Similitude And Approximations In Engineering,
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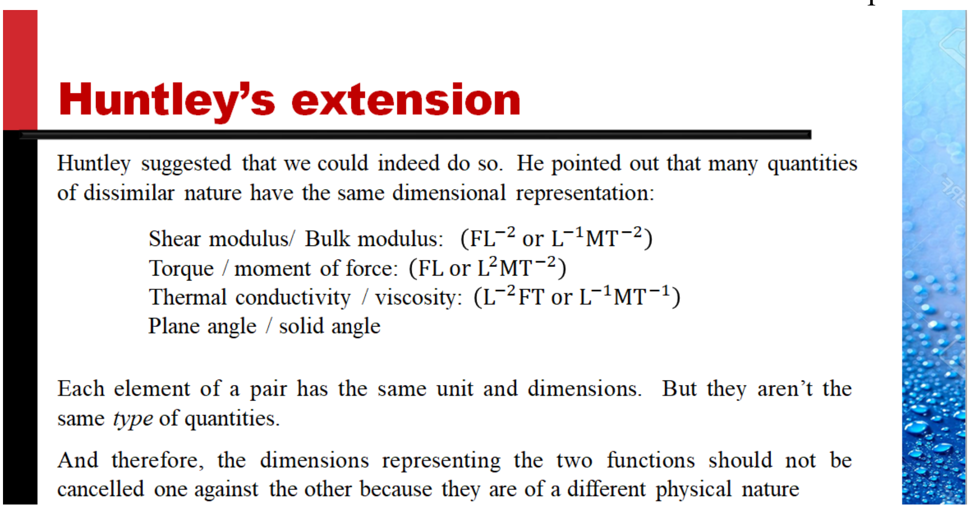
Week - 03
Lecture - 12

Welcome back. In today's lecture, we will discuss a development in dimensional analysis known as Huntley's extension, which is used to extend the dimensional group to reduce the number of non-dimensional pi's. One of the many irritants with the dimensional analysis is that there are too many independent pi numbers. Ideally, there should be one pi number that pi number should be invariant and this would be the best result that is available. If there are two pi numbers, one dependent and one independent, then a dependent pi number can be written as a function of the independent pi number, but we do not know what the functional dependence is. So you need a lot of experiments to determine that functional dependence.

We have seen that for free convection on a plate, the Nusselt number, the non-dimensional dependent parameter Nusselt number is given as a function of Prandtl number and Grashof number. There are three pi numbers involved, one dependent and two independent. Because of this, we have no idea how the functional dependence on Prandtl number and Grashof number would be.

Recall Buckingham pi theorem. The number of independent pi's in the problem is n minus k, where n is the number of independent quantities while k is the number of quantities in the base set, the complete and independent. We had seen in the last lecture how the number n could be reduced by using physical information. In the case of problems in heat transfer, we said we could reduce the number of n by 1 by taking the temperature difference rather than the two individual temperatures as independent parameters. We also combined the density differences and g in one pi number in some problems. This can go only so far.

Is there a way to increase the number k, that is, the number of quantities required in the base set? The number of quantities required in the base set is about the number of dimensions that are used. So can we increase the number of dimensions? So one way to reduce the number of independent pi's is to increase the number of dimensions required to express the independent parameters and variables. Huntley suggested that we could indeed do so. He pointed out that many quantities of dissimilar nature have same dimensional representation. Like the shear modulus and bulk modulus both have the representation of FL^{-2},



Force and two dimensions of length in negative, or in MLT system is $L^{-1}MT^{-2}$. The two quantities are of different nature altogether, but they have the same dimensions. Torque and moment of a force have the same dimensions FL. Thermal conductivity and viscosity have same dimensions too. Plane angles and solid angles also have the same dimensions, no dimensions at all.

Each element of a pair has the same unit and dimensions, but they are not the same type of quantities. And therefore dimensions representing the two functions should not be cancelled one against the other because they are of a different physical nature. Similarly the energy storage function and the inertia function of mass are two different things altogether. Huntley argued that we could differentiate between these two functions and create two separate dimensions. In this lecture and the next, we will do examples where we use this Huntley's extension and increase the number of dimensions required to express the quantities.

He showed that the dimensions representing the two functions may be treated as different and may not be cancelled one against the other. We can represent two dimensions M_m and M_i respectively, for mass. The first one represents the quantity of matter and the second the inertial property of that matter. Consider the flow of a fluid through a pipe. A steady fully-developed flow through a pipe.

The mass flow rate $\dot{m}$ would be a function of p', that is, the pressure gradient along the pipe, the radius of the pipe, the density and μ the viscosity of the fluid. If we express the dimension of these five quantities, one dependent and four independent, in the MLT system, this is the matrix that we get.

	M	*L*	*T*
$\dot{m}$	1		-1
μ	1	-1	-1
p'	1	-2	-2
ρ	1	-3	
R		1	

We have used mass as the same, both for the inertia and for the flow rate. The basic set is p', the pressure gradient, ρ, the density of the fluid, and r, the radius of the pipe. The three we have shown with the yellow tint in the table.

The remaining two parameters, $\dot{m}$, the dependent parameter and μ, the independent parameter. The dimension of this are expressed in terms of p', ρ and r. And, if we do the algebra that was outlined in the last two lectures, we would get $\dot{m}$, the dimension $\dot{m}$ the same as $p'^{1/2}\rho^{1/2}r^{5/2}$. Similarly, for mu: $p'^{1/2}\rho^{1/2}r^{3/2}$. With this table we can write the two non dimensional groups of parameters as $\frac{\dot{m}}{\sqrt{p'\rho R^5}}$ and $\left(\frac{\mu}{\sqrt{p'\rho R^3}}\right)$.

Not a very useful result. We will have to do a lot of experiments to establish this functional relationship. However, Huntley suggested we can improve this by differentiating between the dual roles of mass as quantity of matter and as inertia in this case. If we do this we now have four dimensional set: M_i representing mass as inertia, M_m mass as the quantity of matter and L and T. Then the five parameters $\dot{m}$, p', R, ρ, μ.

Of these, the dimension of $\dot{m}$,, the mass flow rate is easily written as $M_m^{\ 1}T^{-1}$. So, the dimension of the $M_m^{\ 1}T^{-1}$ mass flow rate the quantity of matter flowing. ρ is very clear the role of the mass here is quantity of matter. So, it is written as $M_m^{\ 1}L^{-3}$. Similarly the radius of pipe is easily written as L, but what about μ and p'? We recognize that the dimension of μ should be the dimension of stress divided by dimension of velocity gradient from Newton's law of viscosity.

This stress will be force divided by area, and the force is mass times is the acceleration, but here the role of mass is inertial because inertial mass time acceleration would give you the force. So, the dimension of stress are expressed as $\frac{\left(M_i LT^{-2}\right)}{L^2}$. So, this quantity represents the dimension of stress here the mass has a role of inertia. The velocity gradient is simply LT^{-1}/L: velocity divided by length, and so the dimension of μ would be $M_i L^{-1}T^{-1}$, and this we put in the table. Next, we look at the dimension of pressure gradient, and this would be pressure divided by length, and the pressure is force divided by area.

So, the dimension of p' would be dimension of force divided by L cubed, and then force again is with mass in the inertial role. So, the dimensions of the pressure gradient are $\frac{\left(M_i LT^{-2}\right)}{L^3}= M_i L^{-2}T^{-2}$. This completes the table of dimensions.

	M_i	M_m	L	T
$\dot{m}$		1		-1
ρ		1	-3	
μ				
p'				
R			1	

Now, there are 4 dimensions. So, the basic set would consist of 4 parameters with the yellow tint and so we are left with only one other parameter, the dependent parameter. And we can easily construct the non dimensional parameter $\dot{m}$ as $\frac{\dot{m}\mu}{p'\rho R^4}$, and this should be invariant between the and this should be invariant between the model and the prototype.

So, we can write $\dot{m} = C\frac{p'\rho R^4}{\mu}$. A single experiment would be enough to establish the value of C. Since, there is only one pi number involved in this formulation of the problem, this is a very powerful result. So, it should be apparent now that by increasing the dimensional set required to have one more member, and that we did by splitting mass in its dual role as inertial and as quantity of matter. So, the value of r increased to 4, n was 5. So, there is only one non dimensional group of parameter, one pi, that was needed.

Another thing that Huntley introduced was to consider the role of vectors. Various quantities like vector velocity, acceleration, torque, momentum are vector quantities and so is the area. Consider this object the areas in this object are of two kind. The drag on this object when a fluid flows past this depends on the lateral surface area. The end areas the area of these ends contribute to mass. So, these two areas are of different kinds all together. Again consider two quantities torque and work. A force F at a distance r from a point. The dimension of torque are L, the dimension of r, and MLT^{-2}, the dimension of force, to give you ML^2T^{-2}.

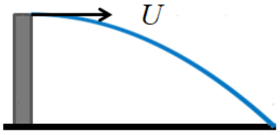
And the dimension of work when object moves through distance x is F x and the dimension are the same ML^2T^{-2}. But the distance involved, in one case r and one case x, are in different directions all together. Torque divided by work or work divided by torque should not be a dimensionless parameter. The dimensions play different roles in there. Huntley argued that the length dimension should be treated as vector dimensions with three independent components Lx, Ly and Lz.

In Huntley's view the measurement of length in the x direction is by specification a different operational procedure from measurement of the length in the y direction. Because the operation are carried out in different planes. The mathematical counterpart of this physical explanation is simply that in a vector equation it is never permissible to cancel an x component against a y or a z component of any vector quantities. Thus we supplement three dimensions m, L and t and instead of L we use Lx, Ly and Lz, three separate dimension instead of one single L. So, the dimensions of the lateral area may be Lx Ly or Lx Lz or Ly Lz. The dimension of volume would then be Lx Ly Lz, product of these three. Area drag on the lateral surface marked A would be μ times the velocity gradient time the area. The velocity gradient if the flow is in vertical direction, the velocity Lz divided by t, and we are talking of the gradient normal to this. So, the velocity gradient is obtained by dividing the velocity by Ly, and the area is of this surface which is Lx Lz. So, the dimensions of this would be $[\mu]\frac{L_zT^{-1}}{L_y} \times L_xL_z$.

On the other hand the pressure on the end face here would be the force which would now be ML_zT^{-2}. Since, you are talking of the z direction, and the area on which it acts has a dimension L_xL_y. So, dimension of this pressure would be $\frac{ML_zT^{-2}}{L_xL_y}$. Torque, which is r F would be function $ML_x^2T^{-2}$, and work would be $ML_x^2T^{-2}$. In the simple MLT system, both would have dimension of ML^2T^{-2}.

But we realize that in torque the two lengths are in different directions Lx and Ly while in the work they are in the same direction Lx squared. Using these directions and writing dimension in terms of this directional dimensions is not easy. We will do a few exercises to get some practice. We start with a very simple example.

Range of projectile



$R = f(U, h, g))$

Basic group: *U* and *h*

	L	T
R	1	
U	1	-1
h	1	
g	1	-2

	U	h
R		1
g	2	-1

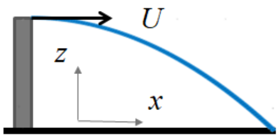
Two pi's: $\frac{R}{h}$ and $\frac{U^2}{gh}$

$$\frac{R}{h} = f\left(\frac{U^2}{gh}\right)$$

A projectile is projected horizontally with the velocity v velocity u from a point a distance h above the ground. We have to find the range, and the range clearly would be a function of the initial velocity U, the height h from which it is projected horizontally and acceleration due to gravity. We do a simple minded dimensional analysis like we have done in the previous classes. This is the dimensional table. There are only two dimensional involved L and T. R and h would have dimension of length only, velocity would have dimension of LT^{-1}, and g of course, has a dimension of LT^{-2}, meters per second squared.

We use the basic group U and h. So, we get R and g as the other parameters: one dependent and the other independent. And using this we get the dimension of R as in terms of U and h as h for R, and U squared divided by h for g. So, that our functional relationship would have 2 pi's: R by h and U^2/gh. So, that R/h, the dependent one can be written as function of U^2/gh. We really have no idea of how this functional relationship should pan out.

Range of projectile



$R = f(U, h, g))$

X Z T system

Basic group: *U, h* and *g*

	X	Z	T
R	1		
U	1		-1
h		1	
g		1	-2

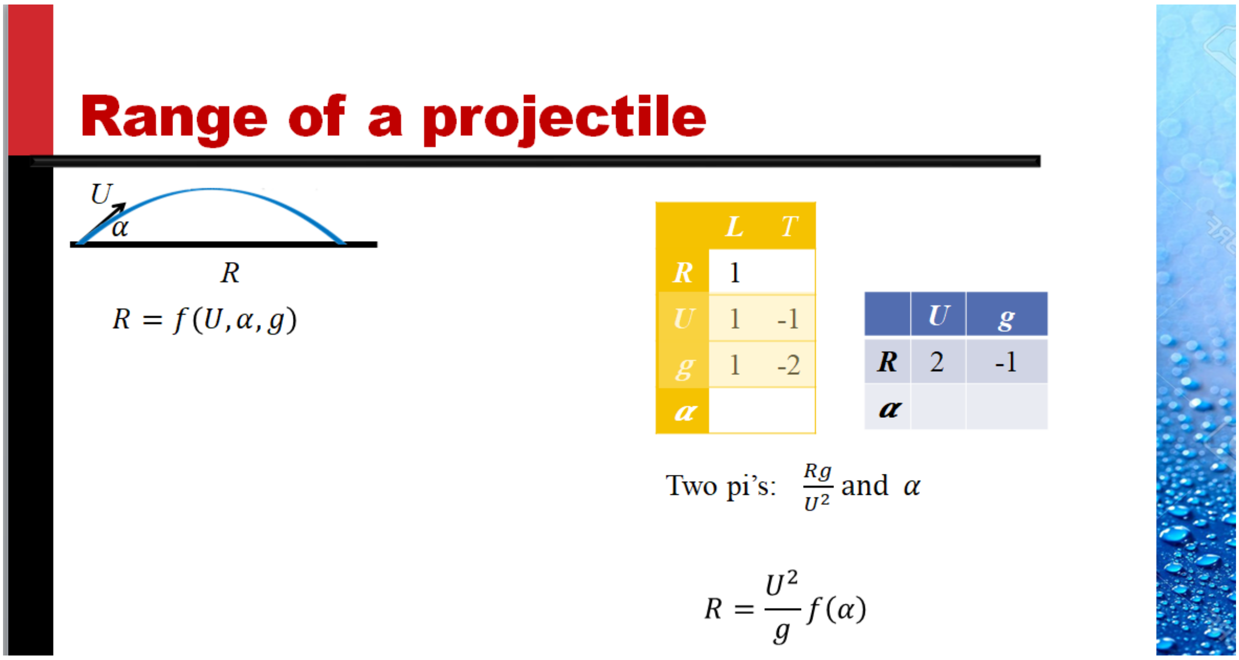
	U	h	g
R	1	1/2	-1/2

Only one pi: $\frac{R}{U}\sqrt{\frac{g}{h}}$

$$\frac{R}{U}\sqrt{\frac{g}{h}} = \text{constant, say } C, \quad \text{or } R = CU\sqrt{\frac{h}{g}}$$

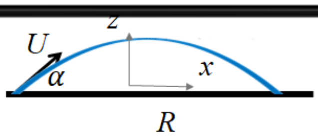
But this is a prime case for using directional analysis for using two different dimensions for two different directions. Here we use X Z and T system, X for Lx and Z for Lz. So, now instead of L and we use X, Z and T system. The range R would be in x direction, the velocity is in x direction. So, it is XT^{-1}, and h is in z direction, and g is in vertical direction. So, it should have a dimension of ZT^{-2}. In this we use the three quantities U h and g as the basic set. And so, there is only one parameter that is left, the dependent parameter R, and the dimension of dependent parameter in terms of U h and g is obtained as $UH^{1/2}g^{-1/2}$. So, that there is only one parameter that is formed, and that parameter can be written as $\frac{R}{U}\sqrt{\frac{g}{h}}$. And this should be invariant should have the same value in every experiment whatever be the value of U, h and g. So, one experiment would establish the value of C in the relation $R = CU\sqrt{\frac{h}{g}}$. This clearly established that using the directions as dimensions could simplify this projectile problem.

We do more of projectile problems. Just get a little more complicated. Now, we have a horizontal plane and we project a projectile with a velocity U at an angle alpha to the horizontal and we want to determine the range.



The range clearly would be a function of U. angle α and g, the acceleration due to gravity. We proceed in a simple minded manner of using only the two dimensions length and time. The dimension of R, U and g are simple. Alpha is dimensionless. So, alpha itself is a parameter. We can use U and g as the basic set. So, there is only one other parameter left R, and we form the dimension of R as U^2g^{-1}, and α. So that the 2 pi numbers that are formed are $\frac{Rg}{U^2}$ *and* α, and we can write $R = \frac{U^2}{g}f(\alpha)$. We have no idea what would be this function of alpha. So, we resort to directional analysis, and while we do the directional analysis, a simple trick of converting U and alpha into Ux and Uz, the velocities in the x direction and in the z direction which will be Ucosα and Usinα, respectively.

Range of a projectile



$R = f(U_x, U_z, g)$

	X	Z	T
R	1		
U_x	1		-1
U_z		1	-1
g		1	-2

	U_x	U_z	g
R	1	1	-1

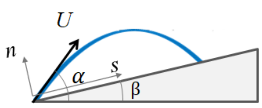
One pi: $\frac{Rg}{U_x U_z}$

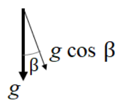
$$R = C\frac{U_x U_y}{g} = C\frac{U^2 \sin 2\alpha}{g}$$

And we use X and Z, this direction, and the relevant directions are these. The dimensions of Ux are clearly XT^{-1}, and dimension of Uz are ZT^{-1}. So, g is in the vertical direction. So, it is ZT^{-2}.

We need 3 elements in the basic group. We use all Ux, Uz and g. Only one pi number would be formed and that pi number has a dimension $\frac{Rg}{U_x U_z}$. Then that should be constant. So, that $R = C\frac{U_x U_y}{g} = C\frac{U^2 \sin\sin 2\alpha}{g}$. Only one experiment would be required to establish the value of C.

Range of a projectile





$\tau = f(U_s, U_n, g_n)$

	s	n	T
τ			1
U_s	1		-1
U_n		1	-1
g_n		1	-2

	U_s	U_n	g_n
τ	0	1	-1

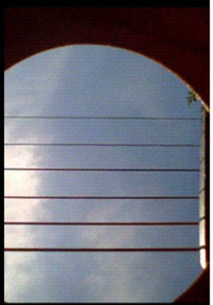
One pi: $\frac{\tau g}{U_n}$

$$\tau = C\frac{U_n}{g_n} = C\frac{U \sin(\alpha - \beta)}{g \cos\beta}$$

We can go a little fancier. Now the same projectile is projected with the velocity capital U at an angle alpha, but now it is not thrown on a horizontal plane, but an inclined plane at an angle beta. And this time our problem is to find the time of flight, the time τ of flight.

For this we use two directions s and n, s along the plane and n normal to the plane. g is resolved a component $g \cos \beta$, normal to the plane, and $g \sin \beta$, along the plane in the negative direction. The time of flight is a function of Us, the initial velocity along the plane s, the initial velocity along the plane s, and initial velocity normal to the plane Un, and it depends upon the acceleration due to gravity in the normal direction. We construct the table with dimension s, n and T. Us has a dimension sT^{-1}, Un has a dimension of nT^{-1}, and gn has a dimension of nT^{-2}. We need three parameters in the basic group, and so we have only one left, and we can construct the dimension of τ in terms of Us, Un and gn using the method outlined in the last couple of lectures: $U_s^{\,0} U_n g_n^{\,-1}$. And from this we get only 1 pi. $\frac{\tau g}{U_n}$. The time of flight $\tau = C\frac{U_n}{g_n} = C\frac{U\sin\sin(\alpha-\beta)}{g\cos\cos\beta}$. Again, a very powerful result in the sense a single experiment would establish the complete functional relationship.

Energy of a vibrating stretched wire



$E = f(n, l, linear\ density\ \rho', amplitude\ A)$

	M	X	Z	T
E	1		2	-2
n				-1
ρ'	1	-1		
A			1	
l		1		

	n	ρ'	A	l
E	2	1	2	1

One pi: $\frac{E}{n^2\rho' l A^2}$ $\quad E = Cn^2\rho' l A^2$

If we had not used vector lengths: $E = Cn^2\rho' l^3 f\left(\frac{A}{l}\right)$

Because A/l is very small, $f\left(\frac{A}{l}\right) = \left(\frac{A}{l}\right)^{\alpha}$, with α undetermined

Let us do one more example energy of a vibrating stretch wire. Stretched wires like the strings of a guitar vibrate when plucked. The energy of these vibrating wires will depend upon the frequency n at which they are vibrating, the length L of the string, the linear density ρ', that is, the mass per unit length, and the amplitude A of the transverse vibrations. We work in M X Z T system. Instead of the length dimension L we use two dimensions one X along the string, and the other Z perpendicular to the string. Then E would have dimension of mass times the velocity squared, the velocity in the vertical direction.

So, it would have a dimension of Z^2T^{-2}. So, that the dimensions of E are MZ^2T^{-2}. Similarly, we can write the dimensions of the other quantities. n, the frequency, ρ', mass per unit length in the X direction, and the amplitude is in the Z direction Length also is in the X direction. There are four dimensions. So, there are four parameters in the basic set. We choose n, ρ', A and L, the only choice we could have made, because the dependent parameter E is never chosen in the basic set. With this we can write the dimension of E in terms of n, ρ', A and L ,and they turn out to be $n^2\rho' lA^2$. So, that there is only one pi that is formed $\frac{E}{n^2\rho' lA^2}$, or we can write that the energy of the wire $EE = Cn^2\rho' lA^2$. There is only one constant C that needs to be determined and one single experiment could do that. If we are not use the vector length there would have been two pi numbers one dependent and one independent. The dependent prime number would have been $Cn^2\rho' l^3$, and the independent prime number would be $\frac{A}{l}$.

Because $\frac{A}{l}$ is small. So, function of $\frac{A}{l}$ would be of the form $f\left(\frac{A}{l}\right) = \left(\frac{A}{l}\right)^{\alpha}$, with alpha undetermined.

Thank you.