Similitude And Approximations In Engineering,
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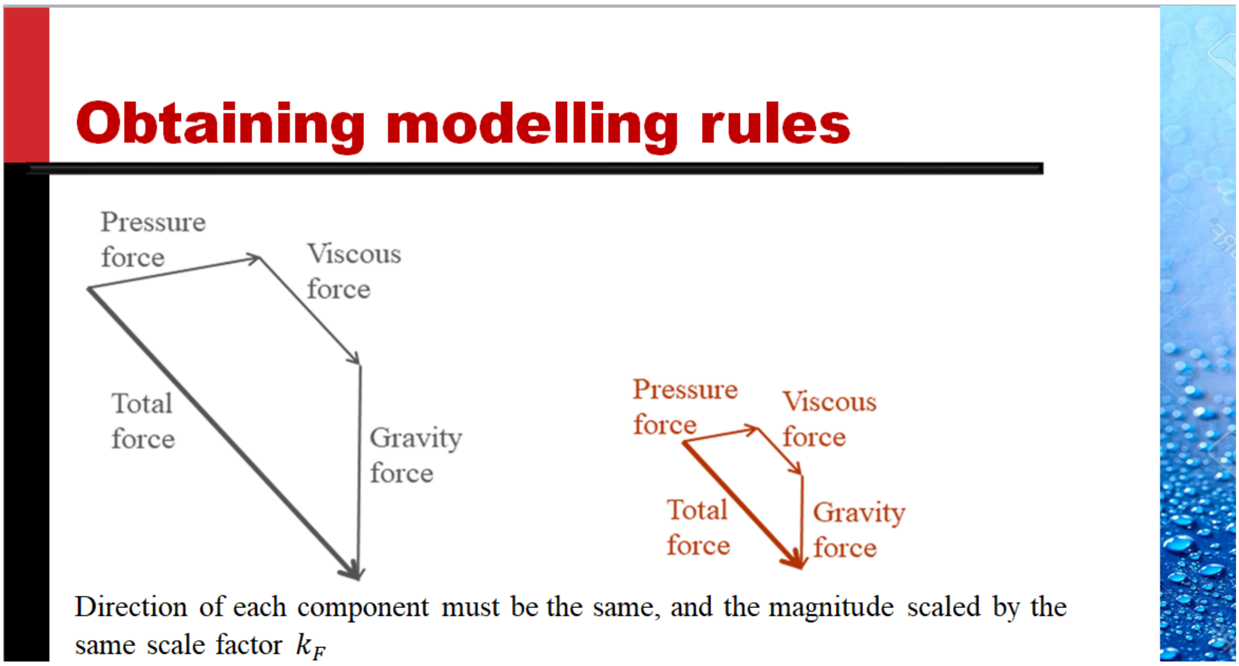
Week - 04
Lecture - 15

Welcome back. In this lecture, we will continue with our discussion of the scale factor approach We will do more examples in where we obtain modeling and prediction rules using the scale factor approach. Similarity requires that not only the total forces on the prototype and the model elements be scaled by a unique scale factor, which depends upon the scale factors for mass and acceleration, but the same factor must also scale every component force that make up the total force vectorially. This is what we discussed in the last class. So, if a pressure force, viscous force and a gravity force make up the total force, then within the prototype and the model there would be the same scale factor and each component would have the same direction and the magnitude of each component would be scaled by the same scale factor k F. The process that we outlined for obtaining the modeling rules consisted of identifying the various force components that determine the net force at a point in the flow field, then obtaining the scale factors for each of these force components using the physical laws that govern the phenomena.



Then equating the scale factor so obtained to the net force scale factor k F obtained from Newton's law of inertia. This is termed as a inertial force factor. And then obtaining the non-dimensional pi numbers in terms of the characteristic independent quantities from the relations established in the step 3 above. These give us the invariance that establish the modeling rules.

As we did in the last class, we first obtained the relationships among the scale factors for each of the forces and then wrote down the corresponding pi numbers. For example, the total or the inertial force Fi is governed by the law that this is equal to mass times acceleration. In writing the scale factor for mass in terms of k_ρ and k_L and for acceleration in terms of k_V and k_L, we get the scale factor for the inertial force as $k_\rho k_L^2 k_V^2$. And from this, we obtain that the $\frac{F_i}{\rho L^2 V^2}$ is the relevant pi number which should be invariant between the prototype and the model. In this and in all subsequent discussion, the characteristic nature of these quantities is understood.

Similarly, we derived the pi numbers for the unsteady flow, for the viscous flow, for the gravity force and for the pressure force. For surface engine force, for the compressibility force and for the centrifugal force which is mass times omega squared r. We equate the scale factors for the various component forces for obtaining the modeling rules. So, for example, if we equate the scale factor for the inertial force and the viscous force, we get this relation among the scale factors for the independent parameters. Then from this we can write $\frac{\rho V L}{\mu}$ as an invariant.

Named Pi numbers

inertial force and viscous force	$\rho V L/\mu$	Re	Re(ynolds)
inertial force and gravity force	V^2/gL	Fr^2	Fr(oude)
inertial force and pressure force	$\rho V^2/\Delta p$	1/Eu	Eu(ler)
inertial force and surface-tension force	$\rho V^2 L/\sigma$	We	We(ber)
inertial force and compressibility force	$\rho V^2/E_s$ V^2/c^2	Ca^2 Ma^2	Ca(uchy) Ma(ch)
centrifugal force and inertial force	$V\tau/L$ or V/fL	1/St	St(rouhal)

Of course, this is recognized as the Reynolds number if $\frac{\rho V L}{\mu}$ are the characteristic quantities for a given flow. In fluid mechanics, if we take the ratio of the inertial and viscous forces that is equate the scale factors for inertial and viscous forces, we get Reynolds number. If we equate the scale factor of inertial gravity forces, we get V^2/gL which is the square of the Froude number. Similarly, the Euler number, Weber number, Cauchy number, Mach number and Strouhal number. This was all done in the last class as well.

Governing pi number

Governing law	Scale-factor relation	Similarity rule	Π-number
Inertia: $F_i \sim ma$	$k_{F_i} = (k_\rho k_L^3) k_V^2 / k_L$		
Gravitation: $F_g \sim mg$	$k_{F_g} = (k_\rho k_L^3) k_g$	$\frac{k_V^2}{k_L k_g} = 1$	$\frac{V^2}{gL} (= \mathrm{Fr}^2)$

Modelling rule: $\frac{V_m^2}{g_m L_m} = \frac{V_p^2}{g_p L_p}$

$$V_m = V_p (0.1)^{1/2} = 100(\mathrm{km/hr}) \times \sqrt{0.1} = 31.6\ \mathrm{kmph}$$

Let us do an example in which we match the Froude number. A sea plane is being designed for a takeoff speed of 100 kilometers per hour. Model tests are to be made on a one tenth scale model. So, k_L is 10, the length scale factor is 10. At what speed should the model be towed to simulate the takeoff condition? And then the force required to drag the model sea plane through the takeoff is 9 Newton.

Predict from this the thrust needed by the prototype at takeoff. So, we identify the forces. Clearly inertia is an important force and in this case the only other force that is dominant is the gravity forces. So, we use two laws. The inertia law Fi is like m a and the gravitational law that the gravitational force mg is like mg where m is the mass.

We write the appropriate scale factors relation mass scale factor is $\left(k_\rho k_L^3\right)$ and acceleration is k_V^2/k_L. Similarly for mass and gravity and from these two laws by equating this scale factor to this scale factor we get $\frac{k_V^2}{k_L k_g}$ is 1 from which the pi number obtained is V^2/gL which is recognized as the square of Froude number. So, this similarity requires that the Froude number of the prototype flow and the model flow match. So, the modeling rule becomes $\frac{V_m^2}{g_m L_m} = \frac{V_p^2}{g_p L_p}$. Clearly, g_m and g_p are the same. L_m by L_p is 1 by 10, and you plug in the relation and we plug in the value of the prototype as 100 kilometers per hour, we get the model speed required for similitude to be 31.6 kilometers per hour. If we tow the model aircraft at this speed then the gravity forces and the inertial forces would be similar, would have the same ratios and hence the similitude is assured. There are no other important forces. And then to determine the scale factor of thrust, we note that k thrust is like any other force.

Prediction rule

We need to determine the scale-factor for thrust

$k_{thrust} = k_F = k_\rho k_L^2 k_V^2$ Converting it to a pi-number: $\frac{\text{Thrust}}{\rho L^2 V^2}$

$$\text{Thrust}_p = \text{Thrust}_m \times \frac{(\rho L^2 V^2)_p}{(\rho L^2 V^2)_m} = 9\text{N} \times (10)^2 \times \left(\frac{100 \text{ MPH}}{31.6 \text{ MPH}}\right)^2 = 9 \text{ kN}$$

A thrust is like any other force. So, the scale factor for thrust would also be equal to the scale factor for any other force, and we choose the inertial force for simplicity, and so the k for thrust would be $k_\rho k_L^2 k_V^2$. And converting it to a pi number we get $\frac{Thrust}{\rho L^2 V^2}$ is invariant, that is, it has a same value for the prototype as for the model. And from this the thrust of the prototype is obtained as 9 kilo Newton a 1000 times more than the thrust for the model which was 9 Newton.

Let us take another example a 1 6th model of an automobile is to be tested in a wind tunnel at the speed corresponding to the prototype speed of 60 kilometers per hour. Determine the model speed for similitude. Then, if the drag on the model at that speed is 510 Newton what is the drag on the prototype and what is the power requirement of the prototype? What power would be required by the prototype at a speed of 60 kmph if the drag on the model at the corresponding speed is 510 Newton. So, as before we decide on what are the important forces and where lies then in this model where there are no free surface involved and where there is no characteristic pressure difference the only forces that need to model are inertia and viscous forces. To obtain viscous shear forces we notice that shear forces are like shear stresses times the area. The shear stress is given by the Newton law shear stress mu dv by dy the velocity gradient.

Example

A one-sixth scale model of an automobile is to be tested in a wind tunnel at a speed corresponding to the prototype speed of 60 kmph. Determine the model speed for similitude. If the drag on the model at that speed is 510 N, what is the drag on the prototype and what is the power requirement of the prototype?

So, the scale factor for shear force is like $k_\mu k_V / k_L$ and the scale factor of area is simply k_L^2 and therefore, the shear force scale factor the viscous force scale factor is $k_\mu k_V k_L$ and the relevant pi number is $\frac{F_\mu}{\mu VL}$. Then we do for the inertial forces and as before the inertial forces are mass times acceleration mass is like rho times the volume. So, k_M is like $k_\rho k_L^3$ and acceleration in terms of v and l can be written as k_a is equal to k_V^2/k_L. So, that the relevant scale factor for the inertial force $k_{F,i}$ is $k_\rho k_V^2 k_L^2$ and from this we get a pi number $\frac{F_i}{\rho V^2 L^2}$. The two scale factors $k_{F,i}$ and $k_{F,\mu}$ should be exactly the same and from this we get $k_\rho k_V k_L = k_\mu$ or as we have done this before we obtain $\frac{\rho VL}{\mu}$ as the pi numbers.

For similitude we need to match the Reynolds number. Thus the modeling rule is the Reynolds number for the model should be same as Reynolds number for the prototype. Since ρ and μ of the model and prototype are identical since the fluid use is the same air the model velocity is the prototype velocity times L_p divided by L_m the characteristic length of the prototype divided by characteristic length of the model. It says the scale factors uses 6 for the length this gives you a velocity of the model to be 360 kmph that is the model should be tested at a wind speed of 360 kmph to higher speed. But there is something in the drag forces on objects that we will just look into that permits us to test at almost any speed.

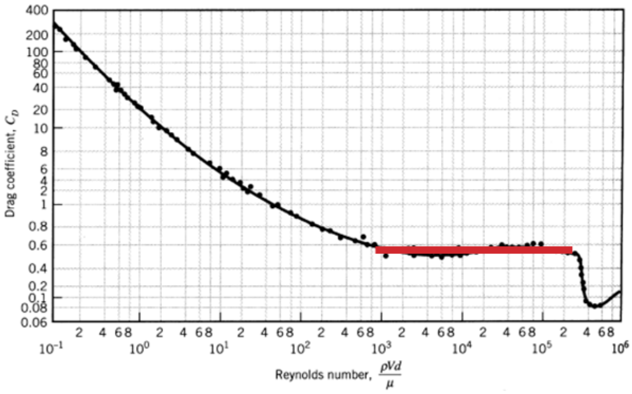
But suppose it is tested at 360 kmph then we obtain the prediction rule for the drag and is obtained from the force scale factor relation. Drag force is like any other force and its scale factor must be equal to the scale factor for all forces. We may choose to equate it to any of the two force scale factors that we obtain. Setting the drag force scale factor equal to the inertia force scale factor we get $k_{drag} = k_\rho k_L^2 k_V^2$ and from this we obtain $\frac{Drag}{\rho L^2 V^2}$ as a pi number, where L is a characteristic length and V is a characteristic velocity. This should be same for the prototype and the model. Thus the drag of the prototype is drag of the model times $\left(\frac{L_p}{L_m}\right)^2 \left(\frac{V_p}{V_m}\right)^2$. The drag force for the model was 510 Newtons. The scale factor length is 6 and the scale factor velocity we just found out is 1 by 6.

So, that the value of the drag force on the prototype would be the same, 510 Newton. Our model is smaller by a factor of 6, but we are testing at a velocity which is 6 times the prototype velocity. So, that the force scale factor is 1, and the drag on the prototype is the same as drag on the model. The power required to overcome the wind drag can be obtained directly from the velocity and the drag as power is equal to force times the velocity. We convert 60 kmph into velocity in meters per second, and we get 8. 5 kilowatt. So, this is the power that will be required by the car to drive at 60 kmph. The pi number $\frac{Drag}{\rho L^2 V^2}$ is a very often used pi number in aerodynamics, though mostly with slight modification. What modification? We use a factor of one half in the denominator and L^2 is replaced by a characteristic area, and this is now given the name drag coefficient. When the flow in the model is similar to flow in the prototype then the drag coefficient is same. The characteristic area depends upon the geometry.

Drag coefficient

Thus, the drag coefficients of any geometric similar shape is going to be the same if the Reynolds numbers match, or

$$C_D = f(\text{Re})$$



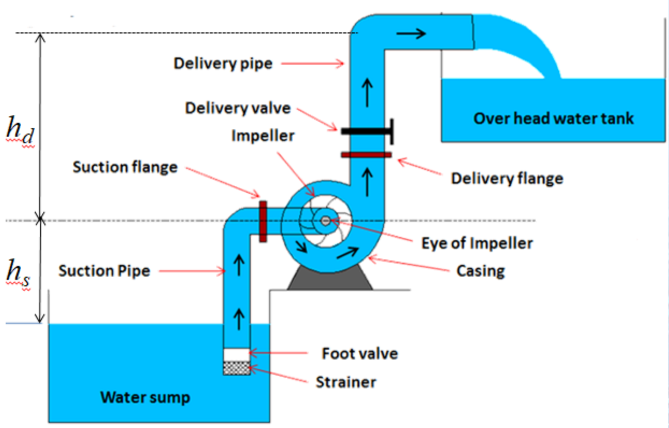
For bluff bodies like automobiles etcetera, the conventionally used characteristic area is the frontal area, the area that meets the flow in the face. While for the aircraft wings, the planform area of the wing is used as the characteristic area. Thus, the drag coefficient of any geometrically similar shape is going to be the same if the Reynolds number match, that is, C_D is a function of Reynolds number for same geometric shape. We plot here the drag coefficient for a sphere at various Reynolds number. The dots represents the measurements by many researchers on different types of fluids, different sizes of spheres, etcetera, and all fall in a single curve, proving the power of the analysis that we carried out.

That the drag coefficient is a function of Reynolds number and the geometric shape is borne out completely from this graph. An interesting thing to notice however is that over a large range of Reynolds number from about 10 raised to power 3 to 2 into 10 raised to power 5 the drag coefficient is effectively independent of Reynolds number. This fact is of great importance. In fact, you would see tables in handbooks for drag coefficients where the Reynolds number is not mentioned. The drag coefficient for a sphere in laminar flow is given as about 0.6. For a flat disk at about 1.1, and so on. Reynolds number is not mentioned because for most engineering applications, the relevant range of Reynolds number is this. So, for this automobile that we had, we did not need to match the Reynolds number. In fact, testing it at the matching Reynolds number which required a speed of 360 kilometers per hour, would create serious problems. The speed is so high that it is tending to be in the regime where compressible effects start becoming significant that we neglect it.

So, if we measure drag coefficient that large a speed, we may not get the right result. So, we measure the drag at any speed, a much lower speed, convert into a drag coefficient, and then using the fact that the drag coefficient is independent of Reynolds number, we say that the same drag coefficient would apply to the prototype automobile, and from this we calculate the drag on the prototype.

Example: Strouhal number and Euler number matching

A pump capable of lifting 5 m^3/s of water against a head of 250 m at 500 RPM is to be designed. A one-ninth scale model is constructed and tested with the same fluid against a 10-m head. At what speed should it run and what should be the volume flow rate? What would be the power scale factor?



Another example of a water pump. This water pump is picking up water from this sump and pumping it to an overhead tank. The delivery point is at a distance h_d. So, this is the head that is being produced by the pump. A pump capable of lifting 5 meter cube per second of water against a head of 250 meter. Here h d is 250 meters, at 500 rpm is to be designed It will be a rather big pump. So, a one ninth scale model is constructed and tested with the same fluid, that is water, against a 10 meter head a much reduced head. At what speed should it run and what would be the volume flow rate if we are going to get similar flows? And then what would be the power scale factor between the prototype pump and the model pump? So, as before, we begin by deciding what are the important forces.

Modelling rules

The pressure and inertial forces are the only ones to be modelled in this problem. Since this pump is a rotary machine, centrifugal inertial forces should also be modelled.

Governing law	Scale-factor relation	Similarity rule	Π-number
Inertia $F_i \sim ma$	$k_{F_i} = (k_\rho k_L^3) k_V^2 / k_L$		
Pressure $F_p = \Delta p \times Area$	$k_{F,p} = k_{\Delta p} k_L^2$	$k_{\Delta p} = k_\rho k_V^2$	$\frac{\Delta p}{\rho V^2} \left(= \frac{1}{\text{Eu}}\right)$
Centrifugal $F_\omega = (mass) \times (\omega^2 L)$	$k_{F_\omega} = k_\rho k_\omega^2 k_L^4$	$k_\omega k_L = k_V$	$\frac{\omega L}{V}$ (= St)

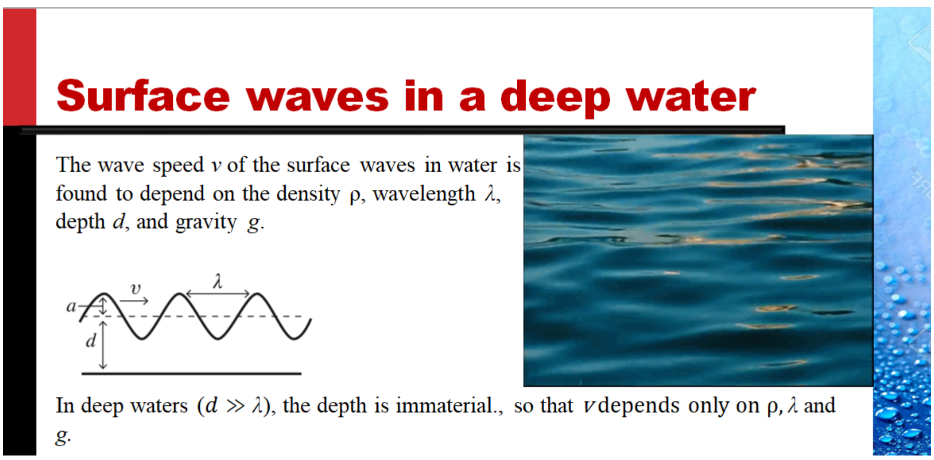
The pressure forces in the inertial forces are the only ones that need to be modeled in this problem. Since this pump is a rotary machine centrifugal inertial forces should also be modeled, that is, the centrifugal forces should also have the same scale factor as the other forces. So, now we have three governing laws: one for the inertia, the second for the pressure and the third for the centrifugal force: mass times omega squared R and R is like a length. So,

$F_{\omega} = (mass) \times (\omega^2 L)$. ω is the angular speed. By using the process that we followed earlier, we write the scale factors for the inertial forces, for the pressure force, and for the centrifugal force. From these two, equating these two, we get $k_{\Delta p} = k_{\rho} k_V^2$. So, $\frac{\Delta p}{\rho V^2}$ is a pi number and we recognize this as 1 over Euler number, that we discussed earlier. By equating the scale factors for the inertial force and the centrifugal force we obtain this similarity rule, that $k_{\omega} k_L = k_V$, and from this we get $\frac{\omega L}{V}$ is a pi number. This pi number is named as Strouhal number and is abbreviated as St.

So, these are the two pi numbers that must match. Now, in our problem we match the Euler numbers we get a relationship for k_V. Rho is the same. The ratio $\frac{\Delta p_m}{\Delta p_p}$ is given as 250 meters for the prototype divided by 10 meters for the model. And so, this is like 5. That is the velocity ratio. This Strouhal number similarity, on the other hand, gives us $\frac{\omega L}{V}$ for the model is the same as $\frac{\omega L}{V}$ for the prototype which gives you that the angular speed of the model should be nine fifth of the angular speed of the prototype. Since the rpm of the prototype is 500, the rpm of the model should be 9 by 5 of 500, which is 900 rpm. So, when the prototype runs at 500 rpm, to obtain similarity we must run the model, the scale model, at 900 rpm. The prediction rule for the discharge is found by the scale factor relations.

The scale factor for discharge. Discharge is like velocity times the area of cross section. Area of cross section would be scaled by k_L^2. Velocity scaled by k_V. So, $k_{\dot{Q}}$ would be like $k_L^2 k_V$. k_L is 9, scale factor of a length, and scale factor of velocity is 5. So, the scale factor for volume flow rate is 405. Thus, the model should deliver only this flow rate instead of 5 meter cubed per second, which is a huge volume flow rate. It needs to deliver only a very small fraction of this, 1 over 405 of that. This scale factor for power consumption is obtained from the relation power is equal to force times velocity. So, $k_{\dot{W}} = k_F \times k_V$, and k_F is like $k_{\rho} k_L^2 k_V^2$. So, $k_F \times k_V = k_{\rho} k_L^2 k_V^3$. Here k_{ρ} is 1. We are using the same fluid, k_L is 9, and k_V is 5. And when we put it all together, we get a power scale factor of more than 10,000. So, prototype will consume 10,125 times the power used by the model pump.

These few examples should demonstrate to you quite clearly the power of this method of scale factors, or the law approach, as it is sometimes called. We can work with the relevant laws and then obtain results very quickly.



Another example: surface waves in deep water, the wave speed V of a surface wave in water is found to depend upon the density ρ of the water, the wavelength λ, the depth d, and gravity g. The waves are moving on the surface of water. The fluid particles are not moving. This is speed of surface waves. In deep waters, where d is much greater than λ, the depth is immaterial, and the speed of the surface waves depends only on ρ, λ and g.

In fact, we will see that rho should not matter because neither V nor λ nor g contains the dimension of mass. And so, the only parameter rho which contains dimension of mass would not occur in any non dimensional relations. Let us determine this function of ρ, λ and g. From the list of independent quantities involved it is easy to see that only the inertial and the gravity forces are likely to be significant. So, we will invoke the law for the inertial and the gravity forces.

Surface waves in a liquid

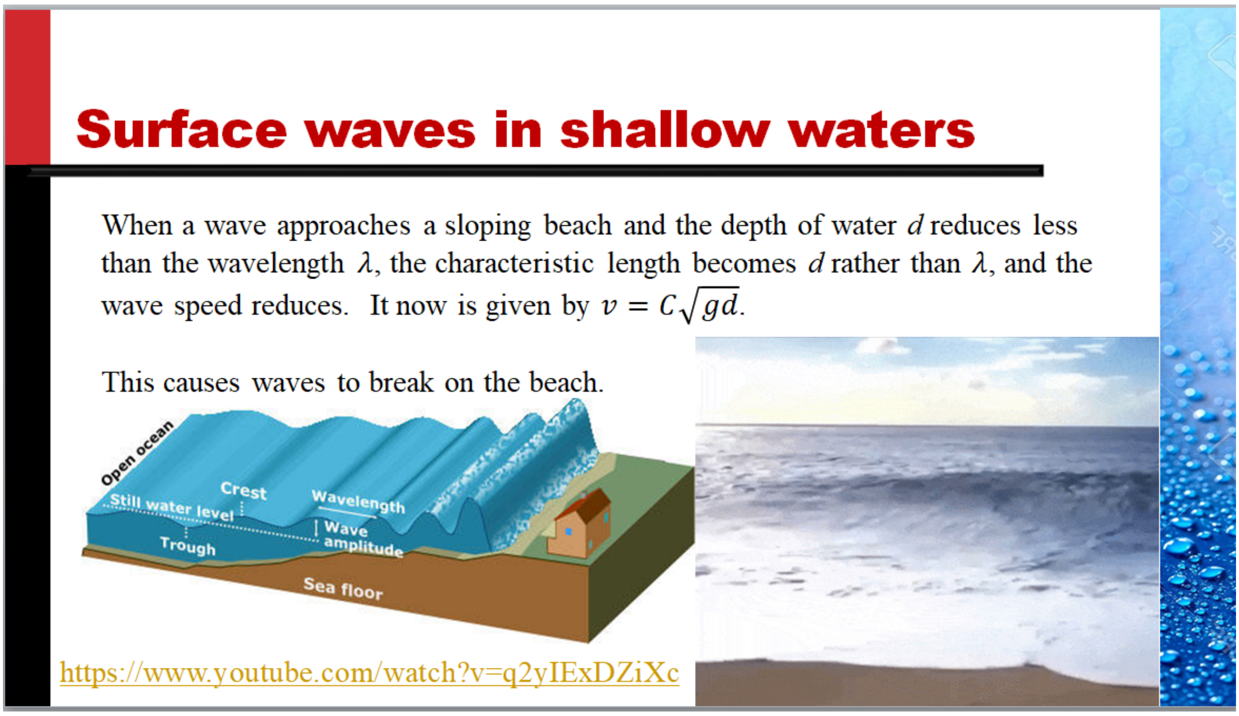
From the list of the independent quantities involved, it is easy to see that inertial and the gravity forces are likely to be significant.

Governing law	Scale-factor relation	Similarity rule	Π-number
Inertia $F_i \sim ma$	$k_{F_i} = (k_\rho k_L^3) k_V^2 / k_L$		
Gravity, $F_g \sim mass \times g$	$k_{F_g} = k_\rho k_L^3 k_g$	$\frac{k_V^2}{k_L k_g}$	$\frac{V^2}{gL} (= \mathrm{Fr}^2)$

It is easy to see that wavelength λ can be used as L_c. However there is no physically relevant independent parameter that characterises the velocity V_c.

These are the relevant scale factor relations, and if we equate these two we get a similarity rule that $\frac{k_V^2}{k_L k_g}$ should be 1. And from this we obtain the pi number $\frac{V^2}{gL}$, the square of Froude number as invariant. It is easy to see that wavelength λ can be used at L_c. However, there is no physical relevant independent parameter that characterizes the velocity V_c. So, the life is simple. This $\frac{V^2}{gL}$ becomes a prediction rule itself. There is no modeling rule required. So, $\frac{V^2}{gL}$ should be order 1. So, V^2 should vary like gL, L is λ. So, V is like $C\sqrt{g\lambda}$. The scale factor for velocity is related to scale factor for time and length. So, that $V_c t_c / L_c$ is invariant, which permits us to use $t_c \sim \frac{L_c}{V_c}$ and so, the time period is like $\sqrt{\frac{\lambda}{g}}$. This is for surface waves in water in deep ocean, where d is much larger than wavelength. When a wave approaches a sloping beach and the depth of water d reduces to less than the wavelength λ, the characteristic length becomes d rather than λ and the wave speed reduces.

It is now given by V is equal to $C\sqrt{gd}$. This is interesting. You see here in this gif, the waves are approaching the beach towards us, the water rises, the waves rise when they approach the beach, and this is happening because the friction at the bottom of the beach slows down the waves. The wave velocity on the top is larger than the wave velocity bottom. So, there is a bunching of the liquid, and the waves rise, causing the phenomena of breaking of waves on the beach. In fact, tsunamis which can have wave heights of a few meters are caused by this phenomena.



Tsunami waves when they are in the open sea have hardly any wave heights. The ships in ocean would not even notice the tsunamis when they pass, but they when they reach the beaches they rise very high. There is a very interesting video available on YouTube regarding the development of waves in oceans. This is the reference that I given here. I recommend highly that you watch it.

Thank you. .