Similitude And Approximations In Engineering,
Vijay Gupta
Applied Mechanics
Indian Institute of Technology Delhi
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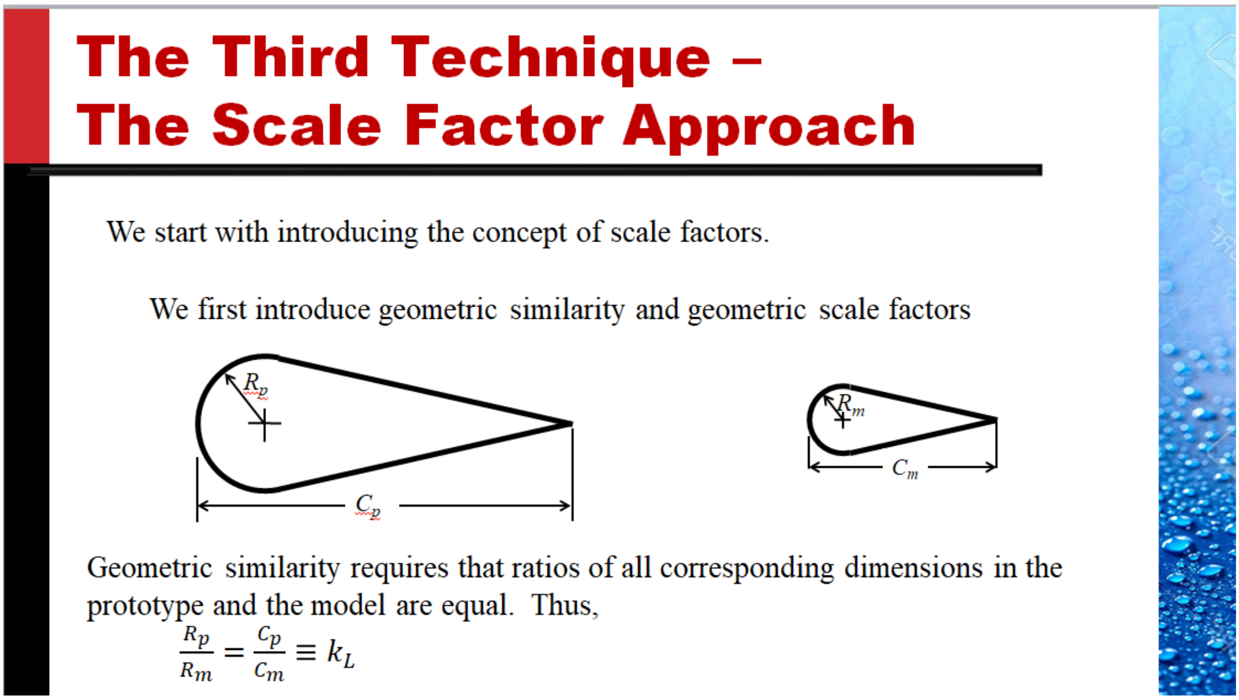
Lecture - 14

Welcome back.

In this lecture, we introduce a new approach to the problem of similitude. This is known as the scale factor approach. The learning outcome of this lecture would be using the scale factor approach to obtain modeling and prediction rules. This approach is far more general and powerful than those discussed earlier. Unlike the process involving non-dimensionalization of the governing equations and boundary conditions, this approach does not require a mathematical model and that is a lot of saving.

And unlike dimensional analysis which operates in a relative vacuum of any physical model, it gives some insight into the physical model, not the whole insight, but some insight and therefore is amenable to reasoned relaxation of the modeling rules. One of the problem of using any similitude approach is there are too many similarity rules and we cannot tailor our materials to make all those similarity rules applicable and so some relaxations have to be made. The scale factor approach permits us to make reasoned relaxation of modeling rules. We will come to this towards the end of the course.



We start with introducing the concept of scale factors. We first introduce geometric similarity and geometric scale factors. Consider two scaled models which are geometrically similar. Let us call the larger one the prototype and the smaller one the model. A model does not need to be always smaller than the prototype, but for discussion today let us assume the smaller is the model and the larger is the prototype.

A geometrically similar model is one in which all the corresponding dimension in the model and prototype bear the same ratio. Thus, the geometrical similarity requires that the ratios of all corresponding dimensions in the prototype and model are equal. If R denotes the radius of the nose of the body, then $\frac{R_p}{R_m}$, the radius in the prototype divide by radius in the model, is equal to the chord Cp of the prototype divide by the chord Cm of the model. Each of these ratios being equal to k_L, read as the length scale factor.

So, scale factor is equal to the value of a parameter in the prototype divided by the value in the corresponding model. Let us consider two areas, one in the prototype and the other in the model, the scaled model. Then the area of this area is equal to $\delta x_p \times \delta y_p$, the length in the x direction of the prototype multiplied by length in the y direction. Similarly area of the model would be $\delta x_m \times \delta y_m$ the two areas and the ratio of these two would give us the area scale factor k_A. Similarly this is equal to k_L^2. The geometric similarity ensures that the scale factor for the area would be the scale factor for the lengths squared. Similarly the scale factor for the volumes k_V would be k_L^3. All volumes in the prototype and all volumes in the model the ratios would be k_L^3 a unique factor.

We introduce here the concept of homologous points again we have talked about that the two points A_p in the prototype and A_m in the model are termed as homologous points. If the ratios of the x coordinates is equal to the ratio of their y coordinates both being equal to k_L, these points are known as homologous points. So, for one point in the prototype we can determine the corresponding location for the homologous point in the model. This will play an important role in later discussion. Concept of similarity implies that the same holds for each of the other quantifiable quantities that is, parameters and variables, both dependent and independent, such as velocity, time, stress, force, power, work done, etcetera. Each and every quantity imaginable. This is to say that value of any quantity at a point in the prototype is related to the value of the same quantity at the homologous point in the model through the corresponding scale factor.

There is a unique scale factor for a unique quantity. There would be a velocity scale factor, a time scale factor, a stress scale factor, a force scale factor etcetera. Let us talk of velocity similarity. On the same body at three points 0, 1 and 2 in the prototype we have three velocities as shown $V_{o,p}$, $V_{1,p}$, $V_{2,p}$, ... We consider the model and we consider the homologous points in this model corresponding to 0, 1 and 2.

These are the homologous points. Let the velocity there be $V_{o,m}$, $V_{1,m}$, $V_{2,m}$, ... Then, the similarity of the two requires that the ratios of the corresponding velocities in the prototype and model is identically, the same. There is a unique scale factor for velocity $k_V = \frac{V_{1,p}}{V_{1,m}} = \frac{V_{2,p}}{V_{2,m}} = \frac{V_{0,p}}{V_{0,m}} = \ldots$ and the ratios of the velocities at homologous points is identical to k_V, whichever point we talk about. This is velocity similarity.

Further it is just not the magnitude of velocity that are so related, but the direction of the velocity at homologous points must also be the same as shown in these two pictures. One a prototype, and one the other a model. The velocities V1, V2 and V3 in the prototype and the model have exactly the same directions. It should be clear that this would require that the individual components of velocities are also scaled by the same scale factor k_V. Thus, the horizontal velocity component at the prototype divided by that for the model is equal to the vertical velocity for the prototype divided by that for the model, both being equal to k_V, which was the ratios of the velocity V1 for the prototype divided by V1 for the model.

$$\frac{u_{1,p}}{u_{1,m}} = \frac{v_{1,p}}{v_{1,m}} = k_V$$

And the same k_V, the same ratio would hold for point 2 or point 3, or any other point in the two fields. The geometrically similar prototypes and models are said to have kinematic similarity if all the kinematic quantities including the velocity, such as frequency f, the rpm N, the angular velocity omega, acceleration a, angular acceleration α, volumetric flow rate $\dot{Q}$, etcetera, have constant scale factors. Different scale factor for each quantity, but for that quantity, a constant value at any pair of homologous points. The basic strategy in obtaining these similarity rules consists of exploiting the fact that the scale factors for various quantities are not all independent. It is alright to say that the scale factors should be constant, but what makes the scale factor constant? That is the problem.

And as discussed earlier we need modeling rules for the purpose. We choose the unique parameters of the model such that they satisfy the similarity requirements, but what are those similarity requirements? The similarity rules and here we state that the basic strategy in obtaining these similarity rules consists of exploiting the fact the scale factors for various quantities are not all independent. Like we have seen the scale factor for area $k_A = k_L^2$. k_A and k_L are not independent. Similarly, the scale factor for volume was seen to be k_L^3, not independent.

The interrelationship between these various scale factors are converted into relation between independent parameters for the prototype and model. Some of these would involve the independent parameters, and they would be related to the independent parameters of the prototype. So these will give the relations between independent parameters, the uniquity parameters of model and the prototype. The remaining part of this lecture and the next lecture would be devoted to how those rules are established. For example, let us start with the scale factor for velocity k_V.

It must be related to scale factor k_L and k_t of length and time interval respectively. k_V the scale factor of velocity is equal to $\frac{V_{prototype}}{V_{model}}$ at the homologous point. $V_{prototype}$ can be written as dxp the distance travelled in the prototype in time dtp divided by dxm, the distance travelled in the model divided by dtm. This is written as $\frac{dx_p}{dx_m} \bullet \frac{dt_m}{dt_p} = \frac{k_L}{k_t}$.

$\frac{dx_p}{dx_m}$ is nothing but the length scale factor k_L and $\frac{dt_m}{dt_p}$ is the reciprocal of the time scale factor k_t. Time scale factor k_t is defined as tp divided by tm, time in the prototype divided by time in the model. So that we obtain the relation $k_V = \frac{k_L}{k_t}$. These three scale factors are not independent of one another $\frac{k_V k_t}{k_L} = 1$. Now, k_V is $\frac{V_{c,p}}{V_{c,m}}$, k_t is $\frac{t_{c,p}}{t_{c,m}}$ divided by k_L. which is $\frac{L_{c,p}}{L_{c,m}}$ and that should be 1.

By rearranging we get $\left(\frac{V_c t_c}{L_c}\right)_p = \left(\frac{V_c t_c}{L_c}\right)_p$. Subscript c denotes that this is a characteristic value. Whatever characterize the velocity time interval and length in the prototype the corresponding value in the model must be related to these through this relation. $\frac{V_c t_c}{L_c}$ is then called a pi number. It is dimensionless, it would be dimensionless the way we have constructed it.

The characteristic value of the velocity times the characteristic value of the time interval divided by the characteristic value of the length would have the same value in the prototype as in the model. So, this is invariant and that is why it is termed a pi number. This is just starting on the road to establish modeling rules. Ensuring similarity consists of obtaining scaling relationships for the values of the independent parameters of the iniquity parameters. So, that given the values of the iniquity parameters for the prototype we can determine the values required for the model for the similarity of the phenomena for ensuring that the two things the prototype and model have similar behavior.

Scale factor for	**In terms of**	
	k_L and k_t	**k_L and k_V**
Time, k_t	k_t	k_L/k_V
Velocity, k_V	k_L/k_t	k_V
Angular speed, frequency and RPM, k_ω, k_f, and k_N	$1/k_t$	k_V/k_L
Acceleration, k_a	k_L/k_t^2	k_V^2/k_L
Angular acceleration, k_α	$1/k_t^2$	k_V^2/k_L^2
Velocity gradient, $k_{dV/dx}$	$1/k_t$	k_V/k_L
Volume flow rate, $k_{\dot{Q}}$	k_L^3/k_t	$k_L^2 k_V$

Once the similarity between the prototype and model is achieved by the proper choice of the independent iniquity parameters, it can be shown that the dependent quantities would also have constant scale factors. This leads us to prediction rules. If we do similar exercises that we do did for the velocity, we would determine this projection of the scale factors of the various kinematic quantities. For time, the scale factor is k_t, and we have written the scale factors in terms of two sets, k_L and k_t in this column, and k_L and k_V in this column. Usually, in fluid mechanics length and velocities are the preferred uniquity parameters.

So, the second column is quite useful in fluid mechanics problem. Elsewhere k_L and k_t may be used. For velocity we just determined that k_V is equal to k_L divided by k_t in terms of k_L and k_t. Of course, in terms of k_L and k_V it will simply be k_V.

Angular speed, frequency, rpm. The scale factors for these quantities k_ω, k_f, and k_N would be $1/k_t$, per unit time, and this would be k_V by k_L in the second formulation. Similarly, for acceleration, rate of change of velocity with time, velocity is k_L by k_t. So, acceleration is k_L by k_t^2, and in terms of k_L and k_V it is k_V^2/k_L. Most frequently used in fluid mechanics.

Angular acceleration, similarly can be written as $1/k_t^2$. Velocity gradient k of dV/dt is $1/k_t$, or k_V/k_L. Volume flow rate $k_{\dot{Q}}$ is like scale factor of volume which is $k_L^2 k_V$, which is k_L^3/k_t. Or it can be used as area of cross section times the velocity. So, scale factor area, k_L^2 and that of velocity is k_V.

You do not need to remember these because they can be obtained in no time whenever you require them. Pi numbers. Given one example for the velocity, let us give one more example. Let us obtain a pi number involving the volume flow rate $k_{\dot{Q}}$. Volume that passes through in one second in one unit time.

So, scale factor for $k_{\dot{Q}}$, we just obtained in the last table that $k_{\dot{Q}}$ can be written in terms of k_L and k_V as $k_L^2 k_V$. The another formulation was k_L^3/k_t, but $k_{\dot{Q}}$ is equal to $k_L^2 k_V$ implies that the ratio of the volume flow rate in prototype and model is equal to the square of the ratio of lengths multiplied by the ratio of the velocities in the prototype and model, respectively. And from this by rearranging we get $\left(\frac{\dot{Q}}{L^2 V}\right)_p = \left(\frac{\dot{Q}}{L^2 V}\right)_m$.

The characteristic nature of these quantities involved is understood. We do not write sub c all the time. It is understood that each of these quantities carried the subscript c, that it is a characteristic quantity. a $\dot{Q}$ that represents the flow, a length that represents the problem situation a velocity that represents the flow situation, or that is typical of the problem involved. So, $\frac{\dot{Q}}{L^2 V}$ is a pi number whose value must be the same in model in the prototype. This may be used as a modeling rule, if all the three quantities $\dot{Q}$, L and V are independent, but if one of them is a dependent, this would be the prediction rule. We will come to this a little later.

So, after obtaining the scale factors that we did earlier, and these are contained in these two columns, we convert them into pi numbers. These pi numbers represent the non dimensional groups of parameters which would have identical values in the prototype and the model. The characteristic nature of the quantities contained here in is understood. Now, let us come to dynamic similarity or the interrelationship between scale factors for dynamic quantities. The prototype and the scale model are said to have dynamic similarity if the net forces at homologous points in the two are related by a constant scale factor k_F, F for force. So, k_F is a scale factor for force. However, the force at a point in a system is related to kinematic quantities. Newton's law of motion relates a net force F on an element to its acceleration a and its inertia. So, if k_F, k_M and k_M are the scale factors for forces, masses and acceleration, respectively, we can write force is equal to m a gives us k_F, the force scale factor would be $k_M k_a$. And as we have converted the relationship between scale factors to a pi number, we can obtain F/ma is a pi number.

So, value of F/Ma is the same in the prototype as in the model, if the phenomena is similar. One way to easily organize these relationship is to construct a flow diagram like this:

$k_{F,i}$ → k_M × k_a

$k_{F,i}$ the inertial force. So, we put a subscript i. Inertial force is one that involves acceleration. So, scale factor for inertial force is like scale factor of mass times a scale factor for acceleration. Now the scale factor of a mass is written from mass is equal to density times the volume. So, $k_M = k_\rho k_L^3$, and k_a, on the other hand, we obtained earlier is k_V^2/k_L. And this gives a relation $k_{F,i} = k_\rho k_V^2 k_L^2$. This is one formulation of the scale factor for inertial forces in terms of the three quantities k_ρ, k_V, k_L. If we wanted to use k_t, this would have a different formulation.

The preferred formulation in fluid mechanics is this, but you looked in similar results if you use the other formulation. You may not recognize the relation immediately, but you looked in very similar relations. And this relation between scale factors would give you a pi number Fi the inertial force characteristic, divided by the density times the characteristic of velocity squared divided by characteristic length squared are identical in the prototype and the model.

Quantity	Scale factors in terms of		Π in terms of	
	k_ρ, k_L and k_t	k_ρ, k_L and k_V	ρ, L and t	ρ, L and V
Force, F	$k_\rho k_L^4/k_t^2$	$k_\rho k_L^2 k_V^2$	$Ft^2/\rho L^4$	$F/\rho L^2 V^2$
Momentum, P	$k_\rho k_L^4/k_t$	$k_\rho k_L^3 k_V$	$Pt/\rho L^4$	$P/\rho L^3 V$
Torque, T	$k_\rho k_L^5/k_t^2$	$k_\rho k_L^3 k_V^2$	$Tt^2/\rho L^5$	$T/\rho L^3 V^2$
Work, W	$k_\rho k_L^5/k_t^2$	$k_\rho k_L^3 k_V^2$	$Wt^2/\rho L^5$	$W/\rho L^3 V^2$
Power, $\dot{W}$	$k_\rho k_L^5/k_t^3$	$k_\rho k_L^2 k_V^3$	$\dot{W}t^3/\rho L^5$	$\dot{W}/\rho L^2 V^3$
Pressure, p	$k_\rho k_L^2/k_t^2$	$k_\rho k_V^2$	$pt^2/\rho L^2$	$p/\rho V^2$
Shear stress, τ	$k_\rho k_L^2/k_t^2$	$k_\rho k_V^2$	$\tau t^2/\rho L^2$	$\tau/\rho V^2$

Once we did for force, we could do for other dynamic quantities like momentum P, the torque T, work done W, the power, work per unit time, W, the pressure p, the shears τ. These columns list the scale factors in terms of k_ρ, k_L and k_t in the first column, and k_ρ, k_L and k_V in the second column, similar to what was done for the kinematic quantities earlier.

And the resultant pi numbers are given in this part. These are just for illustration purposes. We do not need to remember them. We will obtain them as we do the problems because it takes almost no time to obtain these pi numbers. The question that needs to be answered now is how to obtain modeling rules that will ensure full similarity between the behavior of a prototype and its geometrically similar model? This requires specification of the values of the uniquity parameters, that is, of the independent parameters for the model flow, given the values for the prototype flow, if you are dealing with fluid mechanics. Similarly for other sciences: solid mechanics, heat transfer, or any other science.

One approach follows from the recognition that the total force acting on a fluid element which results in its acceleration is the vector sum of the various forces, namely, the pressure forces, gravity forces, and the viscous forces, etcetera. There will be other forces involved surface tension force, compressibility force, but we have listed the typical forces in simple problems. Similarity requires that not only the total forces on the prototype model elements be scaled by unique scale factor which depends on the scale factors for mass and acceleration. But the same factor must scale every component force that make up the total force vectorially. So it is not just the inertial force, but the viscous force, the pressure force and the gravity force, they also must be scaled by the same scale factor k_F.

So if this was the total force which was made up as a sum of three component forces added vectorially the pressure force, viscous force and the gravity force. If this was the picture in the prototype, then the picture in the model should be same. The direction of each component must be same, and the magnitude for each of the component forces should be modeled by the same scale factor k F. This is the crux of the argument that we use for obtaining modeling rules. So it is not only the total force that must be scaled by a constant scale factor k_F, each component must be scaled by the same scale factor k_F.

So the process of obtaining modeling rules consists of identifying the various force components that determine the net force at a point in the flow field. Obtain the scale factor of each of these force components using the physical laws that govern the phenomena. The third step equates the scale factor so obtained to the net force scale factor k_F obtained from the Newton's law of inertia, and this scale factor is termed as the inertial force factor. Then obtaining the non dimensional pi numbers in terms of the characteristic independent quantities from the relationships established in step 3 above.

These give invariants that establish the modeling rules. Example: Let us obtain scale factor for viscous forces. Viscous shear forces would be shear stresses times the area. Shear stresses are obtained from the Newton's law of viscosity: tau is equal to mu times the velocity gradient dV by dy. So the scale factor for shear stresses would be scale factors for viscosity times the scale factor of velocity divided by scale factor for the length. Scale factor for area easily written as k a is equal to k L squared.

So the scale factor for the shear force which you denote by $k_{F,\mu}$, is nothing but $k_\mu k_V k_L$. This gives you a pi number $\frac{F_\mu}{\mu V L}$. The scale factor for viscous forces divided by the characteristic values of the viscosity the velocity and length should be same in the prototype and in the model. We can do this for each of the forces. We already obtained the inertial force factor F_i and from this we would obtain the pi number $\frac{F_i}{\rho L^2 V^2}$. The scale factor for unsteady forces is obtained from the law that unsteady force is mass times the unsteady acceleration $\frac{\partial V}{\partial t}$.

Obtaining modelling rules

Force	Law	Relations among scale factors	Π-number
Total (Inertia), F_i	$F_i \sim mass \times a$	$k_{F_i} = k_\rho k_L^2 k_V^2$	$\frac{F_i}{\rho L^2 V^2}$
Unsteady, F_u	$F_u \sim mass \times \frac{\partial V}{\partial t}$	$k_{F_u} = k_\rho k_L^3 k_V / k_t$	$\frac{F_u}{\rho L^3 V/t}$
Viscous, F_μ	$F_\mu \sim \mu \cdot \left(\frac{dV}{dx}\right) \cdot \text{area}$	$k_{F_\mu} = k_\mu k_L k_V$	$\frac{F_\mu}{\mu L V}$
Gravity, F_g	$F_g \sim mass \times g$	$k_{F_g} = k_\rho k_L^3 k_g$	$\frac{F_g}{\rho g L^3}$
Pressure, F_p	$F_p \sim (\Delta p) \times area$	$k_{F_p} = k_{(\Delta p)} k_L^2$	$\frac{F_p}{(\Delta p) L^2}$

For mass we use $k_\rho k_L^3$, $\frac{\partial V}{\partial t}$ is k_V / k_t, and so the resultant pi number is $\frac{F_u}{\rho L^3 V/t}$. For viscous forces we obtain the pi numbers as $\frac{F_\mu}{\mu L V}$ in the last slide. Similarly, we could write for gravity.

The gravity force is mass times g, and mass is density times the volume, so that $\frac{F_g}{\rho g L^3}$ is the pi number for the gravity force. Similarly, for the pressure force $\frac{F_p}{(\Delta p) L^2}$.

Similarly for surface tension force where the law is the surface tension force is σ times L. For compressibility where the law is E times the area, and for the centrifugal force where the law is mass times $\omega^2 r$. This is the list of pi numbers so obtained. We will learn to use them in the next lecture.

Thank you.