Similitude And Approximations In Engineering,
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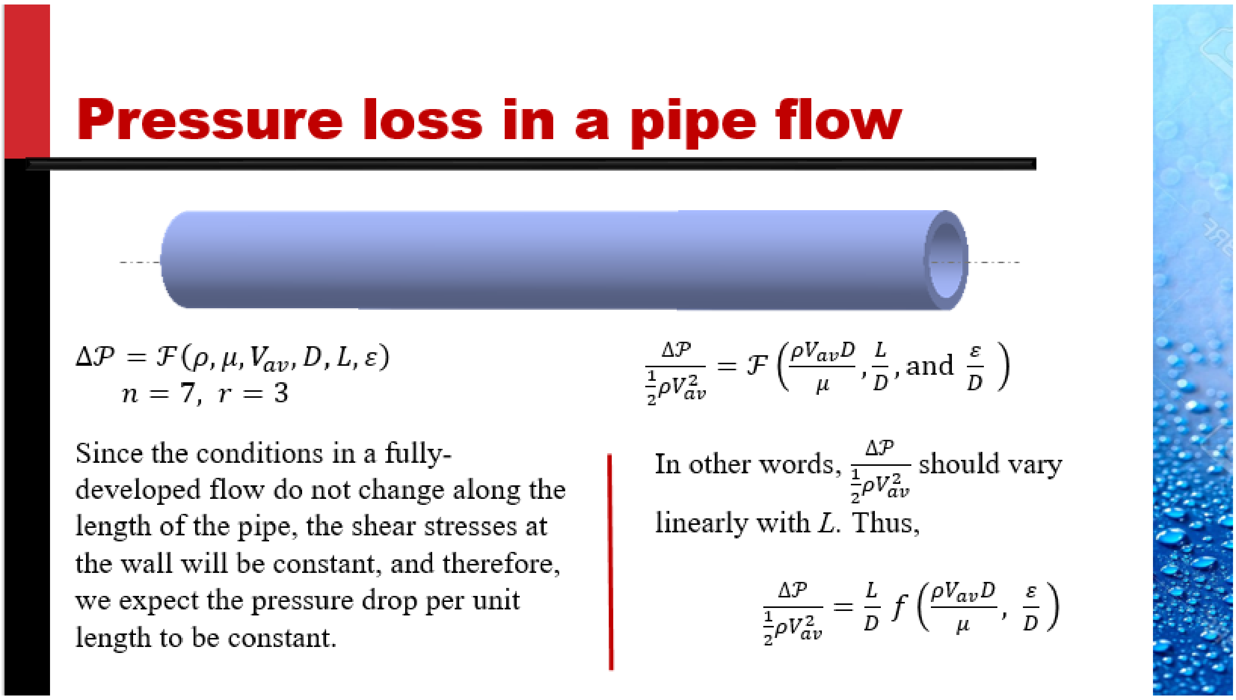
Week - 03
Lecture – 11

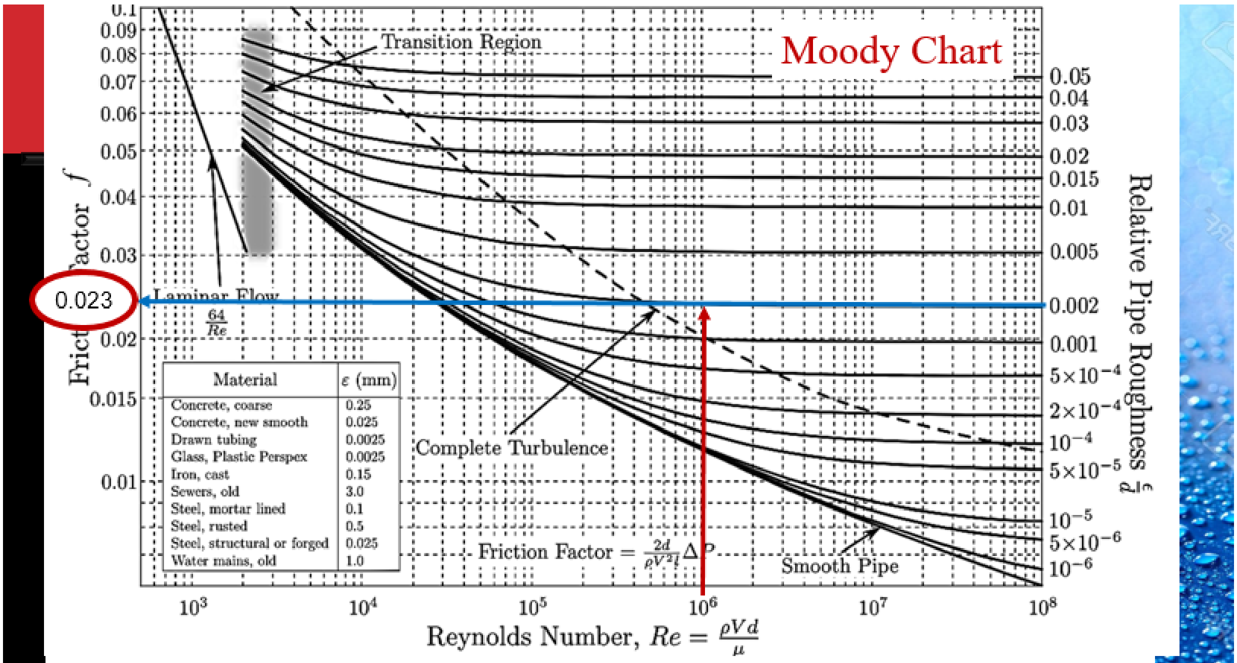
Welcome back. In today's lecture, we will discuss how to use additional physical information for reducing the number of independent pi numbers.



Let us consider the example of pressure loss in a flow of fluid within a pipe. ΔP the pressure difference at the two ends depends upon the density of the fluid, the viscosity of the fluid, the average velocity of the fluid through the pipe, the diameter D of the pipe, the length L of the pipe ,and epsilon, a roughness parameter, typically the average height of roughness on the surface of the pipe, inside surface of the pipe. We see here the total number of parameters involved is 7 and if we work in MLT system, the number of independent dimensions is 3 and with this if we carry out a dimensional analysis like we have carried out in last few classes, the non-dimensional pressure difference parameter $\frac{\Delta P}{\frac{1}{2}\rho V_{av}^2}$ is a function of $\frac{\rho V_{av} D}{\mu}$, which you recognize as Reynolds number, L/D, length over diameter of the pipe, and ε/D, which is called the relative roughness. This is not a very expressive expression because of the three independent pi groups involved on the right hand side.

We like to reduce them. Can we use some additional information to reduce the number of parameters on the right hand side? Yes, we can. Since the condition in a fully-developed flow do not change along the length of the pipe, the shear stresses at the wall would be constant along the pipe and therefore, we expect the pressure drop per unit length to be constant. This is an additional physical insight that we bring in to solve the problem.

If the length of the pipe is very large, large compared to diameter, then we can assume that the things do not change in the z direction that is along the axis of the pipe, and if that does not happen, then the pressure drop would be constant per unit length. In other words, $\frac{\Delta P}{\frac{1}{2}\rho V_{av}^2}$ should vary linearly with L. If it varies linearly with L, then the dependence of the function should be like $\frac{\Delta P}{\frac{1}{2}\rho V_{av}^2}$ $\frac{\Delta P}{\frac{1}{2}\rho V_{av}^2}$ should be $\frac{L}{D}\, f\left(\frac{\rho V_{av} D}{\mu},\ \frac{\varepsilon}{D}\right)$. This is the famous formulation that you must studied in your fluid mechanics course. Now, you notice that the function f is a function of two parameter groups, the Reynolds number and the relative roughness parameter, and so, it is possible to plot a curve between f which is termed as a friction factor, and the Reynolds number Re with epsilon by D used as a parameter to get different curves.



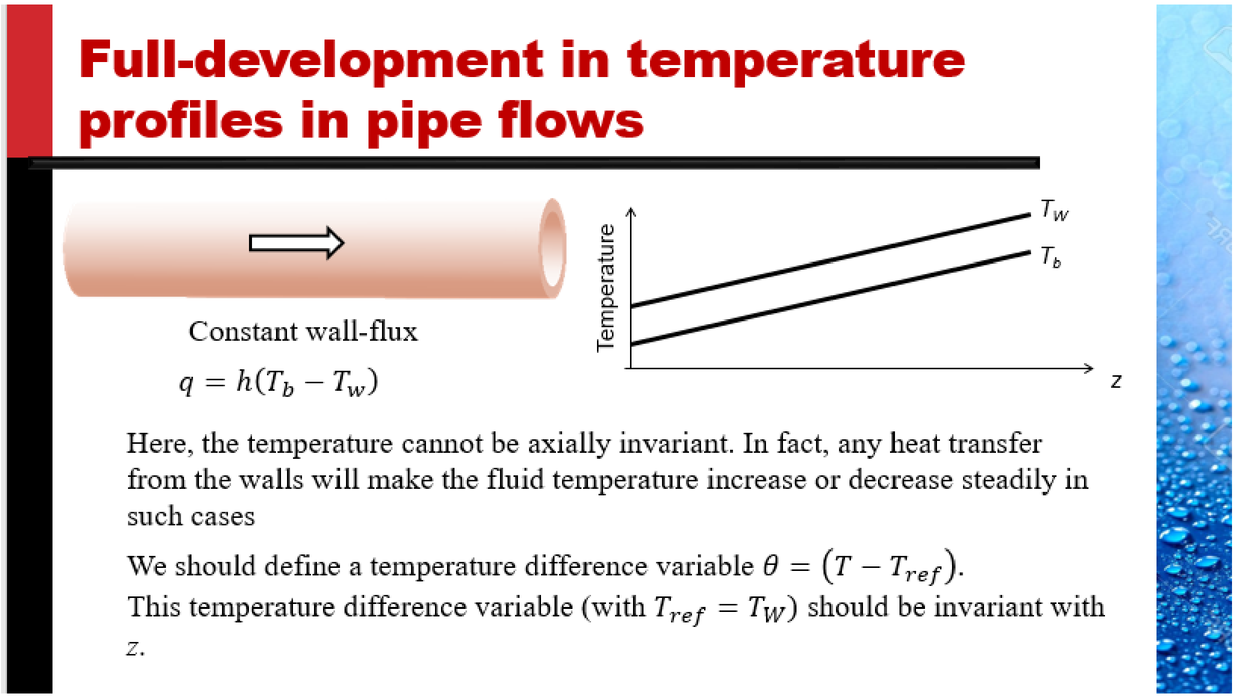
Original diagram: S Beck and R Collins, University of Sheffield

The resulting curve is known as Moody chart and in this Moody chart, we have different lines, each line representing a relative roughness parameter, and then there is Reynolds number along this axis and the friction factor f along this axis. For any pipe, we can calculate the Reynolds number. Let the Reynolds number be, for example, 10^6, 1 million. Then we enter along horizontal axis and suppose the relative roughness of the pipe is 0.02, then we get to this point and then we go to vertical axis, and read the friction factor.

Once we have determined the friction factor, I can determine the pressure drop, delta $\frac{\Delta P}{\frac{1}{2}\rho V_{av}^2}$ is f times L /D. This f we obtain from here and L/D we know. So, using the physical fact that the pressure drop should be proportional to length of the pipe, we have been able to reduce the number of parameters on the right of the equation that is the number of independent parameter groups from 3 to 2. And the 2 group, we are able to plot a graph and from this graph we can determine the value of f. This is substantial improvement on the result.

So, we read the value of f as 0.023 from here. We have used the argument of full development. The full development is a kind of a symmetry argument. The choice of the origin of the axial coordinate system in a infinitely long channel is arbitrary i.e. the change of the origin does not affect either the governing equations for velocity field or the boundary conditions. In this pipe if I choose the origin of z, the coordinate along the axis of the pipe as at this point or this point and if this pipe is truly infinite in length, then it does not matter where we choose the origin. The equations as well as boundary conditions do not change. Do a thought experiment. Suppose you are looking at this pipe and you blink your eyes and when the eyes are closed somebody moves the pipe through a distance, then would you know when you reopen your eyes that the pipe has moved? No.

The should appear to be exactly the same as before. This means that the conditions are independent of the coordinate z. Therefore, the solution of the velocity field should not be a function of the length variable in the axial direction. Thus, the velocity should not be a function of the axial coordinate. This is termed as full development.



For velocity profile this is very simple. The velocity profile is independent of z. But we have a little complication in the temperature profile. Suppose again we have two parallel plates now and these two parallel plates are at two different temperatures with a fluid flowing in between. Let the lower plate be maintained at temperature T_2 and the upper plate at a temperature T_1.

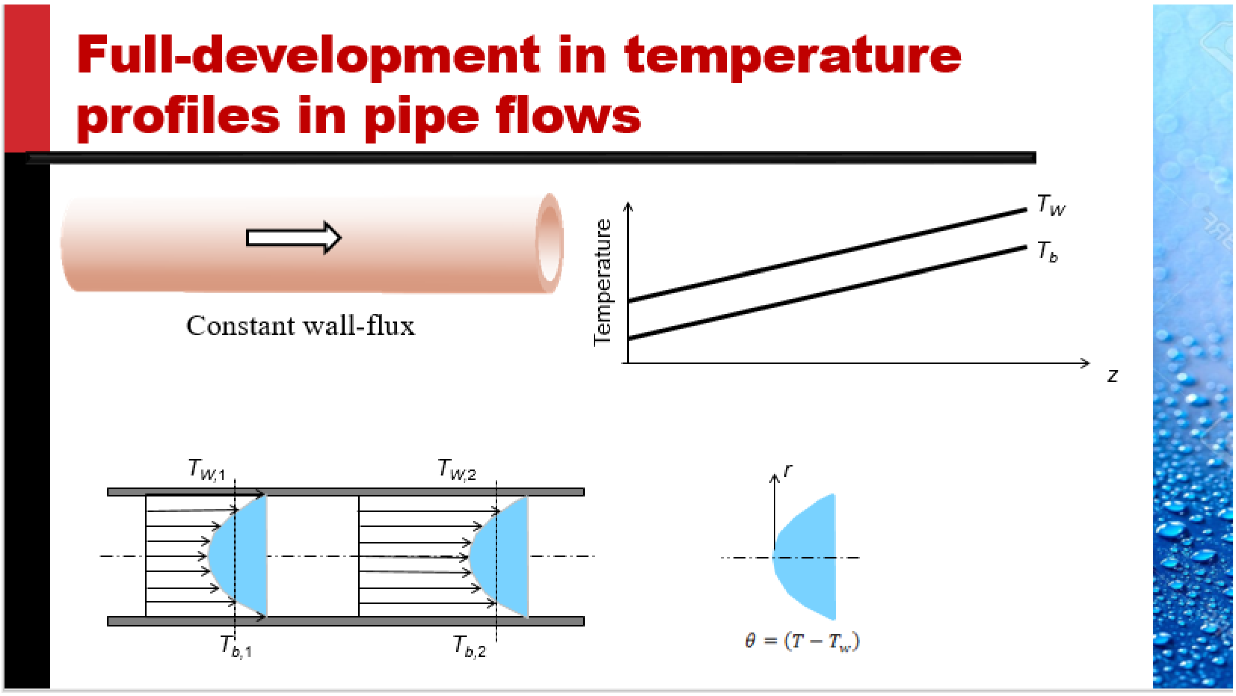
If the plates are of infinite length in the z direction, then as before we can argue that the picture should not change as the origin changes. And because of this the temperature profile should be independent of z. dT/dz should be 0, as shown. The temperature profile at this location and this location are identical. But now let us consider the flow through a pipe.

Now you do not have two walls- only one continuous wall. Let the temperature of this wall be T_w. And let us consider the case when there is a constant wall flux. The heat transfer across the walls of the tube is independent of z, the axial coordinate. And if this happens the heat flux q can be written $q = h(T_b - T_w)q = h(T_b - T_w)$where T_b is the bulk temperature of the fluid.

The bulk temperature, as you recall, is something like average temperature of the fluid. If we take the fluid at a given location and you let it run into a mixing cup and mix it up, then the resulting temperature is called the bulk temperature. So, the heat flux is h times the bulk temperature minus the temperature at wall. Now the temperature cannot be constant since the heat flux is moving continuously into the wall. Then the temperature of the fluid, the bulk temperature of the fluid, must be increasing linearly.

Heat flux is constant, the rate of increase of the bulk temperature must be constant. And from this equation we get if T_b is increasing at a constant rate. T_b minus T_w should also be constant and because of this then T_w must also be increasing at a constant rate. This difference of temperature T_w minus T_b should be constant with z. So, when we look at this picture, it is not the temperature that is invariant with the z, but the temperature difference, $T_w - T_b$, that is invariant with the z direction. So, any heat transfer from the wall will make the fluid temperature increase or decrease steadily in such cases.

We could define a temperature difference variable $\theta = (T - T_{ref})$. The temperature difference variable with T_{ref} is equal to T_w should be invariant with z. Now recall again that when you look at heat transfer equations these are not temperature that occur there it is a temperature difference that occurs there. So, we could introduce a temperature difference variable θ, that is, we can measure T relative to any reference temperature. Here we have chosen reference temperature to be the wall temperature, the local wall temperature.

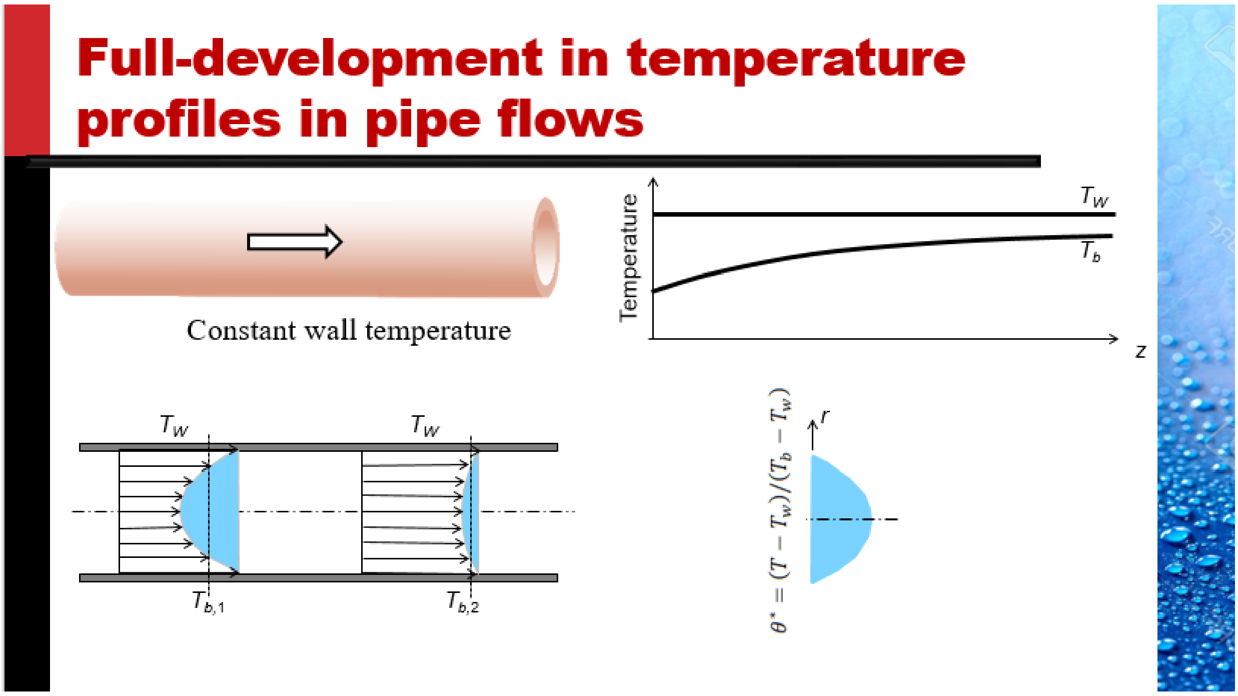


And if we do that, the temperature profiles at two locations will look like this, T_w here is this from here to here at a little low later location T_w is larger. The bulk temperature at the first location is somewhere here, in the second location the bulk temperature is somewhere here. The difference between T w and T b is the same at two locations. And if the flow is fully developed the temperature profile is fully developed then this blue profile which is $T - T_w$, or $T_w - T$ should be invariant. So, the full development in temperature profile means that it is not temperature that is invariant with z, but it is the temperature difference θ which can be defined as T_w minus T_b or T_b minus T_w which is invariant with the z direction.

Thus this is an invariant temperature difference profile. Let us next do the other case of heat transfer in a pipe which is usually done. It is when the wall temperature of the pipe is assumed to be constant. If there is a constant wall temperature then the temperature profiles down the pipe look like this. T_w is constant, but the heat is being transferred into the pipe, that is, to the fluid within the pipe continuously.

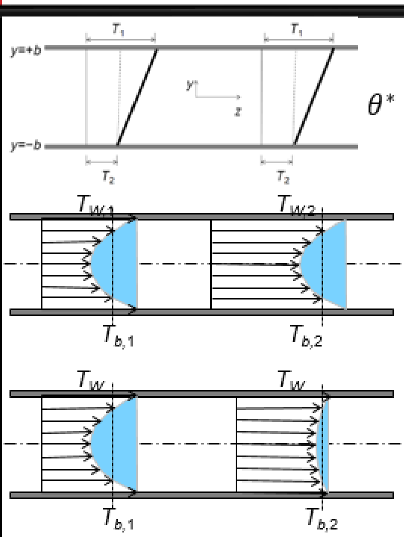
So that the bulk temperature of the fluid is increasing. The temperature difference $\left(T_w - T_b\right)$ is decreasing. What does that mean in full development? Let us look at temperature profile at two different locations T_w is constant, the bulk temperature is increasing. So, the temperature profile, even this blue profile which is the temperature difference profile, is changing shape. But there must be something that is constant, something that is invariant with z.

And what is that? We discover that it is the normalized temperature difference $\theta^* = (T - T_w)/(T_b - T_w)$ that has a constant shape. If I plot this at this location or at this location or at this location, I will get the same picture. So, it is this which is invariant, an interesting argument. I thought I will cover it here because it fascinates me. So, let us summarize the arguments that we have enunciated in the last couple of slides.



The full development in temperature profile, in pipe flow or in flow between two parallel plates, can be expressed by saying that normalized temperature difference θ^* which is defined as $\theta^* = (T - T_w)/(T_b - T_w)$ is independent of z. $\frac{\partial \theta^*}{\partial z} = 0$.

Full-development in temperature profiles in pipe flows



$$\theta^* = \frac{T - T_{ref}}{(\Delta T)_c}$$

$$T_{ref} = T_1 \text{ or } T_2;\ (\Delta T)_c = T_1 - T_2$$

$$T_{ref} = T_w;\ (\Delta T)_c = T_b - T_w$$

$$T_{ref} = T_w;\ (\Delta T)_c = T_b - T_w$$

For two parallel plates with two different temperatures T_1 or T_2, we can choose T_{ref} to be T_1 or T_2, and the characteristic temperature difference as $T_1 - T_2$. And if we do this, we will get invariant normalized temperature difference profile.

For flow with constant heat flux in a pipe, the bulk temperature is changing. Here we use T_{ref} as the wall temperature T_w, and the characteristic temperature difference as $T_b - T_w$. So, T_w is changing with z. delta T_b is changing with z, but $\theta = \left(T - T_{ref}\right)$ divided by $(\Delta T)_c$ is invariant with z.

The final case we did was a constant wall temperature in a pipe. Here again T_{ref} is T_w and the characteristic temperature difference is the local bulk temperature minus the local wall temperature. T_b as well as T_w are functions of z, and then θ^* defined here is invariant in this case as well.

Terminal velocity of a falling sphere in viscous fluid

$V = \mathcal{F}(\rho_s, \text{dia } d, \text{liquid } \rho_f, \text{viscosity } \mu, g)$

Basic group: ρ_s, d, μ

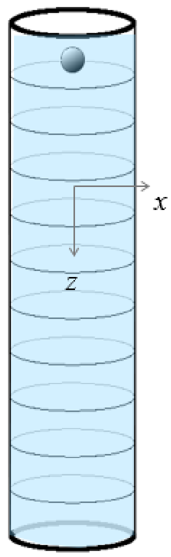
$$\frac{V\rho_s d}{\mu} = \mathcal{F}\left(\frac{\rho_f}{\rho_s}, \frac{g\rho_s^2 d^3}{\mu^2}\right)$$

The only meaningful function of $\frac{\rho_f}{\rho_s}$ is $\left(1-\frac{\rho_f}{\rho_s}\right)$

$$\frac{V\rho_s d}{\mu} = \mathcal{F}\left(\frac{g\rho_s^2 d^3}{\mu^2}\right)\left(1-\frac{\rho_f}{\rho_s}\right)$$

	M	L	T
V		1	-1
ρ_s	1	-3	
d		1	
μ	1	-1	-1
g		1	-2
ρ_f	1	-3	

	ρ_s	d	μ
V	1	1	1
g	2	3	-2
ρ_f	1		



Let us do another example where we will use physical information to simplify the expressions that we obtain through dimensional analysis. Let us consider the terminal velocity of a falling sphere in a viscous liquid. Clearly this terminal velocity V will depend upon the density of the sphere, ρ_s, the diameter D of the sphere, the density of the liquid ρ_f, and the viscosity μ, and of course, the acceleration due to gravity g because gravity is what is causing the ball to fall. If we work in MLT system we write the dimensions of all the parameters involved.

There are 5 independent parameters and one dependent parameter, Dependent parameter V written in red. The dimensions of all of them have been chosen. Then we choose the basic group as the density of the solid sphere, the diameter of the sphere and viscosity. These 3 we choose as the basic group. Now, there are 3 remaining parameters, one dependent and two independent.

So, this will mean that if we work with this, we will get 3 pi groups. To obtain the pi groups we express the dimension of these remaining parameters in terms of ρ_s, d and μ using the method that we discussed in the last few lectures. And if we do this, the dependent parameter results in a dependent group, $\frac{V\rho_s d}{\mu}$, and 2 independent groups of parameters, 2 independent pi's. The ratio of the density $\frac{\rho_f}{\rho_s}$ and $\frac{g\rho_s^2 d^3}{\mu^2}$. This is not a very expressive relationship.

Terminal velocity of a falling sphere in viscous fluid

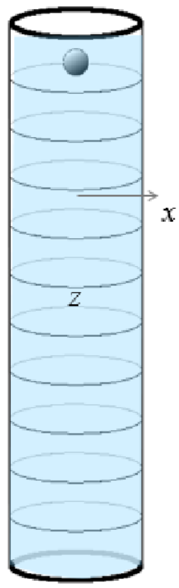
$$\frac{V\rho_s d}{\mu} = \mathcal{F}\left(\frac{g\rho_s{}^2 d^3}{\mu^2}\right)\left(1 - \frac{\rho_f}{\rho_s}\right)$$

We can further assert that g should occur here only in the context of buoyancy force, or in combination $g(\rho_s - \rho_f)$

Therefore, $\frac{V\rho_s d}{\mu} = \mathcal{F}\left(\frac{g\rho_s{}^2 d^3}{\mu^2}\cdot\frac{\rho_s-\rho_f}{\rho_s}\right) = \mathcal{F}\left(\frac{g(\rho_s-\rho_f)\rho_s d^3}{\mu^2}\right)$

We could have started the formulation with

$V = \mathcal{F}(\rho_s, d, \mu, \text{wet weight} \sim (\rho_s - \rho_f)g)$



So, can we bring in some physical information? Yes, the only meaningful function of $\frac{\rho_f}{\rho_s}$ for this problem is $\left(1 - \frac{\rho_f}{\rho_s}\right)$. That is the difference in densities normalized by ρ_s. Actually we should be concerned only with the difference in densities that should derive this. So, we can take this first parameter out and the function of this will have to be of this form. So, this has simplified the problem considerably.

So, now we have only one parameter group whose dependence is unknown. We can do experiments to determine this, but the relation of $\frac{\rho_f}{\rho_s}$ should be of this form. This is from the physics that we bring in. The same result we can further assert that the acceleration g due to gravity should occur here only in the context of buoyancy force or in the combination $g(\rho_s - \rho_f)$. And if this is so, then the functional form should be like this.

The normalized terminal velocity should be a function of this parameter. One parameter. And now we can determine this by conducting experiments on that form. Instead of taking two steps to arrive here, we could have started out with the formulation that velocity is a function of ρ_s, d the diameter, μ, and the wet weight of the sphere, that is, the net weight that is the weight of the sphere minus the buoyancy, the force that moves this sphere downwards. And net weight is proportional to the density difference times g.

So, instead of using ρ_f and g separately, we could have used this in this combination. And if we had done that, we would have arrived at this equation directly. This again illustrates how physical information can be used to augment dimensional analysis.

Let us consider a thermal problem where we will bring in additional information. Consider the total heat transfer rate from a long thin wire of diameter D in length L. Long thin wire means L is much larger than D and it has very high conductivity. The wire is initially temperature T_o and is immersed in a fluid with specific heat capacity C_p, the density ρ, conductivity k, and the temperature T_f, and moving past the wire with the velocity V. So, the rate of heat transfer from the wire $\dot{Q}$ is a function of the properties of the fluid, k conductivity, ρ density, C_p the specific heat, the geometry parameters of the wire, the diameter and the length. The two temperatures T_o the initial temperature of the wire, and T_f the temperature of the fluid bath, and V the velocity of the fluid relative to the wire.

There are total of 9 parameters. For physical problem where there is no or negligible conversion of mechanical energy into thermal energy, it is convenient to just introduce two additional dimensions besides M L and T. The dimensions are Q for heat energy and θ for temperature that is what thermal energy or heat and temperature respectively. So, that gives the number of dimension involved r as 5. n is 9, r is 5. So, we expect to have 4 groups of parameters, 4 pi's, 1 dependent and 3 independent.

Dimensions of various quantities in Q θ M L T systems are these. If we had used M L T system, the dimensions would have been these, but we are not concerned with this here. In this system this table represent dimensions of all the 9 quantities involved Q dot the rate of heat transfer has a dimension of Q 1 and T minus 1. We can deduce the dimensions of k and C p, which have been deduced here, and transferred here.

Forced Convection

$\dot{Q} = \mathcal{F}(k, \rho, C_p, D, L, T_o, T_f, V)$ $n = 9,\ r = 5$ number of pi's = 4

	$Q\ \Theta\ M L T$	$M L T$
Heat	Q	$M L^2 T^{-2}$
Temperature	Θ	$L^2 T^{-2}$
Thermal capacity, C_p	$Q\ \Theta^{-1} M^{-1}$	M
Conductivity, k	$Q\ L^{-1}\ T^{-1}\ \Theta^{-1}$	$M\ L^{-1}\ T^{-1}$

	Q	Θ	M	L	T
$\dot{Q}$	1				-1
ρ			1	-3	
k	1	-1		-1	-1
C_p	1	-1	-1		
D				1	
L				1	
T_o		1			
T_f		1			
V				1	-1

These are all the dimensions involved. It can be argued that the heat transfer should depend on temperature differences rather than temperatures as we had done earlier in this lecture. So, T_o and T_f should be together to define T_f. Thus, number of independent parameters is reduced by 1, from T_o and T_f separately, to a combination ΔT. Similarly, the sole function of density should be to define mass that provides heat capacity. So, that ρ and C_p should occur as a product ρC_p, not independently. ρ should not occur anywhere else. There is no function of inertia that it serves. C_p also should not occur anywhere else except in the combination ρC_p. Thus we can express $\dot{Q}$ as a function only of 6 independent parameters k, ρ C_p together, D, L and ΔT and V. We can determine when we do this the basic set will consist only of 4 quantities and not of 5, and so that number of pi's will be 7 minus 4, or 3. So, there is a reduction of 1 pi that we have been able to achieve by bringing in this physical information.

Supplementing with Physical Information

$$\dot{Q} = \mathcal{F}(k, \rho C_p, D, L, \Delta T, V)$$

Basis set: $\rho C_p, k, D, \Delta T$

$$\frac{\dot{Q}}{kD\Delta T} = \mathcal{F}\left(\frac{VD\rho C_p}{k}, \frac{L}{D}\right)$$

Raleigh in 1918 went further: he invoked that the heat transfer from a wire should be similar to that from a flat plate where $\dot{Q} \sim V^{1/2}$. He, therefore, postulated that

$$\frac{\dot{Q}}{kD\Delta T} = \left(\frac{VD\rho C_p}{k}\right)^{1/2} \mathcal{F}\left(\frac{L}{D}\right)$$

	Q	Θ	M	L	T
$\dot{Q}$	1				-1
ρC_p	1	-1		-3	
k	1	-1		-1	-1
D				1	
ΔT		1			
L				1	
V				1	-1

	ρC_p	k	D	ΔT
$\dot{Q}$		1	1	1
L			1	
V	1	-1	1	

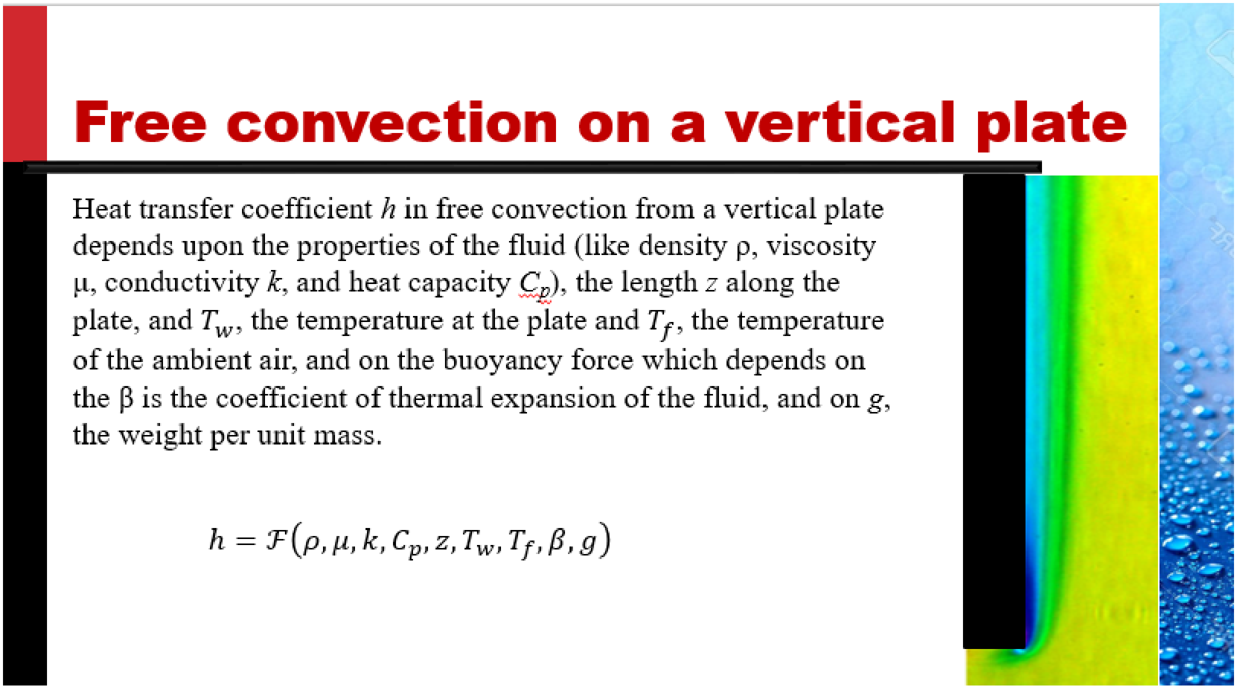
Here we carry out the dimensional analysis. We construct the table of dimensions again. This therefore, basic quantities involved in the basic set we choose ρ C_p k D and ΔT as the basic quantities highlighted with the yellow tint here. There are 3 other quantities left, 1 dependent and 2 independent, $\dot{Q}$, L and V. So, it will result in 3 pi's. We express the dimension of $\dot{Q}$, L and V in terms of the basic set, and this is what we get.

	ρC_p	k	D	ΔT
$\dot{Q}$		1	1	1
L			1	
V	1	-1	1	

And from this we get a dependent non dimensional parameter $\frac{\dot{Q}}{kD\Delta T}$, that is, we get $\frac{\dot{Q}}{kD\Delta T} = F\left(\frac{VD\rho Cp}{k}, \frac{L}{D}\right)$. Raleigh in 1918 went further. He invoked that the heat transfer from the wire should be similar to that from a flat plate where $\dot{Q}$ varies like $\sqrt{V}$. He, therefore, postulated that wherever the velocity occurs it should be to the power one half. So, that he was able to write this again based on additional physical information.

$$\frac{\dot{Q}}{kD\Delta T} = \left(\frac{VD\rho Cp}{k}\right)^{1/2} F\left(\frac{L}{D}\right)$$

This is of the form $Nu = C(Re \cdot Pr)^{1/2}$. From this we can get this. This combines with this to produce Nusselt number. This is the standard correlation that is used in convection.



Let us do another example of free convection on a vertical plate. Shown here is a vertical plate, hot plate in a cold fluid. Plate heats up the fluid next to it, the heated fluid is now less dense. So, it experiences a buoyancy force. The fluid starts moving up and as it moves up it is replaced by the colder fluid from below. The heat travels in that direction and this fluid is swept up in that direction. So, that limits the region which is affected by heat. So, this shows the extent of what is termed as a free convection boundary layer on a vertical plate. The heat transfer coefficient h in free convection from vertical plate depends upon the properties of fluid like density ρ, viscosity μ, conductivity k and the heat capacity C_p of the fluid. The length z along the plate, T_w, the wall temperature, and T_f the temperature of the ambient air. And it also depends on the buoyancy force which depends upon β, the coefficient of thermal expansion of the fluid, the fractional rate of change of density with temperature, β. And of course, on g, the weight per unit mass the acceleration due to gravity. So, h can be written in this functional form

Depends upon the properties of the fluid. This again is a property of the fluid. The two temperatures, the length of the plate in the z direction, and g, the acceleration due to gravity. There are 9 independent parameters.

The expression here has very large number of parameters. We should use whatever physical information that we have to reduce the number of parameters as much as possible. Note that heat transfer depends not on individual temperatures. We use that argument a couple of times earlier, but on the temperature difference. So, that it is not T_w and T_f that matter, but ΔT is equal to $T_w - T_f$ that should enter the list of parameters. So, these two parameters would combine to give only one parameter $T_w - T_f$.

This reduces the value of n by 1. Further, the fluid motion in the vertical direction depends upon the buoyancy force which in turn depends upon the combination gβ rather than g and β independently. β would result in a density difference and that multiplied by g will give you the buoyancy force. So, this means that these two combine to give βg and that further reduces n by 1. So, that the original equation modifies into this equation 7 independent parameters and 1 dependent parameter.

Total N is equal to 8 and work in MLTQθ system. We need r is equal to 5, which gives 3 non dimensional pi's. The table in the next slide facilitates the organization of this information. The dependent parameter h is never selected in the basic group of parameters. So, we have chosen the basic group at z, ρ, μ, k, and ΔT.

Free convection on a vertical plate

	Variables							
	Basic					Others		
	z	ρ	μ	k	ΔT	h	C_p	$g\beta$
Dimensions	L	ML^{-3}	$ML^{-1}T^{-1}$	$L^{-1}T^{-1}Q\Theta^{-1}$	Θ	$L^{-2}T^{-1}Q\Theta^{-1}$	$M^{-1}Q\Theta^{-1}$	$LT^{-2}\Theta^{-1}$
Non-dimensional	z^a	ρ^b	μ^c	k^d	ΔT^e			
	Values of exponents							
Π_0(= Nu)	1	0	0	-1	0	hz/k		
Π_1 (= Pr)	0	0	1	-1	0		$\mu C_p/k$	
Π_2 (= Gr)	3	1	-2	0	1			$\frac{g\beta z^3\rho^2\Delta T}{\mu^2}$

5 of them, independent and complete. Independent in the sense that you cannot express any one of them as a product of power of the rest within this group. And complete in the sense all dimensions that are relevant the 5 dimensions here are contained here in. These are the 3 other parameters which remain: h the dependent parameter C_p and gβ together. The dimensions of these are written here and you can verify. And now we want to write each of these other parameters as a product, of z^a, ρ^b, μ^c, k^d, ΔT^e.

And when we do this, the dimension of these products should match the dimension of the parameter that we selected from the others group. So, with h, the dependent parameter, we get a dependent pi, and it is z/k. And from this we get a parameter hz/k as a dependent non dimensional parameter. This is termed as Nusselt's number and abbreviated by Nu in heat transfer literature. From C_p, I get the dimension of C p as a dimension of μ/k.

So, that the non dimensional parameter that we formulate $\mu C_p/k$ is this and this is recognized as Prandtl number which is abbreviated as Pr. And gβ involves z ρ μ and ΔT, , which results in $\frac{g\beta z^3\rho^2\Delta T}{\mu^2}\frac{g\beta z^3\rho^2\Delta T}{\mu^2}$. And this is recognized as what is termed as Grashof number, which is important in free convections alone.

So, the result can be expressed as Nusselt number is a function of Prandtl number and Grashof number. Those of you who have completed a course in heat transfer know this empirical relation very well.

It is possible to improve this result by resorting to what has been termed as directional analysis, where the length of dimension L is broken up into three mutually perpendicular dimensions Lx, Ly and Lz. This we will do a little later.

Let us go further and look at what else can we do. Asymptotic behavior of the function is something that we must consider. When there are only two parameters involved in a problem, then we get an expression like $Y = f(X)$, a dependent parameter Y as a function of a single independent parameter X, where X is either very small or very large compared to 1. Since we are talking of non-dimensional parameters and normalized, so we expect them of order 1. But suppose these parameters are very small or very large compared to 1. In such cases there are three possibilities.

One is $f(X)$ approaches a dimensionless constant value C. X and Y both are dimensionless. So, function $f(X)$ approaches a dimensionless constant value C. The second possibility, this approaches a power law $f(X) = CX^n$, where n is non-zero. And third, of course, none of these. Now first and second are about the same. The first is a special case of the second possibility with the value of n as 0. So, first and second can be combined to say that $f(X)$ approaches a power law CX^n, where n could be 0 or non-zero. Of course, dimensional analysis alone does not reveal the asymptotic behavior. We will have to draw on physical knowledge.

For this, consider the case of a cable which is hanging, and under its own weight. It extends.

Extension of a hanging cable

$\delta = f(l, \rho', g, E)$

$\rho' g$ may be considered together

F L is quite convenient for this problem

	F	L
δ		1
l		1
$\rho' g$	1	-1
E	1	-2

$\frac{\delta}{l} = f\left(\frac{E}{\rho' g/l}\right)$ or $\delta = Clf\left(\frac{\rho' g}{El}\right)$

	l	$\rho' g$
δ	1	
E	-1	1

For most cables, the non-dimensional parameter $\frac{\rho' g}{El}$ has very small values compared to 1, so the functional form would be $f\left(\frac{\rho' g}{El}\right) = \left(\frac{\rho' g}{El}\right)^n$

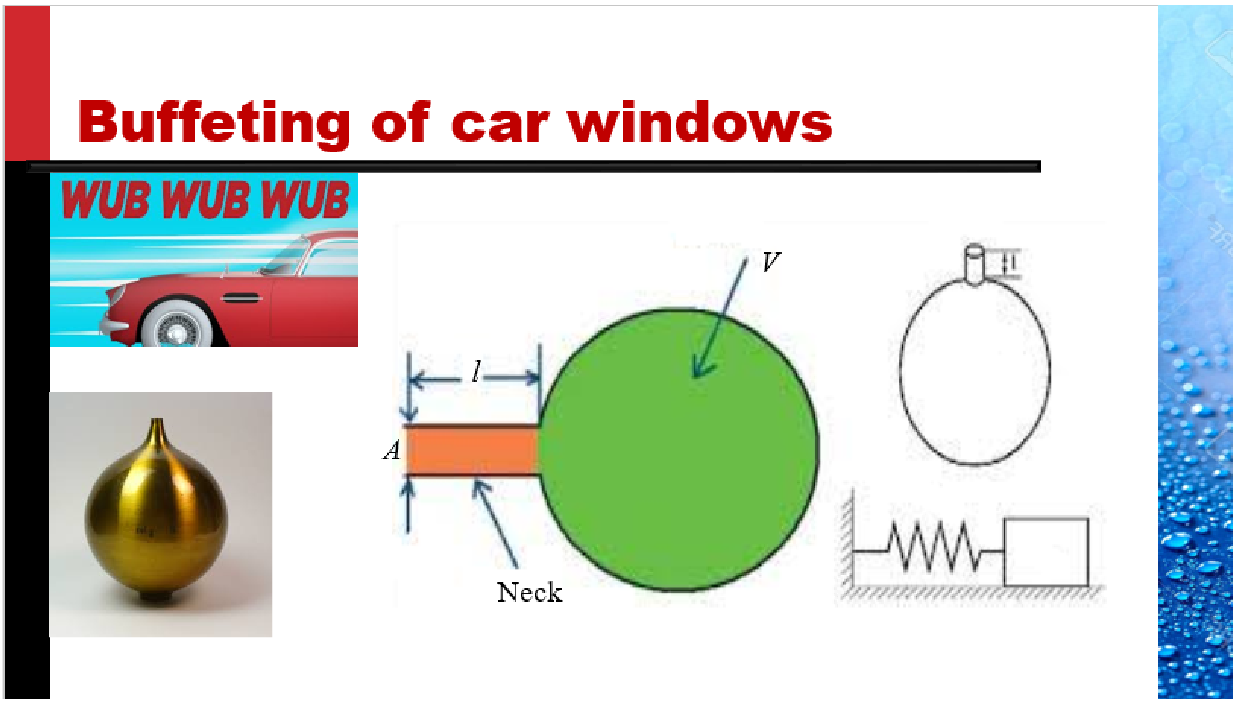
So, from the ceiling we have this long cable, long thin cable. So, the length of the cable L is much greater than its diameter. The extension of this cable δwould clearly be function of L the length ρ' which is the mass per unit length of the cable. So, a density times cross sectional area ρ' mass per unit length, g the acceleration due to gravity, or weight per unit mass, and E the elasticity of the cable material. Now, we bring in the physical fact that since the cable extends under its own weight, the $\rho' g$ may be considered to be together, not independent of each other.

So, that instead of four independent parameters, we have only three independent parameters. Now, there is no need to work in MLT system. Force and length are the only two dimensions that are involved and we construct the dimensional table, that is what it looks like. Two dimensional F L. δ has a dimension of length, length of course has dimensions of length, $\rho' g$ has a dimensional force per unit length, and E is force per unit area. So, we choose L and $\rho' g$ g as the basic set, and in terms of this basic set, we obtain the dimension of delta as L raised to power 1, and of E as L raised to power minus 1 and $\rho' g$ raised to power 1. And this gives us $\frac{\delta}{l} = f\left(\frac{E}{\frac{\rho' g}{l}}\right)$, or delta can be written as $Clf\left(\frac{\rho' g}{El}\right)$.

Now, for most cables, the non-dimensional parameter $\frac{\rho' g}{El}$ has very small value. ρ' would typically have a value in SI system is 10^3 or so, g has a value about 10. So, numerator is about 10^4. The value of E, the elasticity for most material is of the order of 10^8 to 10^{10}, and L, the length of the cable would definitely be in meters. So, this parameter, $\frac{\rho' g}{El}$ has a value at most 10^{-4} or 10^{-5}, a very small value compared to 1.

So, the functional form should be like this: $f\left(\frac{\rho' g}{El}\right) = \left(\frac{\rho' g}{El}\right)^n$, y as a function of x, x becoming very small. So, the functional form should be like x raised to power n. Now, we could draw some argument about the value of n. Here the n must be positive because if n is not positive, then delta would not increase with the changed mass ρ'. That means n must be positive. physical information brought in, and some opinion formed about the value of n, n must be positive. Further because the longer cables should stretch more than the shorter ones, delta should increase as a positive power of L. So, L so the value of n should be less than 1, otherwise L in the numerator and L raised to power n in the denominator. If n is greater than 1, then delta would decrease with L. It is not possible. So n should be less than 1. Therefore, n must have a value between 0 and 1, both ends excluded. Can neither 0 nor 1. In fact, there are reason to believe that the value of n is about one half.

Let us do one more example, an interesting example. If we sit in a car with one window open. Then at certain speed you hear a very low bass sound within the car. One of the car windows is open, not all four windows, only one window is open and you hear a very low bass sound, low frequency sound. And this could be very irritating. The only way to stop the sound is either to close all windows or to open all windows.



If only one window is open, then you hear this wub, wub, wub, wub sound which is very irritating. And why is this happening? It is happening because of an interesting phenomenon known as the resonance in Helmholtz resonator. Helmholtz resonator is a volume with a neck. The volume serves as a spring and the mass in the neck oscillates. Given a volume and given the neck dimensions, there is a particular frequency at which this resonates and so you hear that sound.

Helmholtz has analyzed this: a volume V with a neck of cross sectional area A and length L. It is like a mass on a spring. This air volume serves as a kind of a spring. This is a mass that oscillates.

Helmholtz found the frequency for this mass. This is a very important innovation. It has been used in old churches to produce sound damping effects. It is also used in a muffler. If this is exhaust or the air stream, then if we can make a Helmholtz resonator with a frequency of the sound carried in here, then the energy from this stream, acoustic energy, would go into the resonator and if you put some damping material with the resonator, then energy would be taken out and so this could work as a muffler. But in a car with the window open, this is producing noise.

Buffeting of car windows

$\omega = f(V, p, \rho, A, l)$

Basic set: V, p, ρ

$$\frac{\omega V^{1/3}}{\sqrt{p/\rho}} = f\left(\frac{A}{V^{2/3}}, \frac{l}{V^{1/3}}\right) \text{ or } \omega = C\frac{c}{V^{1/3}} f\left(\frac{A}{V^{2/3}}, \frac{l}{V^{1/3}}\right)$$

	M	L	T
ω			-1
V		3	
p	1	-1	-2
ρ	1	-3	
A		2	
l		1	

	V	p	ρ
ω	-1/3	1/2	-1/2
A	2/3		
l	1/3		

So the frequency of this resonator is a function of V, the volume. In our case, it would be the volume of the cabin of the car, p the air pressure, ρ the air density, A the area of the open window, and L the thickness of the neck that is formed by the open window. So, if we write the dimensions of all these parameters in the M, L and T system, this is the table that we get. We choose the basic set as V the volume, the pressure and the density of air, and these are the three groups that are formed from ω, A and L. So that this is the functional equation that we get. $\omega = C\frac{c}{V^{1/3}} f\left(\frac{A}{V^{2/3}}, \frac{l}{V^{1/3}}\right)$

Can we get further information from this? That functional relation repeated here. The resonator with the same shape, yet of different sizes that is having the same ratio $\frac{A}{V^{2/3}}$, and $\frac{l}{V^{1/3}}$, this and this same values, different sizes, but same shape. So, if these two parameters are same, then omega is a function of $V^{1/3}$ with a negative sign. Large volume of the car gives you very low frequency. So, a base sound that you get, the common experience. However, we also know that the value of $\frac{A}{V^{2/3}}$ and $\frac{l}{V^{1/3}}$ are very small for a typical Helmholtz resonator, and definitely for the passenger compartment of a car.

The volume of the car, the area of the window and the thickness of the window opening. Thickness of the window opening area are much smaller than the volume of the car raised to power one-third or two-thirds. So, these two parameters are tending to be very small. And so, we can write omega in this form according to the discussion that we had previously. So, we have learned here in today's lecture about how to simplify the results obtained from dimensional analysis by supplementing with additional physical information.

Thank you.