

Similitude And Approximations In Engineering,
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Week - 03
Lecture - 13

Welcome back.

In this lecture, we continue with the discussion of Huntley's extension that is, we would be using the extension of dimensional groups to reduce the number of non dimensional pi numbers.

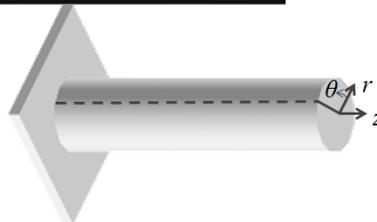
Torsion of a shaft

$$\varphi = \mathcal{F}(T, G, R, L)$$

Basis set: T, G, R

$$\varphi = \mathcal{F}\left(\frac{T}{GR^3}, \frac{L}{R}\right)$$

	M	L	T
φ			
T	1	2	-2
G	1	-1	-2
R		1	
L		1	



Φ should vary linearly with L, so that $\varphi = \frac{L}{R} \mathcal{F}\left(\frac{T}{GR^3}\right)$

Let us consider a simple example of a torsion of a shaft. A shaft is subjected to a torque T at this end, it is attached to a wall at this end. The twist φ of the shaft is a function of T, G, R and L . We use the dimensions M, L and T , the time. We use the basic set of T, G and R , and then we obtain a functional relationship that φ is a function of $\frac{T}{GR^3}$, and $\frac{L}{R}$. Then, as in the last lecture, we could use the information that twist should be directly proportional to the length of the shaft. More the length, more would be the twist, linearly proportional. So, the relationship between φ and L should be linear, and that is why the relationship should be: $\varphi = \frac{L}{R} F\left(\frac{T}{GR^3}\right)$. We know from our theory of mechanics of material that this function of T divided by G by R^3 is simply T by $G R^3$, and so that the twist is $L T$ divided by $G R^4$ times a constant. But can we improve the result from dimensional analysis alone?

And we do it breaking them into the component length dimensions. Let us take dimensions in x, y and z coordinates. Writing the dimension is not easy. So, please pay attention.

Torsion of a shaft

$$\varphi = \mathcal{F}(T, G, R, L)$$

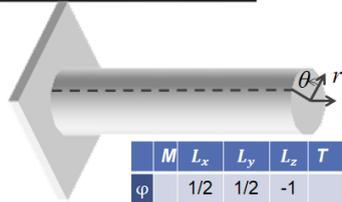
$$\text{Force} \sim ML_x T^{-2} \text{ or } ML_y T^{-2}$$

To preserve axial symmetry,

$$\text{we take this as } ML_x^{\frac{1}{2}} L_y^{\frac{1}{2}} T^{-2}$$

$$\text{Similarly, } [R] = L_x^{\frac{1}{2}} L_y^{\frac{1}{2}}, \text{ and then, } [T] = ML_x L_y T^{-2}$$

$$\text{And } [G] = \left[\frac{\text{Force/Area}}{\text{Angular vstrain}} \right] = \frac{(ML_x^{\frac{1}{2}} L_y^{\frac{1}{2}} T^{-2}) / L_x L_y}{L_x^{\frac{1}{2}} L_y^{\frac{1}{2}} / L_z} = ML_x^{-1} L_y^{-1} L_z T^{-2}$$



	M	L _x	L _y	L _z	T
φ		1/2	1/2	-1	
T	1	1	1		-2
G	1	-1	-1	1	-2
R		1/2	1/2		
L				1	

The force that is twisting the shaft, in fact is a torque that is twisting the shaft, the torque arises because of force that we apply. And the force would have a dimension $ML_x T^{-2}$ or $ML_y T^{-2}$. In fact, the whole problem is symmetric in x and y. z is a special dimension, but not x and y. x and y interchangeable. So, to preserve axial symmetry, we can take the dimension of force as $ML_x^{\frac{1}{2}} L_y^{\frac{1}{2}} T^{-2}$. So, instead of L_x or L_y, we write $L_x^{\frac{1}{2}} L_y^{\frac{1}{2}}$. This preserves the axial symmetry, symmetry between L_x and L_y. z is along the axis of the shaft. Similarly, the radius of the shaft which is anywhere in x and y plane, and to preserve the axial symmetry, we write the dimension of R is $L_x^{\frac{1}{2}} L_y^{\frac{1}{2}}$, and then the dimension of the torque would be force times the radius, and we can write it as $[T] = ML_x L_y T^{-2}$.

And then the dimension of G, the shear modulus, shear force per unit area divided by the angular strain. Force, as we see is $ML_x^{\frac{1}{2}} L_y^{\frac{1}{2}} T^{-2}$, the area is $L_x L_y$ and the angular strain, we write as R divided by length in the z direction. So, the angular strain is written as $L_x^{\frac{1}{2}} L_y^{\frac{1}{2}} / L_z$. If you are not using direction, this will be simply be dimensionless, but it is actually in the R direction divided by in the z direction. So, it is written as $L_x^{\frac{1}{2}} L_y^{\frac{1}{2}} / L_z$.

So, this gives the dimension of G, the shear modulus as $ML_x^{-1} L_y^{-1} L_z T^{-2}$. Notice that as in everything else earlier, L_x and L_y carry the same dimensions denoting that the problem has symmetry between x and y directions. So, this is the table of dimension that we construct in M L_x L_y L_z and T system. The dimension that we have calculated on the left has been transferred to this table. We need we use these four quantities as the basic set and so, we get only one pi number.

	T	G	R	L
φ	1	-1	-4	1

φ has a dimension of torque raised to power 1, G raised to power minus 1, R raised to power minus 4, L raised to power 1 and so, we get $\frac{\varphi GR^4}{TL} = \text{Constant}$. A very powerful result single

constant can be obtained by twisting a single shaft and measuring its twist. We can obtain the value of C and this will give you a relationship between twist and torque for any circular shaft of radius R length L and shear modulus G.

Terminal velocity of a falling sphere in viscous fluid

$$V = f(\rho_s, \text{dia } d, \text{liquid } \rho_f, \text{viscosity } \mu, g)$$

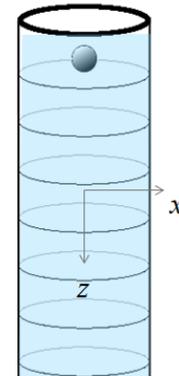
$$[V] = Z T^{-1}$$

$$[\rho_s] = M X^{-1} Y^{-1} Z^{-1}$$

$$[\rho_f] = M X^{-1} Y^{-1} Z^{-1}$$

$$[d] = X || Y \rightarrow X^{-1/2} Y^{-1/2}$$

$$[g] = Z T^{-2}$$



Let us consider next an example from fluid mechanics. Let us consider a terminal velocity of a falling sphere in a viscous fluid.

As a sphere falls in a column of viscous liquid, it accelerates initially under the combined influence of the weight and the buoyancy force. As it accelerates the drag develops in the opposite direction which increases with velocity and when the velocity reaches such a value that the drag balances the net weight of the sphere, that is, the weight of the sphere minus the buoyancy, then the sphere is said to have acquired a terminal velocity, a constant velocity which does not change. The terminal velocity has a great value in problems involving, say, a parachute. So, this terminal velocity would be a function of the density of the solid sphere, the diameter D of the sphere, the density of the liquid ρ_f in which it is falling, the viscosity μ of the liquid and of course, the acceleration due to gravity. In terms of x and z and y coordinates, the velocity v has dimensions of $Z T^{-1}$. The density $[\rho_s]$ of the solid has a dimensions of $M X^{-1} Y^{-1} Z^{-1}$. So, these are the dimensions of the solid, same are the dimensions of the density of the fluid. The diameter of this sphere has x y symmetry. So, we write this as $[d] = X || Y \rightarrow X^{-1/2} Y^{-1/2}$. g is in the z direction. So, it is $[g] = Z T^{-2}$.

We have to worry about dimension of viscosity. How do we obtain the dimension of viscosity? We obtain with the relationship of the viscous forces from Newton's law of viscosity. The viscous force F is μ times area times the velocity gradient, dV/dz . And so, the dimension of μ would be dimension of force in the z direction $M Z T^{-2}$, divided by the area which would be $Z Y$, the side area of these, and the velocity gradient in that direction of the velocity field. So, that would be $\frac{Z T^{-1}}{X}$. So, these are the dimension that are obtained as $Z^{-1} T^{-1}$. These are the two formulations and from this we get this result. All of these we can form 2 pi numbers. One pi number would be $\frac{\rho_f}{\rho_s}$ which is obvious is the dimension of F and S are same and the other pi number is $\frac{V \mu}{\rho_s g d^2}$. So, that this pi number which contains the dependent parameter, terminal velocity V, should be a function of this parameter which is written here.

So, V is $C \frac{\rho_s g d^2}{\mu} f\left(\frac{\rho_f}{\rho_s}\right)$. As we had argued in the previous lecture that in this problem of terminal velocity, the only thing that matters is the wet weight of the sphere. So, it is not $\frac{\rho_f}{\rho_s}$ independently in this, but should be the difference $(\rho_s - \rho_f)$. So, this parameter should have a form of $\left(1 - \frac{\rho_f}{\rho_s}\right)$. So, that the terminal velocity V can be written as $C \frac{\rho_s g d^2}{\mu} \left(1 - \frac{\rho_f}{\rho_s}\right)$ only one pi number involved. And we can do the experiments and then can easily find that this constant is 8 by 9.

So, this is the relationship for the terminal velocity of a falling sphere in a viscous liquid. Notice again that it is the use of dimensions that decreased the number of pi numbers to the ideal number 1. So, ended up only with one pi number. Of course, we have used the fact that the density of the fluid should enter only in the combination $\left(1 - \frac{\rho_f}{\rho_s}\right)$.

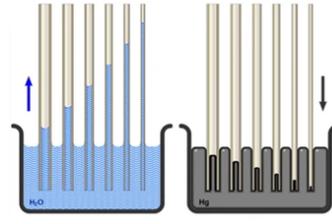
Capillary rise

$$h = \mathcal{F}(\sigma, D, \rho, g, \theta)$$

$$h = \mathcal{F}(\sigma \cos \theta, D, \rho, g)$$

Basis set: $\sigma \cos \theta, D, \rho, g$

	L_x	L_y	L_z	M	T
h			-1		
$\sigma \cos \theta$	-1/2	-1/2	1	1	-2
ρ	-1	-1	-1	1	
D	1/2	1/2			
g			1		-2



Another example capillary rise of a fluid, we know that the surface tension causes the fluid to rise in a capillary for fluids that wet a surface for fluids which do not wet a surface like mercury in glass. There is a depression, but that is the same phenomena. The h the capillary rise or capillary depression is a function of σ the surface tension, the diameter of the capillary, ρ the density of the liquid, g acceleration due to gravity and θ the wetting angle, the angle that the surface makes with the horizontal. θ is 0 for a liquid like water or glass and is 180 degrees for mercury and glass. Actually if you work in a direction coordinate, then this force for example, if this is the meniscus the force is at this angle. The component of surface tension force which is responsible for the capillary rise is the vertical component.

This is the angle of contact. The horizontal component has nothing to do with the capillary rise. So, it is not σ and θ independently, but $\sigma \cos \theta$ that should enter our formulation. So, capillary rise h should be a function of $\sigma \cos \theta, D, \rho, g$. We work in L_x, L_y, L_z and M T system and we write the various dimensions.

These are easy to write. We use the basic set that consists of these parameters, and then there is only one other variable left. The dependent variable h , the distance through which the fluid rises, and in terms of the basic set is

	$\sigma \cos \theta$	ρ	D	g
H	1	-1	-1	-1

So, this $\frac{h\rho Dg}{\sigma \cos \theta}$ is a constant or $h = C \frac{\sigma \cos \theta}{\rho Dg}$. This is a formula that you would obtain in your fluid mechanics course. For water $\cos \theta$ is 1, because theta is 0. The wetting angle, the contact angle is 0 for water.

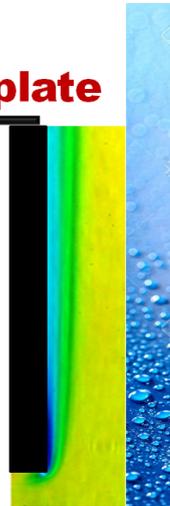
Free convection on a vertical plate

$$h = \mathcal{F}(\rho, \mu, k, C_p, z, T_w, T_f, \beta, g)$$

$$h = \mathcal{F}(\rho, \mu, k, C_p, z, \Delta T, g\beta)$$

$$\frac{hz}{k} = \mathcal{F}\left(\frac{g\beta z^3 \rho^2 \Delta T}{\mu^2}, \frac{\mu C_p}{k}\right) \text{ or, } Nu = F(Pr, Gr).$$

It is possible to improve this result by resorting to directional analysis, where the length dimension L is broken up into three mutually perpendicular dimensions, L_x , L_y and L_z .



Let us do this last example of free convection on a vertical plate. We have done this example earlier and obtained some results. The heat transfer coefficient h is a function of the fluid property ρ , μ , k , C_p , the length of the plate z in the vertical direction, the temperatures T_w and T_f the wall temperature and the fluid temperature far away from the plate, β , the coefficient of expansion of the fluid with temperature, and g the acceleration due to gravity. We had argued earlier that in a heat transfer problem it is not the temperatures that matter, but the temperature difference that matters.

So, T_w and T_f were replaced by ΔT , the temperature difference, $T_w - T_f$. Then we also argued that the expansion of the fluid contributes to free convection through the action of gravity. And so, it is not β and g independently that should enter the formulation, but together. So, this should be $g\beta$. We had reduced the number of parameters by 2 previously.

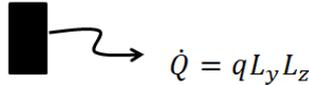
This is how we did it. And when we solved this problem last we obtain $\frac{hz}{k}$, which is Nusselt number, is a function of Grashof number and Prandtl number. This is the functional relationship that we obtained two classes earlier. Can we improve the result by directional analysis? It is possible to improve this result when the length dimension L is broken up into 3 mutually perpendicular dimensions L_x , L_y and L_z . So, the problem with the reduced number of parameters that is on using ΔT and $g\beta$ together. Now, the dimension of h in terms of directions are obtained from the relationship q is equal to $h \Delta T$.

Free convection on a vertical plate

$$h = \mathcal{F}(\rho, \mu, k, C_p, z, \Delta T, g\beta)$$

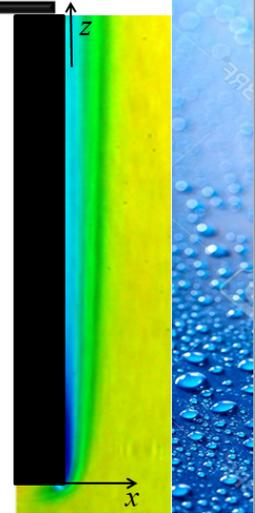
Dimensions of h are obtained from

$$q = h\Delta T \text{ with } \dot{Q} = qL_yL_z,$$



$$\dot{Q} = qL_yL_z$$

$$\text{so that } [h] = \left[\frac{\dot{Q}}{L_yL_z\Theta} \right] = QL_y^{-1}L_z^{-1}T^{-1}\Theta^{-1}$$



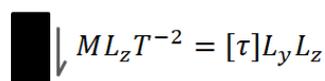
q is the heat flux, the heat transfer per unit area, h is the heat transfer coefficient, \dot{Q} is heat flux times the area, and the area would be like L_yL_z this on the plate. It will have the dimension L_yL_z not x . So that, the dimension h would be given by $QL_y^{-1}L_z^{-1}T^{-1}\Theta^{-1}$.

Free convection on a vertical plate

$$h = \mathcal{F}(\rho, \mu, k, C_p, z, \Delta T, g\beta)$$

Dimensions of μ are obtained from

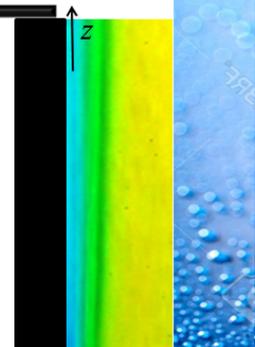
$$\tau = -\mu dV/dx$$



$$ML_zT^{-2} = [\tau]L_yL_z$$

$$[dV/dx] = L_zT^{-1}L_x^{-1}$$

$$\text{And therefore, } [\mu] = ML_xL_y^{-1}L_z^{-1}T^{-1}$$



Dimension of μ are obtained from the Newton's law of viscosity . $\tau = -\mu dV/dx$

This is the wall. The viscosity: viscous force would be in vertical direction. So, dimensions in direction in terms of directions would be ML_zT^{-2} and which would be τ , the stress times the area, which is L_yL_z . So, dimension of τ the stress would be $ML_xL_y^{-1}L_z^{-1}T^{-1}$.

Free convection on a vertical plate

$$h = \mathcal{F}(\rho, \mu, k, C_p, z, \Delta T, g\beta)$$

Dimensions of k are obtained from

$$q = -kdT/dx$$

Dimensions of q has already been obtained as $QL_y^{-1}L_z^{-1}T^{-1}$ so that the dimensions of k are obtained as

$$Q\Theta^{-1}L_xL_y^{-1}L_z^{-1}T^{-1}$$



The dimensions of k , the conductivity, are obtained from this equation. The heat flux is $q = -kdT/dx$. Dimension of q have already been obtained as $QL_y^{-1}L_z^{-1}T^{-1}$, and so, dimension of k are obtained as $Q\Theta^{-1}L_xL_y^{-1}L_z^{-1}T^{-1}$. You can verify. So, with a bit of patience, we can construct a matrix of dimension of all our parameters here, and the dimensions here. These are the thermal quantities Q and θ , thermal energy and temperature. The length is broken up into these three, and mass and time.

	Q	Θ	L_x	L_y	L_z	M	T
h	1	-1		-1	-1		-1
μ			1	-1	-1	1	-1
ρ			-1	-1	-1	1	
k	1	-1	1	-1	-1		-1
C_p	1	-1				-1	
z					1		
ΔT		1					
$g\beta$		-1			1		-2

A total of 7 dimensions. We write the dimensions here. All of these have been obtained in the last few slides. We use the 6 quantities as the basic set, and then we have only two other quantities left: one dependent and one independent. And we can formulate the dimensions in terms of the basic set as in this table. And then can write the functional relationship. The dependent pi number is a function of independent pi number.

	ρ	k	C_p	z	ΔT	$g\beta$
h	1/2	1/2	1/2	-1/4	1/4	1/4
μ		1	-1			

This is simplified to give $\frac{hz}{k}$ which is Nusselt number, and this is equal to Grashof number raise to power one-fourth into a function of Prandtl number. So, that this is the relation: $Nu = Gr^{1/4} F(Pr)$. The last time we did it, we obtained Nusselt number as a function of Grashof number and Prandtl number. How it depends upon Grashof number was not obtained

by dimensional analysis. But using the directional analysis, as presented here, we have been able to establish that Grashof number should enter the relationship as one-fourth power.

So, the Nusselt number is like Grashof number raised to power 1/4 times a function of Prandtl number. This will also require a number of experiments to establish the functionality of Prandtl number, but the number of experiments required now would be far fewer. This illustrates the power of directional analysis.

Thank you.