Similitude And Approximations In Engineering,
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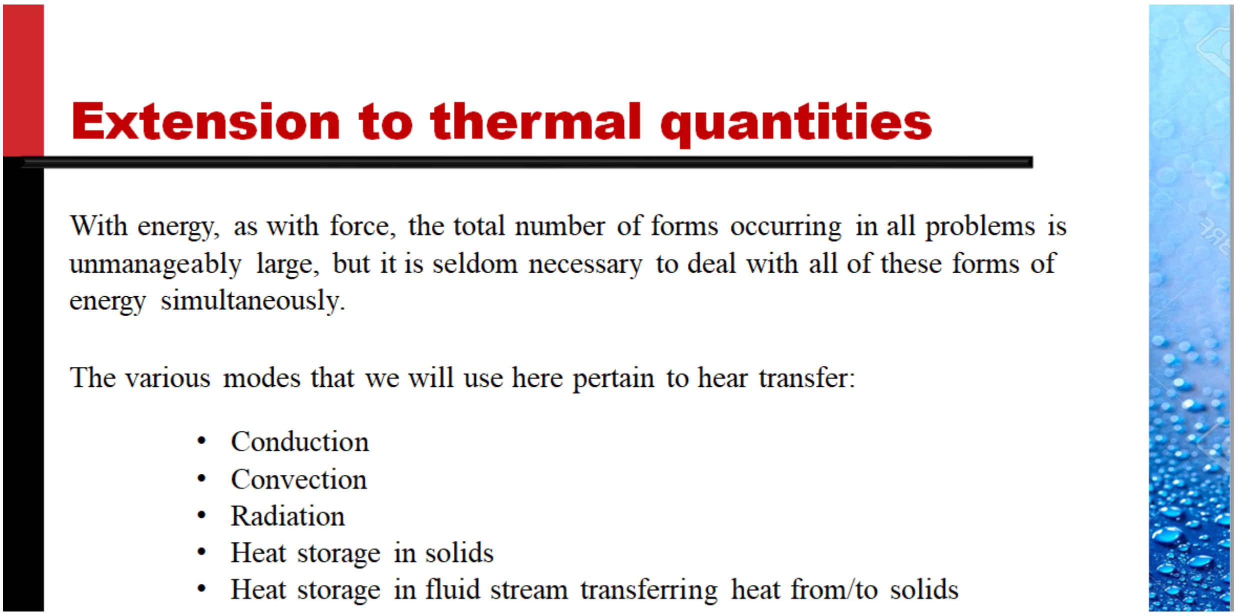
Week - 04
Lecture - 16

Welcome back. Another lecture on Scale Factor Approach where we would be using scale factor approach to obtain modeling and prediction rules for problems in thermal sciences, heat and mass transfer, for example. In the previous lectures, we had employed the force ratios to develop the invariant pi numbers. These are quite useful in the field of mechanics, but need to be supplemented when we deal with thermal quantities. Especially when employing similitude in heat and mass transfer, two fields that famously employ similitude for quite a bit of work. It is pertinent to extend the method of similitude by use of scale factors for energy, energy flow and mass fluxes.

We will not deal with mass fluxes here, but we will deal with energy flow. With energy, as with force, the total number of forms occurring in all problems is unmanageably large. But it is seldom necessary to deal with all of these forms of energy simultaneously. The various modes of energy transfer that we will use here pertain to heat transfer, conduction, convection, radiation, rate of heat storage in solids, and the rate of heat storage in fluid streams transferring heat from or to solids.



So, let us consider them one by one. The rule applicable to conduction of solids is the Fourier law. The rate of heat transfer $\dot{Q}$ is area times the heat flux and the heat flux is given by the conductivity of solid k_s times the temperature gradient $d\theta/dx$, Fourier law of heat conduction. Converting this into a scale factor relation, we get $k_{c,s}$, scale factor of conduction in solids is $k_{k_s} k_L k_{\Delta\theta}$. As before, we note that it is not the temperatures, but the temperature differences that are relevant in the heat transfer problem.

The resultant pi number is $\frac{\dot{Q}_{c,s}}{k_s L\Delta\theta}$. Conduction of fluid near solid boundaries: it is within the fluid at the solid boundary. The heat is transferred through conduction alone because the velocity at the boundary is zero. So, the applicable relation is still the Fourier law, except that the conductivity of the fluid is used, and $d\theta/dx$ is on the fluid side of the boundary. We get a similar pi number, the only difference is k_s is replaced by k_f, the conductivity of the fluid.

Rates of heat transfer

	Applicable relations	Scale factor relations	Pi-numbers
Conduction in solid	$\dot{Q} = Aq = Ak_s d\theta/dx$	$k_{c,s} = k_{k_s} k_L k_{\Delta\theta}$	$\frac{\dot{Q}_{c,s}}{k_s L\Delta\theta}$
Conduction in fluid at solid boundary	$\dot{Q} = Aq = Ak_f d\theta/dx$	$k_{c,f} = k_{k_f} k_L k_{\Delta\theta}$	$\frac{\dot{Q}_{c,f}}{k_f L\Delta\theta}$
Convection	$\dot{Q} = Aq = Ah\Delta\theta$	$k_{cov} = k_h {k_L}^2 k_{\Delta\theta}$	$\frac{\dot{Q}_{cov}}{hL^2\Delta\theta}$
Radiation	$\dot{Q} = Aq_r = Ah_r\Delta\theta$	$k_{rad} = k_{h_r} {k_L}^2 k_{\Delta\theta}$	$\frac{\dot{Q}_{rad}}{h_r L^2\Delta\theta}$
Heat storage in solids	$\dot{Q} = (\rho V\!\!\!/)c_v(d\theta/dt)$	$k_u = k_\rho {k_L}^3 k_{c_v} k_{\Delta\theta} {k_t}^{-1}$	$\frac{\dot{Q}_u t}{\rho L^3 c_v \Delta\theta}$
Heat storage in fluid stream	$\dot{Q} = (\rho_f AV)c_p\Delta\theta$	$k_f = k_{\rho_f} {k_L}^2 k_V k_{c_p} k_{\Delta\theta}$	$\frac{\dot{Q}_f}{\rho_f L^2 V c_p \Delta\theta}$

The law governing the convection is the Newton's law of cooling: $\dot{Q}$ is area times h times Δθ, where h is the heat transfer coefficient, Δθ is the temperature difference between the fluid and the solid boundary. So, $k_{cov} = k_h {k_L}^2 k_{\Delta\theta}$, and $\frac{\dot{Q}_{cov}}{hL^2\Delta\theta}$ is the relevant pi number.

Radiation is written in terms of h_r, the equivalent heat transfer coefficient for radiation and we get a similar pi number. We will not deal with radiation in this course.

Then the rate of heat storage in solids is the mass of the solid times specific heat times rate of change of temperature. We call it the scale factor for unsteady temperature in the solid, and this is $k_u = k_\rho {k_L}^3 k_{c_v} k_{\Delta\theta} {k_t}^{-1}$, the time scale factor in the denominator, and $\frac{\dot{Q}_u t}{\rho L^3 c_v \Delta\theta}$ is the relevant pi number.

Similarly, for the fluid stream. Heat storage in the fluid stream, the mass flow rate is $(\rho_f AV$, c_p is the specific heat, and $)\Delta\theta$ d is a temperature difference, and so, we get the heat storage in fluid stream scale factor having $\frac{\dot{Q}_f}{\rho_f L^2 V c_p \Delta\theta}$ pi number. This is using the laws to determine the pi numbers.

Pi numbers

	Pi-numbers
Conduction in solid	$\frac{\dot{Q}_{c,s}}{k_s L \Delta\theta}$
Conduction in fluid at solid boundary	$\frac{\dot{Q}_{c,f}}{k_f L \Delta\theta}$
Convection	$\frac{\dot{Q}_{cov}}{h L^2 \Delta\theta}$
Radiation	$\frac{\dot{Q}_{rad}}{h_r L^2 \Delta\theta}$
Heat storage in solids	$\frac{\dot{Q}_u \tau}{\rho L^3 c_v \Delta\theta}$
Heat storage in fluid stream	$\frac{\dot{Q}_f}{\rho_f L^2 V c_p \Delta\theta}$

$\frac{hL}{k_s}$, Biot number

$\frac{hL}{k_f}$, Nusselt number

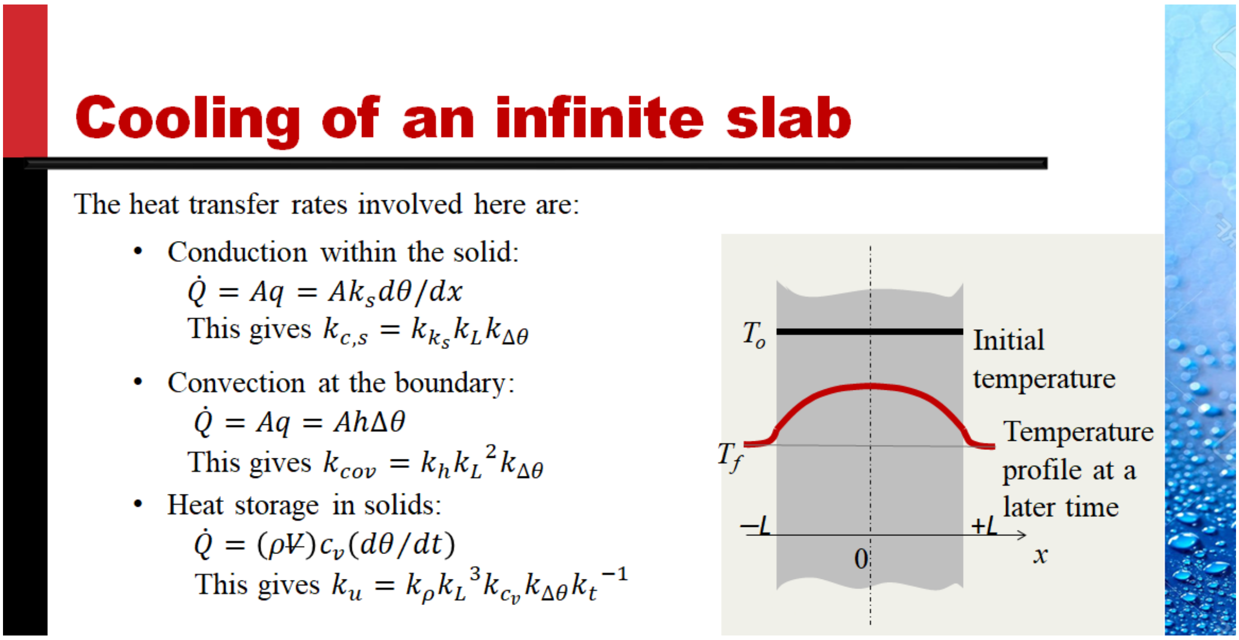
$\frac{L^2 \rho c_v}{k_s \tau} = \frac{L^2}{\alpha\tau}$, Fourier number

$\frac{h}{\rho_f c_p V}$, Stanton number

We copied them here from the last table and then we equate the scale factors for Conduction in solid and Convection, then we get from this a pi number $\frac{hL}{k_s}$, heat transfer coefficient times a characteristic length divided by the conductivity of solids. This is named Biot number and is abbreviated as capital B lower case I, Bi.

If we equate the scale factor for conduction in fluid at the solid boundary with the rate of convection, then we get an equivalent formulation $\frac{hL}{k_f}$, which is termed as Nusselt number where k_f is the conductivity of fluid. Note the difference between Biot number and Nusselt number. h and L are the same, the conductivity in Biot number is of the solid, and in the Nusselt number is for the fluid.

If we equate the scale factors for conduction in fluid at solid boundaries with the heat storage in solids relevant to unsteady conduction in fluid. For a solid immersed in a stream of fluid, the pi number that we get is $\frac{L^2}{\alpha\tau}$, which is termed as Fourier number, and abbreviated as Fo, where α is k_s divided by ρc_v, thermal diffusivity. And one more convection and heat storage in the fluid stream: if we combine this we get $\frac{h}{\rho_f c_p V}$ as an invariant and this pi number is termed as Stanton number, in which the rate of heat convection at the boundary is related to the heat storage capacity of the fluid stream. Let us do some examples in heat transfer.



In an example we did earlier using a different approach, cooling of an infinite slab, an infinite slab of thickness 2 L in the x direction is immersed in a fluid of temperature T_f. The initial temperature of the slab is T_o everywhere. The slab loses heat to the adjacent fluid and the temperature profile changes. This temperature of the slab is coming down regularly, but it is highest at the center, it varies with x.

This is the drop of temperature within the solid at the given moment and this is the drop in temperature within the fluid at the boundary layer, thermal boundary layer at the boundary of the slab within the fluid. The heat transfer rates involved here are conduction within the solid, and for this we had obtained the scale factor as $k_{c,s} = k_{k_s} k_L k_{\Delta\theta}$. Conduction within the solid in convection at the boundary $\dot{Q}$ is like Aq, heat flux. By the Newton's law of cooling this is $Ah\Delta\theta$, which gives you $k_{cov} = k_h k_L^2 k_{\Delta\theta}$. The two laws, the Fourier law and the Newton's law of cooling.

There is a third law that is relevant. Heat storage in solids: $\dot{Q} = (\rho V)c_v\left(\frac{d\theta}{dt}\right)$. This gives you $k_u = k_\rho k_L^3 k_{c_v} k_{\Delta\theta} k_t^{-1}$ as a scale factor for unsteady changes in the slab. These three factors should be equal. That is the requirement of similarity.

Cooling of an infinite slab

Similarity requires that the three heat transfer rate factors must be the same, i.e., $k_{c,s} = k_{cov} = k_u$

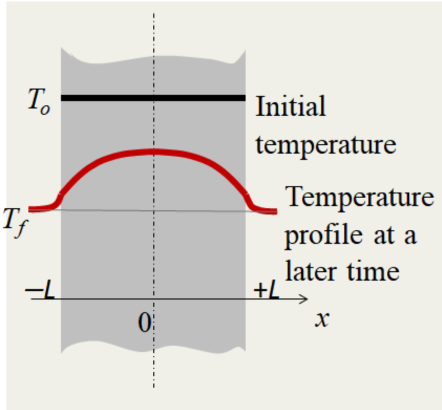
$$k_{c,s} = k_{k_s} k_L k_{\Delta\theta}$$

$$k_{cov} = k_h {k_L}^2 k_{\Delta\theta}$$

$$k_u = k_\rho {k_L}^3 k_{c_v} k_{\Delta\theta} {k_t}^{-1}$$

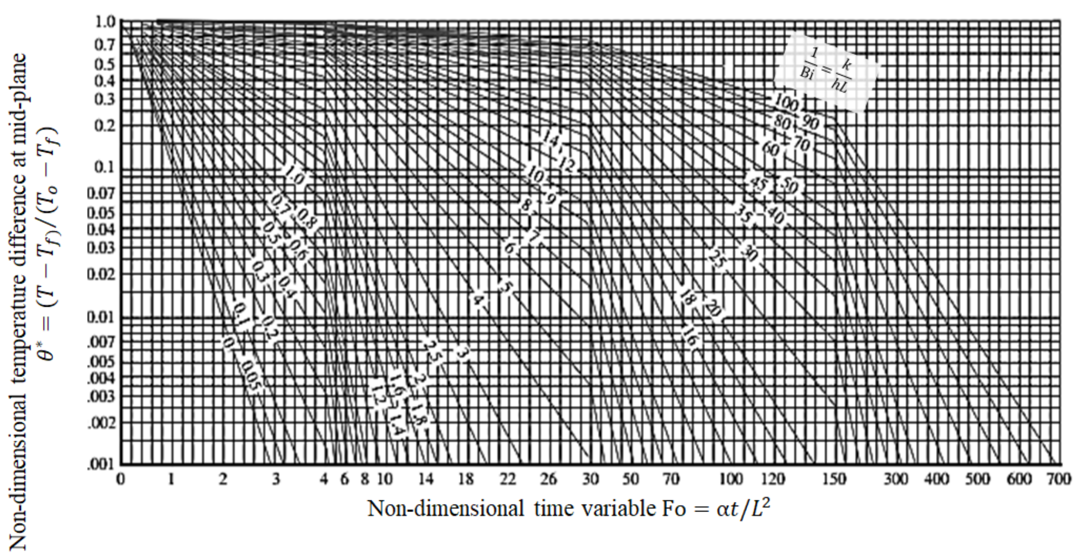
$$\frac{hL}{k_s} = \text{Biot}$$

$$\frac{k_s\tau}{\rho c_v L^2} = \frac{\alpha\tau}{L^2} = \text{Fo}$$



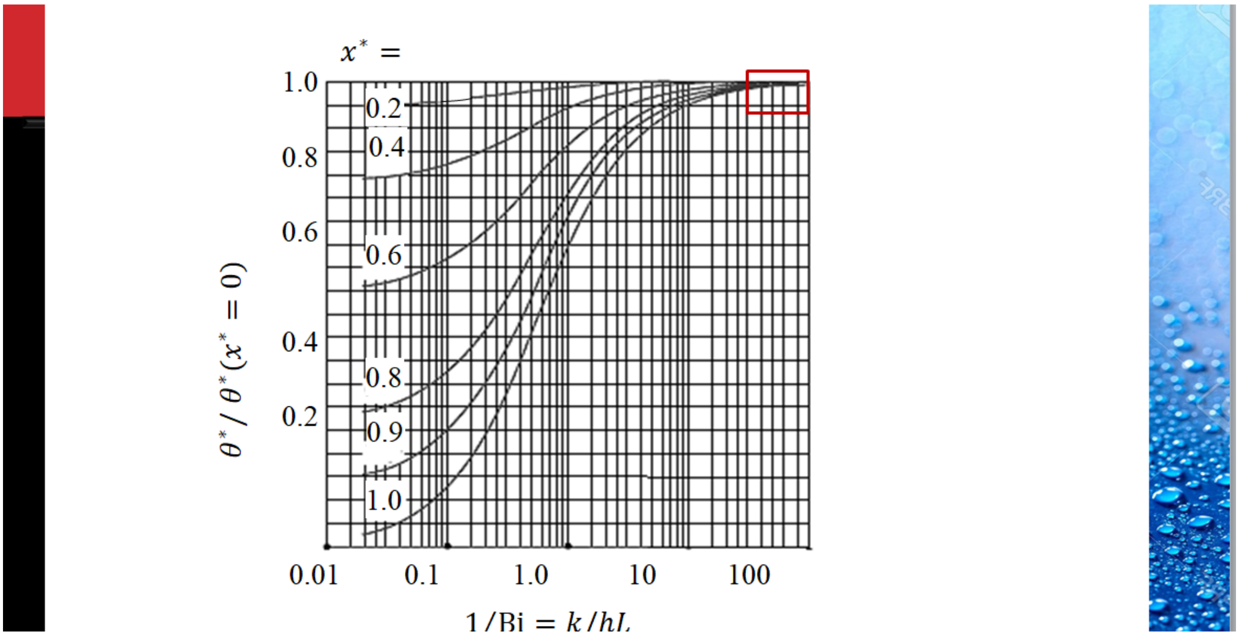
$$\frac{T - T_{ref}}{(\Delta T)_c} = \frac{T - T_f}{T_o - T_f} = f(x^*;\ \text{Bi}, \text{Fo})$$

The three scale factors are given here by equating the first two I obtain the Biot number $\frac{hL}{k_s}$, and by equating the first and third I get $\frac{k_s\tau}{\rho c_v L^2}$, and using the thermal diffusivity α for $\frac{k_s}{\rho c_v}$, this is $\frac{\alpha\tau}{L^2}$, and this is named the Fourier number, Fo. So, the dependent parameter, the normalized dependent parameter, which here would be $\frac{T-T_{ref}}{(\Delta T)_c}$. The reference temperature is conveniently taken as the fluid temperature. So, this normalized dependent variable $\frac{T-T_f}{T_o-T_f}$ is a function of the normalized independent variable x^*, the location within the slab, and two parameters, Biot number and Fourier number. This is still quite complicated as was discussed earlier when we did this problem, this problem has been solved by writing this non dimensional temperature difference, dependent variable, as a product of two factors.



One, a factor where we determine this function at x^* equal to 0, and you multiply this by a correction factor which depends upon x^* and Biot number. The first one is given by what is termed as the Heisler chart. Fourier number, $\alpha t/L^2$ is given here. The non dimensional temperature difference $\theta^* = \frac{(T-T_f)}{(T_o-T_f)}$ is here, and these are the curves for the different values of Biot number or as is written here as $\frac{1}{Bi} = \frac{k}{hL}$. This is the temperature at mid plane T is at x^* is equal to 0, at mid plane, at the center of the slab.

So, for a given time, I can calculate the Fourier number, I can go up, I go to the relevant value of $\frac{1}{Bi}$ and read the non dimensional temperature difference at mid plane from this scale. The Heisler charts, that you would have done in a course in conduction. These of course, are corrected by the position correction chart.



For different value x star, the correction factor θ^* divided by θ^* at the mid plane for various values of Biot number. It is interesting to note that for very small values of Biot number compared to 1, that is when 1 over Biot number is large, all the curves for different values x^* or r^* collapse into one line.

That means, there is very little temperature difference within the solid. The solid could be assumed to be at a relatively uniform temperature and all the temperature difference occurs within the thermal boundary layers at the two boundaries. The case of low by numbers, low internal resistance.

Let us next do an interesting problem. A 1 kilogram roast or meat requires 60 minutes of cooking time in an oven. The chicken that is being roasted in an oven it requires 60 minutes if the weight is 1 kilogram. How many hours would a 2 kilogram roast require at the same temperature? We assume geometrical similarity between the 1 kilogram model chicken and the 2 kilogram prototype chicken. If we normally cook a 1 kilogram roast and on a particular day we want to cook a 2 kilogram roast, how much longer should I bake? Ok, there are two laws that apply: conduction in solids within the chicken meat: Fourier law. $k_{c,s} = k_{k_s} k_L k_{\Delta\theta}$ and $k_u = k_\rho k_L^3 k_{c_v} k_{\Delta\theta} k_t^{-1}$ are the relationship for the scale factors.

We assume the temperature at the surface is constant in either case. Equating the two scale factors for heat flow rates, we get $k_{k_s} k_L k_{\Delta\theta} = k_\rho k_L^3 k_{c_v} k_{\Delta\theta} k_t^{-1}$, which gives us the time scale factor as $k_t = \frac{k_\rho k_L^2 k_{c_v}}{k_{k_s}}$. We can very reasonably assume that the density of the chicken meat, the specific heat of the chicken meat, and the conductivity of the chicken meat would be same that is k_ρ, k_{c_v} and k_{k_s} would be equal to 1. Scale factors for these quantity would be 1 being the same material. So, this gives us $k_t = k_L^2$.

The time scale factor is like square of length scale factor. And what is length scale factor? Length scale factor is related to mass scale factor. We are given mass or we are given mass scale factors. $k_m = k_\rho k_L^3$, density times the volume, which gives $k_L = k_m^{1/3}$. So, that k_t, the time to roast, should vary like $k_m^{2/3}$.

k_m here is 2 from 1 kilogram which we treated model to 2 kilogram that we treated as prototype. So, k_m is 2. So, that k_t is 1.59, and this implies that we would require 95 minutes, instead of 60 minutes to cook twice the mass of the roast. 1.6 times about. Very interesting.

Convection

In a heat convection problem, motion of fluid is also involved. This means that besides the pi-numbers discussed above, the pi-numbers related to fluid motions should also be invariant.

We had earlier obtained the following pi-numbers relating to fluid motion:

Reynolds number: $\mathrm{Re} = \frac{\rho V L}{\mu}$

Euler number: $\mathrm{Eu} = \frac{\Delta p}{\rho V^2}$

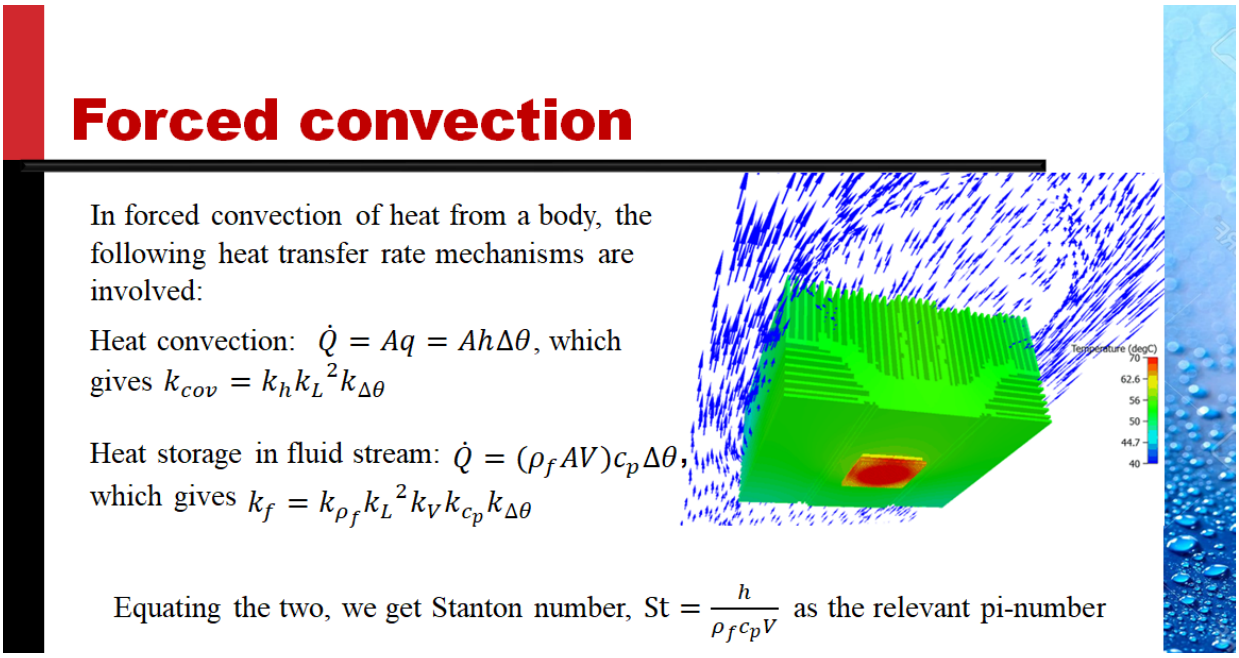
Froude number: $\mathrm{Fr} = \frac{V}{\sqrt{gL}}$

There are many other pi-numbers, but not all are relevant to a given problem

Let us apply this law approach or the scale factor approach to problems in convection. In a heat convection problem, the motion of the fluid is also involved. This means that besides the pi numbers discussed above that related to thermal transfer rates, the pi numbers related to fluid motion should also be invariant in a convection problem. We had earlier obtained the following pi numbers related to fluid motion.

We spent considerable time on these. Reynolds number Re, Euler number Eu, and the Froude number Fr. We discussed that in problems where there are no free surfaces and where there is no cavitation involved, the Euler number and Froude number can be dropped, and only the Reynolds number matter. But if there is cavitation involved then Euler number comes into play and if the free surface of a liquid is there then the Froude number comes into play. That is all we discussed earlier in quite some details. There are many other pi numbers, but all are not relevant to a given problem.

Let us consider forced convection in which a stream of a fluid is passing past a hot object and is picking up heat from there. The stream is heated up and moves away, and this motion is forced. The following heat transfer mechanisms are involved, the heat convection, the law is the Newton's law of cooling which gives you this scale factor for the convective rate, heat storage in a fluid stream. The fluid is picking up the heat and is getting heated up. So, that we have for the heat storage in the fluid stream this is the scale factor obtained earlier in this lecture.



Equating the two k_{cov} and $k_{fluid\ stream}$ we get Stanton number. And this is a relevant pi number that should have the same value in the model as in the prototype. ρ_f is the density of the fluid, C_f is the specific heat at constant pressure of the fluid, V is the velocity at which it is flowing and h is the heat transfer coefficient. In addition, we need to model the fluid motion. The only relevant forces that control fluid motion are the inertial and viscous forces.

Gravity forces do not need to be considered in the absence of free surface. The pressure forces also need not be considered because the cavitation not involved so Euler number is not important. We have seen earlier that the pi number that ensures similarity of these two forces is the Reynolds number $\frac{\rho VL}{\mu}$. So, that we can write Stanton number as a function of Reynolds number very simple relation, but convention makes things little difficult. It is conventional these days to express results in terms of Nusselt number Nu is equal to $\frac{hL}{k_f}$, with k_f as the conductivity of the fluid.

It can be shown the Stanton number which is $St = \frac{h}{\rho_f c_p V}$ can be written as $\frac{hL}{k_f}$ into $\frac{\mu}{\rho_f LV}$ times $\frac{k_f}{\mu c_p}$. So, this is $\frac{h}{\rho_f c_p V}$ same thing as this and what is this? This is Nusselt number, this is one over Reynolds number and this is one over Prandtl number, $\frac{\mu c_p}{k_f}$ of the fluid is the Prandtl number of fluid. This Stanton number is nothing but Nusselt number divided by Reynolds number times Prandtl number.

Just another way of expressing the Nusselt number. And if we do this we can write Nusselt number as a function Reynolds number and Prandtl number. This is the relation that you used to write the correlations for various geometries in convective heat transfer, forced convective heat transfer in your course in heat transfer. For various geometries how does the Nusselt number change with Reynolds number and Prandtl number? Let us do free convection. An object, heated object sitting in an atmosphere with no draft.

Because of the heat convection from the object, the density of the liquid changes. It becomes lighter. A buoyancy force acts on this, and it moves up, and is replaced by colder fluid from below. This is termed as free convection. Forces causing the fluid flow the inertia force, the viscous force, and in addition, the buoyancy force.

The buoyancy force scale factor would depend upon the scale factor for gravity, the scale factor for density differences Δand volume k_L^3. Now, $k_{\Delta\rho}$can be written as $k_\rho k_g k_\beta k_{\Delta T}$ where β is the thermal expansion coefficient of a liquid. B is how does the density change with temperature. Matching the scale factors for buoyancy force with the inertial force, which $k_\rho k_L^2 k_V^2$, we get $\frac{g\beta\Delta TL}{V^2}$ as invariant. This is matching the scale factors of the buoyancy force and the inertial force. Thus, this Stanton number now is a function of the Reynolds number and $\frac{g\beta\Delta TL}{V^2}$. The heat transfer portion is same, but now the fluid motion is governed by two factors not just Reynolds number, but Reynolds number and $\frac{g\beta\Delta TL}{V^2}$. But there is no independent parameter V that can be used at the characteristic velocity. In fact, the resulting velocity is the result of the phenomenon, result of the free convection. It is not something that can be specified independently.

So, since is a result, we can set Reynolds number to be order 1. So, that, it gives you an idea of what should be the characteristic velocity. If we set Reynolds number is equal to 1, the characteristic velocity should be like $\frac{\mu}{\rho L}$. With this, this parameter which was obtained from equating the scale factor of inertial forces and the buoyancy forces, would become $\frac{\rho^2 g\beta\Delta TL^3}{\mu^2}$. This is given the name Grashof number, and is abbreviated as Gr.

So, the functional relationship can be written as: Nusselt number is a function of Grashof number and Prandtl number. A very interesting result obtained with so little work. Of course, this can be improved if we write scale factors in terms of the directions, like we have done before. And it can be shown that Nusselt number varies like one fourth power of the Grashof number.

Free-burning fires

A free-burning fire is a cyclic process in which some of the heat generated by combustion is transmitted back to the fuel bed to produce combustible fuel, which then mixes with air and liberates more heat. The combustion products are removed from the reaction zone, drawing in fresh air.

One last case study on free burning fires. Fires are very important for human existence. In fact, most of the history of early development of human civilization can be traced to the mastery of keeping the fires burning. A free burning fire is cyclic process in which some of the heat generated by combustion is transmitted back to the fuel bed to produce combustible fuel, gaseous fuel which then mixes with the air reacts and liberates more heat. The combustion products are removed from the reaction zone by drawing in fresh air that further promotes combustion.

Let us do a little study of this fire. The basic process of fire can be summarized as the generation of combustible fuel vapor from the fuel bed by heat supplied from the combustion zone. If we are using a fossil fuel like coal or even wood then heating of the wood releases combustible gases. If we have a liquid fuel like a diesel or kerosene then as it is heated it vaporizes and that vapor is combustible. Then vapor and oxygen react in the combustion zone at the elevated temperature to liberate heat. This heat forms a hot plume above the flame as the hot gaseous product are driven upward by buoyancy.

And as this gaseous product move up the cold air from all sides are drawn in and mixes with the flame creating turbulent diffusion. A rather complex process for which the analysis is quite complex. Let us look at what are the forces that are important. Clearly initial forces and the gravitational forces through buoyancy are the dominant forces.

Viscous forces can be neglected. Conduction is far slower than the mixing of gases involved here. Therefore, conduction can be neglected. Radiation too can be neglected relatively small energy in radiation. Only two laws need to be modelled for the forces. Inertial force which gives you the standard form of $k_{F,i}$, and the buoyancy force $k_\rho k_L^{3}(k_V^{2}/k_L)$, gives you two pi numbers, and from this, if we equate the scale factors $k_{F,i}$ and $k_{F,b}$, we get this pi number.

For gases, $\frac{\Delta\rho}{\rho}$. is like $\frac{\Delta\theta}{\theta}$, and the resulting pi number is like a Froude number. It is called the densimetric Froude number, or even Richardson number Ri in literature. Now, we want to conduct test on a small fire to model a large fire. One of the simple ways to conduct model test involves keeping delta theta by theta a constant. And if we can keep $\frac{\Delta\theta}{\theta}$ is a constant, then $\frac{Lg}{V^2}$ is a constant.

The temperature difference between a hot flame and the cold air, that is Δθ, depends largely on fuel type, the fuel configuration, and the cold air temperature. If the cold air temperature theta is the same then using the same fuel and fuel configuration will ensure the delta theta by theta is a constant, and if that is so, then $\frac{Lg}{V^2}$ would be a constant.

One of the phenomena that you see in open fires which is burning freely, is the periodic pulsation. You look at this GIF, the fire bulges out, and then sudden contraction, and goes up, and then do this. This repeats. There is a periodic pulsation and there is a time period, according to this. Now, if we do experiments in which we kept $\frac{\Delta\theta}{\theta}$ constant, then $\frac{Lg}{V^2}$ is a constant and V^2 can be written as L^2/τ^2, where τ is the characteristic time. So, that for constant $\frac{\Delta\theta}{\theta}$, this value of Richardson number reduces to $\frac{g\tau^2}{L}$ as a constant between similar fires, or that is the time period is like $\sqrt{\frac{L}{g}}$. Brian and Nelson conducted experiments by using a liquid fuel on circular pans of different diameters from about 8 centimeter diameter to a meter in diameter, and they observed the pulsation period in seconds. And they get these dark points as a pulsation period versus pan diameter on the log log scale.

This red line is the line of the rule that we obtained earlier. So, we see clearly that our modeling rule is quite valid. Thank you very much.

Thank you.