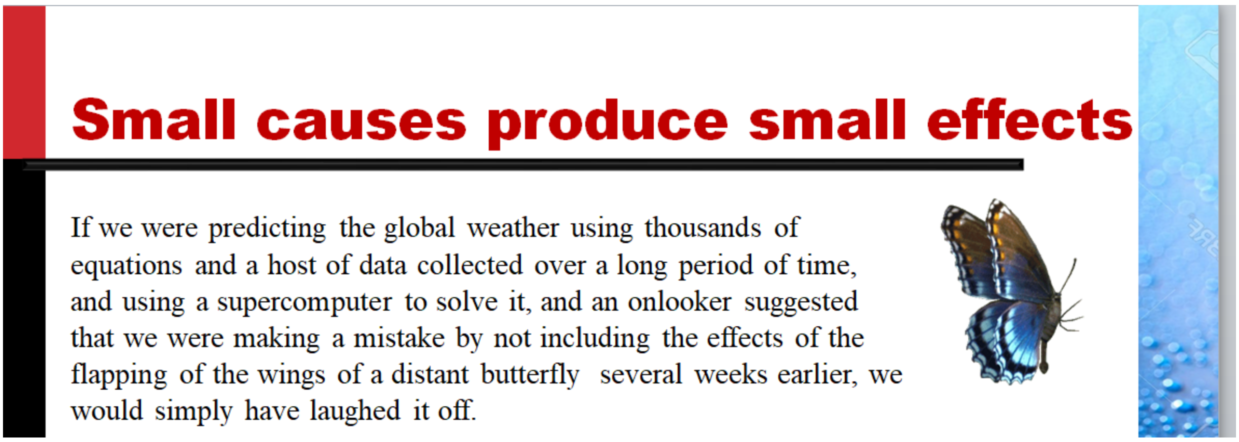
Similitude And Approximations In Engineering,
Vijay Gupta
Applied Mechanics
Indian Institute of Technology Delhi
Week - 05
Lecture - 17

Welcome back. In today's lecture, we will start the consideration of approximations in engineering. We will try to understand how to make and justify approximation in engineering sciences. The mathematical equations and boundary conditions that govern most of the engineering situations are quite complex and defy closed form analytic solutions. The limited number of exact solutions available have been obtained for some very simple geometries and for very restrictive conditions. In such cases, one or more of the terms from the governing equations drop out because of some or the other simplifying feature, such that in fluid mechanics and heat transfer, it is common to make assumptions of steady flows, fully developed flows, one dimensional or two dimensional flows, low viscosity flows, negligible viscosity flows, but such features are not encountered in general.

The solutions obtained through these simplifications are so cherished that we like to extend them to realistic situation where those conditions are not met exactly. This is what we mean by approximate solutions. If you are predicting the global weather using thousands of equations and a host of data collected over long period of time and using a supercomputer to solve it, and an onlooker suggested that we are making a mistake by not including the effects of the flapping of the wings of a distant butterfly several weeks earlier, we would simply have laughed it off. Because we believe that small causes produce small effects and we believe that the flapping of wings of a butterfly is a small cause.



But is it a small cause? Edward Lorenz in 1961 found out that a very simple rounding off of initial conditions while predicting the weather through a numerical model resulted in a massive deviation in prediction of weather about two months later. Here are some results. We obtained two solutions to a given problem where the two initial points differ by only 0.00001 such a little difference. And if we have two runs, one a blue run and a green run, then the difference of the values predicted by the two runs with two different initial conditions differing only by a very small amount 0.00001. Looks like this. Initially, for a very long period, the two solutions are the same. But then they start deviating. And the deviations are quite big showing that the solutions can be very different even if the initial conditions differ only very slightly.

One meteorologist when he looked at the results like these remarked that if the theory was correct, one flap of a seagulls wing would be enough to alter the course of the weather forever.

The controversy has not yet been resolved. But as someone said that the most recent evidence seemed to flavor the seagulls. Let us look at a double pendulum, two pendulums in series. In this video, we see that initially we have same configuration in all the six setups.

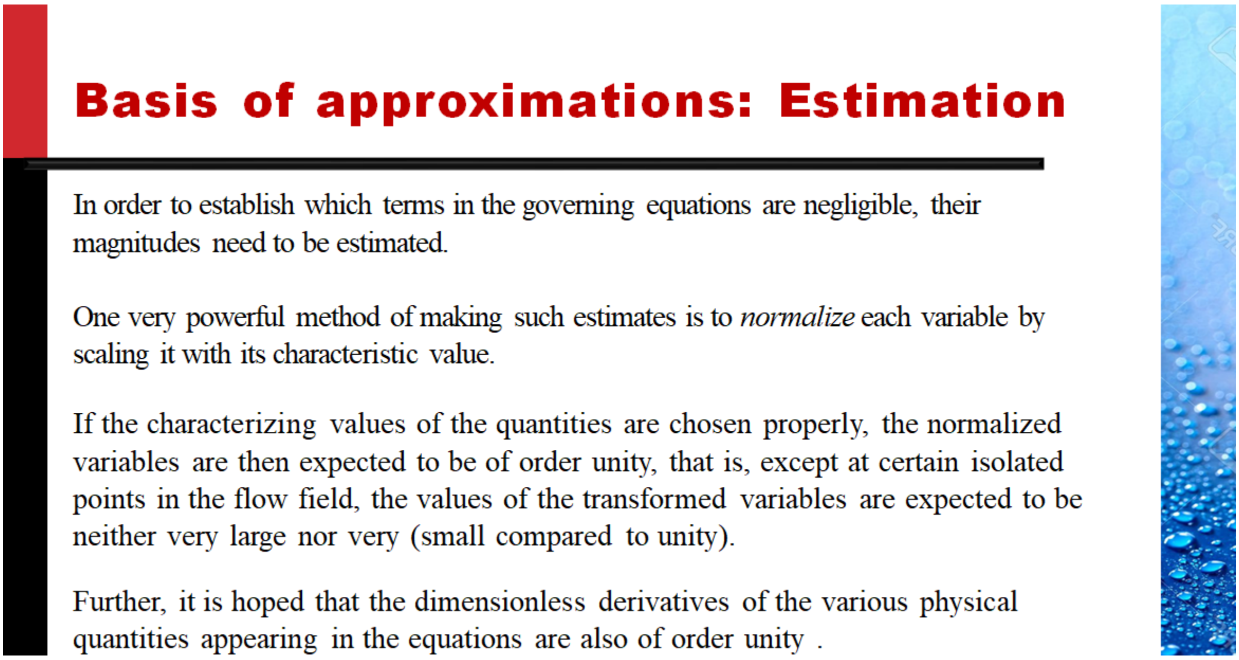
The setups are identical and when they are released, they have the same initial conditions. But as the time passes, the small deviations start appearing. And even before 20 seconds have elapsed, the motion of all six pendulums is entirely different from one another. There is a whole theory of deterministic chaos that can work on this. Chaos theory states that within the apparent randomness of chaotic complex systems, there are underlying patterns, interconnections, constant feedback loops, repetitions, self-similarity, fractals and self-organization.

This deterministic chaos is where the present determines the future. But the approximate present does not approximately determine the future. Of course, the present determines the future. The initial conditions determine what is going to take place later. But slightest approximation in the present condition can lead to significant departures.

Again, approximate present does not approximately represent the future. But still, the practice of engineering proceeds on the principle that small causes produce small effects. We assume this to be true, and we shall persist with this here till something breaks down. And then we will investigate what is happening. It soon will and we will see many cases where this does.

We must approximate. We cannot find exact results because we cannot model all effects. The kind of forces that are present in a simple phenomena cannot all be written down in the equations. We also cannot model all boundary conditions exactly. In fact, we cannot model the geometry of a problem exactly.

There is always some assumptions. If nothing else, we assume the material is uniform. So, we must approximate. So, we must have a basis on which we can decide if our approximations are going to work or they are not going to work. And we start with the approximations of the governing equations.



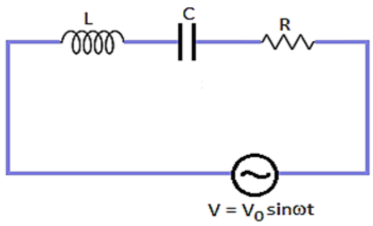
In order to establish which terms in the governing equations are negligible, their magnitudes need to be estimated. One powerful method of making such estimate is to normalize each variable by scaling it with its characteristic value. If the characterizing values of the quantities are chosen properly, the normalized variables are then expected to be of order unity, that is, except at certain isolated locations in the flow field, the values of the transformed variables are expected to be neither large nor very small compared to unity. Further, it is hoped that the dimensionless derivatives of the various physical quantities appearing in the equations are also of order unity. This last is a very important hope, very important requirement.

Basis of approximations: Estimation

If this happens, then the coefficients of the various terms will reflect the estimate of the corresponding term in the equation.

$$L\frac{di}{dt} + Ri + \frac{q}{C} = V_o \sin \omega t$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V_o \sin \omega t$$



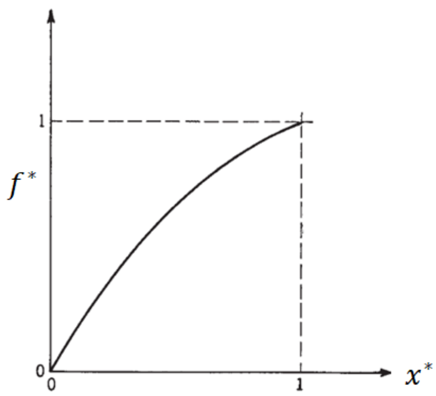
Consider an LCR circuit, the voltage drop across the inductance plus the voltage drop across the capacitance plus that across the resistance must sum up to the alternative voltage that is applied to the system. The resulting equation is $L\frac{di}{dt} + Ri + \frac{q}{C} = V_o \sin sin\, \omega t$. If I write the current as $\frac{dq}{dt}$, where q is the charge on the capacitor, then this equation converts to $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V_o \sin sin\, \omega t$. Now, we start with non-dimensionalizing it, or rather normalizing it. This equation, the solution of this equation is of the form that $q = q_o \sin sin\, (\omega t - \varphi)$, where q_o is the amplitude of the varying charge on the capacitor.

So while normalizing, we introduce the non-dimensional variables q^* *as* q divided by some characteristic value q_c for the dependent variable, and for the independent variable time, we introduce $t^* = t/t_c$. If we choose q_c and t_c correctly, then the resulting q^* and t^* should be of order 1 throughout the phenomena that we are discussing. The characteristic time t_c must be related to ω, the frequency of the voltage that we apply. We may choose t_c as 1/ω so that $t^* = \omega t$. q_c may be the amplitude q_o of this instantaneous charge on the capacitor, and if we do this, the equation becomes $LC\omega^2 \frac{d^2q^*}{dt^{*2}} + RC\omega\frac{dq^*}{dt^*} + q^* = \frac{CV_o}{q_o}\sin sin\, t^*$.

Besides introducing the two normalized variables, we have manipulated the equation such that the coefficient of one term has become unity. Now, we hope that $q^* = q/q_c$, the non-dimensional dependent variable, $t^* = \omega t$, the non-dimensional independent variable and the non-dimensional derivatives of the dependent variable are of order 1. If they are of order 1, then the coefficient $LC\omega^2$ and the coefficient $RC\omega$, and the coefficient $\frac{CV_o}{q_o}$ measures the relative magnitude of the various quantities in these terms. So, the ratio of the voltage drop across the inductance to the voltage drop across the capacitance is $LC\omega^2$. But let us look at this expectation that the derivatives are also of order 1 in little more detail.

Conditions required

Consider a problem in which a function f is given by an ordinary differential equation.



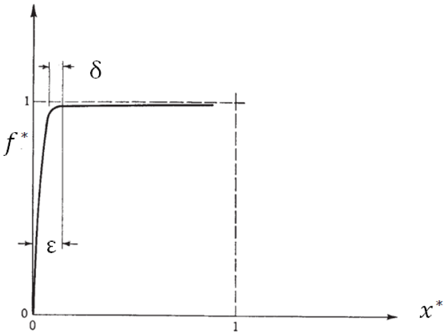
- the function and its derivatives must be continuous and differentiable up to an order one lower than that appearing in the differential equation.

This means that the function can be normalized, since it is bounded, and that the domain of integration can be made approximately unity by normalization. There are often good grounds for assuming these conditions.

Consider a problem in which a function f is given by an ordinary differential equation. If we normalize f and x by their maximum values, we may get a plot of f^* and x^* like shown here. It can be shown that for the function and the derivative to be of order 1, the function and its derivatives must be continuous and differentiable up to an order 1 lower than that appearing in the differential equation. This means that the function can be normalized, since it is bounded and that the domain of integration can be made approximately unity by normalization. There are often good grounds for assuming these conditions.

Further, look at $\frac{df^*}{dx^*}$. If the function is well behaved as shown, then $\frac{df^*}{dx^*} \approx \frac{1-0}{1-0} = 1$ everywhere. Similarly, the second derivative of f^* with x^* would definitely be less than the typical value of $\frac{df^*}{dx^*}$, 1 −minus 0 divided by 1 minus 0, the range of x^*, which is 1. So, that the second derivative is also of order 1. Similarly, if the function is differentiable, the higher order derivatives normalized would also be of order 1.

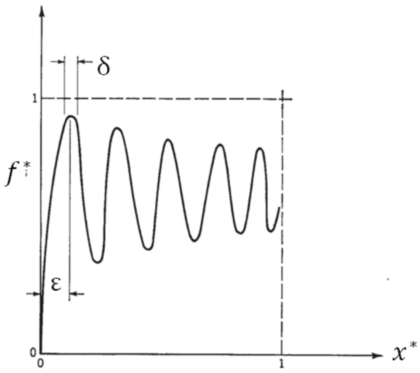
This implies that the nth derivative of function f, the original function f non normalized with respect to x, is less than the characteristic value of f divided by the characteristic value of x raised to power n. Thus, if the function is well behaved, not only the variables, but the derivatives are also rendered of order 1, when we normalize the variables with the quantities that are truly characteristic of these variables. Such estimates are termed as smooth estimates.



But not all estimates are smooth. f^*, x^* star plot could be like this, and if this is the case, there is no single estimate that is valid for the entire range of x^*.

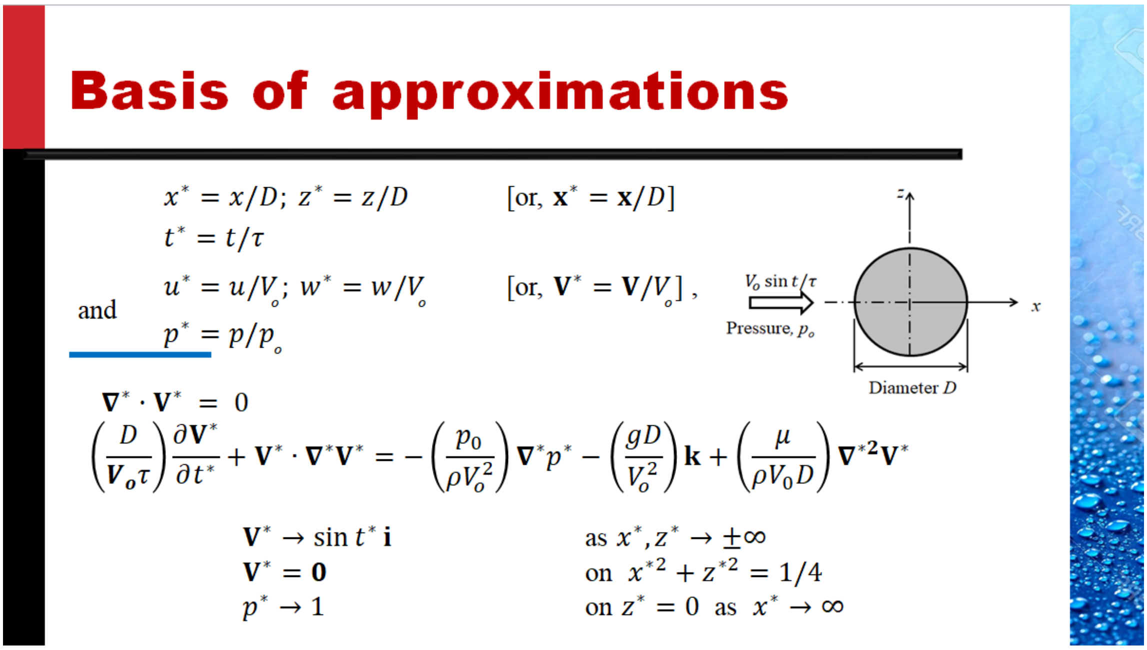
For the range 0 to epsilon, $\frac{df^*}{dx^*}$ is like $\frac{1}{\varepsilon}$. If ε is small compared to 1, this is pretty large, and one can show quite easily that the second derivative $\left|\frac{d^2f^*}{dx^{*2}}\right|$ should be like $\frac{1}{\varepsilon\delta}$ here δ is the width of this transition, and that would be like 1 over epsilon squared. But in this larger range from epsilon to 1, the first derivative is quite small as is the second derivative. This is an example where one single estimate of the derivatives would not work for the entire range or the values of x star. We would assume, in most discussions, that the function is well behaved like in the previous case, till we run into a problem.

And then we look if something like this is happening. This was famously done by Ludwig Prandtl at the turn of the last century when he introduced the concept of boundary layer. A thin layer with the gradients are very large when for the rest of the regions the gradients are well behaved. We will discuss this a little later.



We could have variation of f star with x star that look something like this. Here again, normalizing the variables does not ensure that the derivatives are normalized too. So what do we do then? We assume that the functions are of first type, well behaved and that the estimates are smooth, but we should be constantly on the lookout for trouble. And we are sure to find it often enough.

Let us do one more example where we introduce the concept of approximations. This is the flow past a circular cylinder of an incompressible fluid.



The free stream has a pulsating velocity $V_o \sin \sin t/\tau$, where τ is a given time period. The pressure for our stream is p_o. The diameter of the cylinder is D. It is a 2 dimensional flow.

Independent variables are x, z and t. Let us use D the diameter as the characteristic length. So that $x^* = x/D$, $z^* = \frac{z}{D}$ and we can use τ as the characteristic time. So that $t^* = t/\tau$. The dependent variable u and w along x and z directions are normalized with respect to the amplitude of the free stream velocity V_o. So that we define $u^* = u/V_o$, $w^* = w/V_o$, and we define a non dimensional pressure, another dependent quantity, as $p^* = p/p_o$.

If we introduce this transformation in the equations for the governing flow, the equation for the governing flow include continuity equation and the Navier Stokes equation. The equations become $\nabla^* \cdot V^* = 0$ for incompressible flows. This is the continuity equation.

$\left(\frac{D}{V_o\tau}\right)\frac{\partial V^*}{\partial t^*} + V^* \bullet \nabla^* V^* = -\left(\frac{p_0}{\rho V_o^2}\right)\nabla^* p^* - \left(\frac{gD}{V_o^2}\right)k + \left(\frac{\mu}{\rho V_0 D}\right)\nabla^{*2} V^*$ is the Navier Stokes equation in the normalized variables x^*, z^*, t^*, u^*, w^* and p^*. The first term is the unsteady flow term, the second is the convective acceleration term, the third is the pressure force term, the fourth is the gravity force term, and the last is the viscous force term. We have to normalize the boundary conditions as well.

And these are the resulting boundary conditions.

$\mathbf{V}^* \to \sin t^* \mathbf{i}$ as $x^*, z^* \to \pm\infty$

$\mathbf{V}^* = \mathbf{0}$ on $x^{*2} + z^{*2} = 1/4$

$p^* \to 1$ on $z^* = 0$ as $x^* \to \infty$

So, this is what we had. Now, if we have chosen our characterizing quantities properly, then the variables and their derivatives are expected to be order 1. As I said before, the first term is the unsteady force term, the second is the convective acceleration term, the third term is the pressure term, the gravity term and the viscous terms.

Now, if the derivatives are of order 1, the coefficient of the unsteady term can then be seen as the ratio of the characteristic unsteady force divided by the convective force where the coefficient is 1. So, $\frac{D}{V_o\tau}$ can be interpreted as the ratio of the unsteady acceleration term divided by the convective acceleration term in this equation where the coefficient of the convective acceleration is 1. Similarly, this is interpreted as the ratio of the pressure term to the inertial term, the convective inertial term and as had been discussed earlier, this is given the name 1 over Euler number. The next can be interpreted as the ratio of the gravity force to the inertial force and this is seen as 1 over Froude number squared. And the last coefficient is the inverse of Reynolds number and is the ratio of viscous force to inertial force.

So, if any of these coefficients is small compared to 1 that is compared to the convective acceleration term, then the corresponding term can be neglected from the governing equation. This is the primary tool that we used for estimating the terms and making approximations. Here we have made We have the coefficient of the last term, the viscous force term, to be 1, and then this is the ratio inertial force to viscous force, this is the ratio of inertial force convective to the viscous force, this for the pressure to viscous force and this gravity to viscous force. We had discussed earlier that in situations where the cavitation and the free surface effects are not present, we could introduce a modified pressure script P by this expression.

And so that the resulting equations absorbs the pressure and gravity term into one term, and this is the resulting equation. This then is the ratio of the unsteady term to the viscous term, this is inertial acceleration to the viscous force term, this is pressure and gravity or non gravitational pressure term to the viscous term, and the coefficient of viscous term is 1. So, this is the equation that we obtained last. Now, if we have a steady low Reynolds number flow.

Of course, if the flow steady. So, the unsteady term, which was the first term in the last equation, drops out, and this is the equation that results: $\left(\frac{\rho V_o D}{\mu}\right)V^* \bullet \nabla^* V^* = -\left(\frac{(\Delta p)_0 D}{\mu V_o}\right)\nabla^* P^* + \nabla^{*2} V^*$. Now, the flow Reynolds number is low, low compared to 1. If it is small compared to 1, then this term, $\left(\frac{\rho V_o D}{\mu}\right)V^* \bullet \nabla^* V^*$ which is the Reynolds number itself, this coefficient is less than 1. So, we can drop this in comparison to $\nabla^{*2} V^*$. And if we do this, this is what the resulting equation is: $\left(\frac{(\Delta p)_0 D}{\mu V_o}\right)\nabla^* P^* = \nabla^{*2} V^*$.

The pressure forces balance the viscous forces. In terms of dimensional variables, this equation becomes $\nabla p - gk = \mu\nabla^2 V$, the Stokes equation, and this is the form that you see in the Stokes equation, which is valid for Reynolds number much less than 1. You may note that this equation is linear in the velocity variable. The Navier Stokes equation is non-linear in the convective acceleration term $V\cdot\nabla V$, and that term drops out in this low Reynolds number flow equation. So, this equation is linear in the velocity variable which affords enormous simplification in its solution. The linearity of this equation along with those of the applicable boundary conditions permits us the superposition of elementary solutions to obtain more complex solutions.

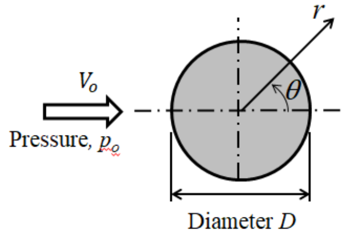
Stokes flow: Creeping flows

Incompressible steady flow past a sphere at Re << 1

$$V_r(r,\theta) = V_o\left[1 - \frac{3}{2}\cdot\frac{R}{r} + \frac{1}{2}\left(\frac{R}{r}\right)^3\right]\cos\theta,$$

$$V_\theta(r,\theta) = -V_o\left[1 - \frac{3}{4}\cdot\frac{R}{r} - \frac{1}{4}\left(\frac{R}{r}\right)^3\right]\sin\theta,$$

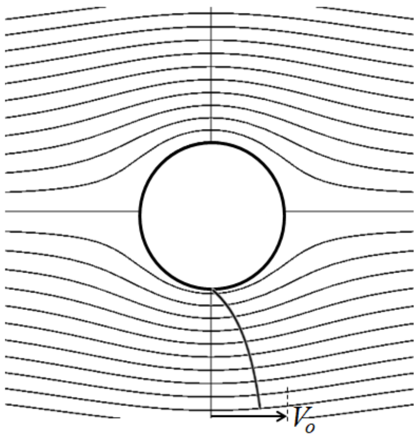
and $$\mathcal{P}(r,\theta) = -\frac{3}{2}\cdot\frac{\mu V_o}{R}\left(\frac{R}{r}\right)^2\cos\theta$$

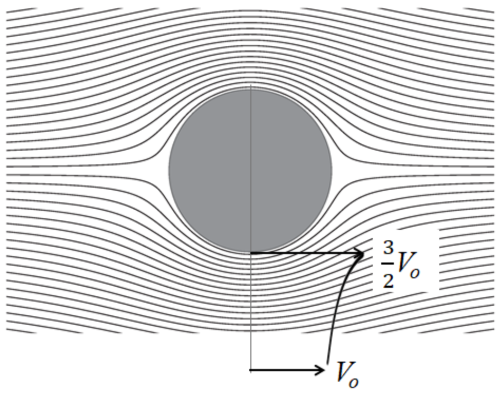


The Stokes law is the lower Reynolds number flow, Reynolds number much less than 1 past a sphere is called a creeping flow equivalent mechanics. The solution is possible, but it is not easy. We give here the solution in the spherical polar coordinates. The r velocity is like $V_r(r,\theta) = V_o\left[1 - \frac{3}{2}\bullet\frac{R}{r} + \frac{1}{2}\left(\frac{R}{r}\right)^3\right]\cos cos\,\theta$, the theta velocity is like $V_\theta(r,\theta) =- V_o\left[1 - \frac{3}{4}\bullet\frac{R}{r} - \frac{1}{4}\left(\frac{R}{r}\right)^3\right]\sin sin\,\theta$, and the pressure is given by $P(r,\theta) =- \frac{3}{2}\bullet\frac{\mu V_o}{R}\left(\frac{R}{r}\right)^2\cos cos\,\theta$. This Stokes law at lower Reynolds number are very different from the inviscid flows.

Look at the velocity profile on the shoulder in the Stokes law. This is very different from the velocity profiles in an inviscid flow where viscosity was negligible. The Stokes law viscosity dominated flow while the this flow is a flow in which the viscosity is 0. The nature of the velocity profile at the shoulder is entirely different and this can be obtained as exact solution. This also can be obtained in exact solutions under this specified conditions.

Stokes flow





The flow pattern in Stokes law is completely symmetrical. The streamlines are the same whether the flow approaches the sphere from the left or from the right. Even though the flow pattern is symmetrical, the pressure distribution is anti symmetric. Thus while the pressure is maximum at the nose of the sphere, it is minimum at the tail even though the velocity is the same at two points. This observation is also in marked contrast to the behavior at large Reynolds number flow.

Bernoulli equation does not apply. Even though the velocities are the same at two points, the pressures are entirely different. The fluid velocity is significantly lower than the free stream value of V_o over a considerable distance from the shoulder. In fact, we need to go out to 16 time the radius before the velocity approaches 99 percent of the free stream velocity V_o. This means that the effect of the presence of sphere is felt over large distances. This too is unlike that in the case of high Reynolds number flow where the velocity approaches V_o in very short distances, less than two radius of the sphere.

Because of this influence of the sphere over large distances, small particles moving at low Reynolds number interact strongly with one another. That is the motion of a particle is affected by the presence of other particles even when the separation is large compared to their diameters. This too is in sharp contrast to the motion of larger particles which have almost no such interaction. Also look at this equation: $\left(\frac{(\Delta p)_0 D}{\mu V_o}\right)\nabla^* P^* = \nabla^{*2} V^*$. Since the two terms must be retained in this equation, the coefficient $\left(\frac{(\Delta p)_0 D}{\mu V_o}\right)$ should be order 1.

$(\Delta p)_0$ is the characteristic pressure difference. So, to make this coefficient 1, $(\Delta p)_0$ should be like $\mu V_o / D$. From fluid mechanics, you may recall that the characteristic pressure difference in large Reynolds number flow is $\frac{1}{2}\rho V_o^2$. Here it is $\mu V_o / D$. This was to be expected since the pressure forces balance the viscous forces.

Hence the two terms in the equation must be of the same order. Further, the total pressure and shear forces acting on the sphere in the horizontal direction are obtained as $2\pi\mu R V_o$ and

$4\pi\mu RV_o$, respectively. So, that the total drag experienced by the sphere is the sum of the two, that is $6\pi\mu RV_o$. And the drag coefficient defined as $C_D = \frac{D}{\left(\frac{1}{2}\rho V_o^2 A\right)}$, the traditional definition of the drag coefficient is like $\frac{24}{Re_D}$, Reynolds number based on diameter. So, the approximation of flows where the velocity is very small which is marked by the values of Reynolds number to be much less than 1. If that is the case, we can drop the convective acceleration term and then the viscous forces are balanced by the pressure forces.

The resulting equations can be solved with a bit of advanced mathematics and results can be obtained which turn out to be correct representation of experimentally obtained results at such low values of Reynolds numbers. Note that the drag in the Stokes formula $6\pi\mu RV_o$ does not depend upon the density, density of the fluid, confirming that the inertia of the fluid is unimportant for the low Reynolds number flows where the inertial term was dropped from the equation. That is why the density does not occur in the formula for drag. This should give you some confidence in the way we propose to make approximations. For studying the large Reynolds number flow, we manipulate the terms such that the coefficient of the convective acceleration is 1, and then the coefficient of the viscous term becomes 1/Re.

When Reynolds number is large, clearly this drops out. The boundary conditions are these:

$$\mathbf{V}^* \to \mathbf{i} \qquad \text{as } x^*, z^* \to \pm\infty$$
$$\mathbf{V}^* = \mathbf{0} \qquad \text{on } x^{*2} + z^{*2} = 1/4$$
$$p^* \to 1 \qquad \text{on } z^* = 0 \text{ as } x^* \to \infty$$

But we sense trouble. There are two conditions, boundary conditions on velocity, but this equation has only one derivative of velocity. This cannot take two boundary conditions in velocity, it can take only one boundary condition in velocity. A boundary conditions upstream and a boundary condition on the surface of the sphere. So, the simple minded approximation method that we introduced has broken down so quickly.

The resulting equation does not satisfy the boundary conditions. In fact, if we drop the viscosity term, then this boundary conditions are no slip on the surface of the body cannot be satisfied, but it is an essential boundary condition. So, something has broken down. This is what is irregular perturbation, and this is what Prandtl responded to, and we will see later that we can handle this, and we will develop methods for doing so.

Thank you very much.