Similitude And Approximations In Engineering,
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Week - 02
Lecture – 10

Welcome back. In the last lecture we had introduced the classical dimensional analysis using the Buckingham pi theorem. In this lecture, we will practice on this method by doing some simple examples.

Some simple examples

Sag of wire stretched between two points
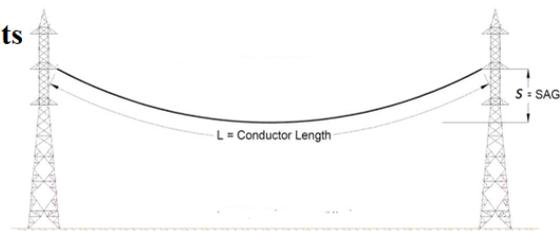
$$s = \mathcal{F}(T, \rho, L, g)$$

Basis set: T, ρ, L

$$[g] = [T]^\alpha [\rho]^\beta [L]^\gamma$$

$$[LT^{-2}] = [MLT^{-2}]^\alpha [ML^{-1}]^\beta [L]^\gamma$$

$$0 = \alpha + \beta$$
$$1 = \alpha - \beta + \gamma \qquad \beta = -1$$
$$-2 = -2\alpha \qquad \gamma = -1$$
$$\alpha = 1$$

$$\frac{s}{L} = \mathcal{F}\left(\frac{g}{T/\rho L}\right) = \mathcal{F}'\left(\frac{T}{\rho g L}\right)$$

	T	ρ	L
s			1
g	1	-1	-1

	M	L	T
S		1	
T	1	1	-2
ρ	1	-1	
L		1	
g		1	-2

The first example discusses the sag of a wire stretched between two points like an electrical conductor of length L between two transmission towers. The sag s will be a function of tension T in the wire, ρ, the density of the wire, L the length of the wire and of course on the acceleration due to gravity because the sag would be because of the weight of the wire which depends upon g. In the first step we take all the parameters, the dependent parameter s shown in red and T, ρ, L and g, independent uniquity parameters.

We use MLT systems of dimension and write the dimensions of these. The density rho is here the density per unit length of the wire. We don't need to worry about the density per unit volume since the weight would be a function of length so we work with density per unit length. So the dimension of ρ would be simply mass $M\,L^{-1}$.

Tension would be like force, so MLT^{-2}, and g acceleration due to gravity would have the same dimension as acceleration and that would be LT^{-2}. We take a basic set as T, ρ and L. This basic set. Make sure that it is complete and independent. We cannot form a non dimensional group out of these three.

Powers of these three multiplied together will not give you a non dimensional group. So they are independent. They are complete because from this we can extract the dimension of mass,

length and time. So the value of k here is 3. The value of n was 4, the number of independent parameters.

The basic group is 3. So the number of independent groups of parameter non dimensional parameter they can form would be 1 based on g, and a non dimensional dependent parameter based on S. S would have dimension of length. So if you write S as $[T]^{\alpha}[\rho]^{\beta}[L]^{\gamma}$. So the value of gamma would be 1 and alpha and beta would be 0.

So S has a same dimension L. So S by L is dimensionless, and then we have to form the dimension of g. g we write as $[T]^{\alpha}[\rho]^{\beta}[L]^{\gamma}$ in terms of the basic set. g has a dimension of LT^{-2}. T has a dimension tension T is a tension here not the time.

Tension of the dimension of MLT^{-2}, $MLT^{-}2$ raised to power alpha. ρ is a dimension ML^{-1} raised to power beta, and the length of the wire raised to power gamma. From this, equating the dimensions of mass on left and right sides we get mass is 0 on the left, mass at dimension alpha in the first term and beta in the second term. So 0 is equal to alpha plus beta. Then for equating the dimension of length on the left hand side, length is 1 dimension, on the right hand side in the first factor L has a dimension alpha, in the second factor L has a dimension minus beta, and in third it has a dimension plus gamma.

So $1 = \alpha - \beta + \gamma$. Similarly, for time: $-2 =- 2\alpha$. From these we can solve the value of alpha beta and gamma. Alpha comes out to 1. From this we get beta is equal to minus 1, and then gamma is equal to minus 1. So that the dimensions of g are $\frac{T}{\rho L}$.

So that we can write $\frac{s}{L} = F\left(\frac{g}{\frac{T}{\rho L}}\right)$. It does not matter whether you write inverse or any power of a parameter. Non dimensional parameter can have any powers. So we used a power of minus 1 here. So $\frac{s}{L} = F'\left(\frac{T}{\rho g L}\right)$.

So it depends only on 1 parameter. We need to do 1 experiment for 1 wire we measure T, ρ, g and L, and then obtain the value of s. Once we know this, I can find out what this functional relationship is, and then for any wire with any tension any material, I could predict the sag. This is the power of dimension analysis. In fact this is the power of similitude.

Some simple examples

Sag of wire stretched between two points
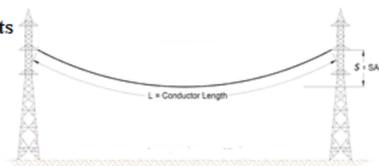
$s = \mathcal{F}(T, \rho, L, g)$

Basis set: T, ρ, L

$[g] = [T]^{\alpha}[\rho]^{\beta}[L]^{\gamma}$

$[LT^{-2}] = [MLT^{-2}]^{\alpha}[ML^{-1}]^{\beta}[L]^{\gamma}$

$0 = \alpha + \beta$
$1 = \alpha - \beta + \gamma$ $\beta = -1$
$-2 = -2\alpha$ $\alpha = 1$ $\gamma = -1$

Analytical result:

$\frac{s}{L} = \frac{\rho g L}{8T}$

$\frac{s}{L} = \mathcal{F}\left(\frac{g}{T/\rho L}\right) = \mathcal{F}'\left(\frac{T}{\rho g L}\right)$

L = Conductor Length

S = SAG

Dimension analysis is only 1 method of carrying out the similitude. This problem can be solved very easily. It is done in many books on statics and the result is $\frac{s}{L} = \frac{\rho g L}{8T}$. It is in the same form as a functional relationship we have obtained by dimensional analysis except that a prime is the function $\frac{T}{\rho g L}$ is to power minus 1, inverse of this. Clearly the sag should be inversely proportional to the tension in the wire.

More the tension the less is the slack. The factor 8 cannot come from the dimensional analysis. It can be obtained only from experiments or from analytic results.

Some simple examples

Time period of vibration of a Nicholson hydrometer

$$\tau = \mathcal{F}(M, S, \rho, g)$$

Basis set: S, ρ, g

$$T = (L^2)^\alpha (ML^{-3})^\beta (LT^{-2})^\gamma$$
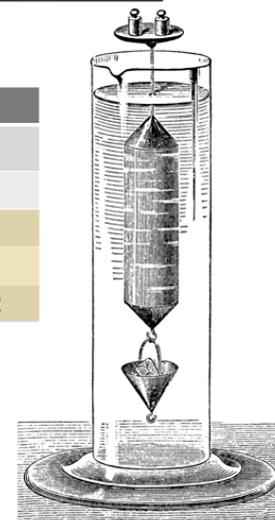
$$0 = \beta$$
$$0 = 2\alpha - 3\beta + \gamma$$
$$1 = \beta - 2\gamma$$

$$\frac{\tau g^{1/2}}{S^{1/4}} = \mathcal{F}\left(\frac{M}{\rho S^{3/2}}\right)$$

	M	L	T
τ			1
M	1		
S		2	
ρ	1	-3	
g		1	-2

	S	ρ	g
τ	1/4		-1/2
M	3/2	1	

Let us do another example. Time period of vibration of a Nicholson hydrometer. Hydrometer was an instrument used in earlier days to find the density of a liquid. It consists of a weighted cylinder with a long thin neck of cross sectional area S. If the liquid were denser the hydrometer would pop up, if the liquid was less dense hydrometer would sink more. And this hydrometer was designed so that very small difference in densities could be measured. This oscillated, not something that you wanted it to do, but it did oscillate when you put it in a liquid.

The time period of vibrations of the Nicholson hydrometer is given as function of the mass of the hydrometer, S the cross sectional area of the neck where the free surface of the liquid is, ρ the density of the liquid that was that is there in the jar, and g, the acceleration due to gravity. So the value of n is 4. There are 3 units available if we use MLT system. So that we can write the dimension of the dependent parameter τ and the independent parameter M, S, ρ and g in terms of MLT and this table gives it. rho the dimension of density is mass per unit volume ML^{-3}, g is like acceleration, LT^{-2}, S the area is like L^2. n is 4, k the basic set that we choose is 3, and we choose S, ρ, and g as the basic set.

These are independent and complete. We can write the dimensions of the remaining quantities τ and M in terms of S, ρ, and g and they themselves do not form a dimensionless group. So the time period τ would have the dimension $T = \left(L^2\right)^\alpha \left(ML^{-3}\right)^\beta \left(LT^{-2}\right)^\gamma$.

In matching the dimensions of M, L and T. On the left hand side there are no dimensional M and L, only T on the left hand side the dimension of M is only beta in the middle term. For length, the first S contributes 2 alpha, rho contributes minus 3 beta and g contributes gamma. So the equation is $0 = \beta$, $0 = 2\alpha - 3\beta + \gamma$, and $1 = \beta - 2\gamma$.

From this we can find out the dimensions of tau as $S^{1/4}\rho^0 g^{1/4}$. This is for the dependent parameter. And the remaining independent parameter is M and M we will find out if we do the same thing we will find out the dimension will be $S^{3/2}\rho$. So that the non-dimensional dependent group is $\dfrac{\tau g^{1/2}}{S^{1/4}}$ and that will be a function of $\left(\dfrac{M}{\rho S^{3/2}}\right)$. Doing a few experiments we can find out what should be the nature of this functional relationship and once we discover this relationship we can find out the time period of any hydrometer.

Another example

Frequency of oscillation of a *small* drop of liquid pendant from a circular orifice

$$f = \mathcal{F}(\sigma, \rho, D, g)$$

Basis set: σ, ρ, D

$$T^{-1} = (MT^{-2})^\alpha (ML^{-3})^\beta (L)^\gamma$$

	M	L	T
f			-1
σ	1		-2
ρ	1	-3	
D		1	
g		1	-2

$$\frac{f}{\sqrt{\sigma/\rho D^3}} \;;\; \frac{\rho g D^2}{\sigma}$$

	σ	ρ	D
f	1/2	-1/2	-3/2
g	1	-1	-2

$$\frac{f}{\sqrt{\sigma/\rho D^3}} = \mathcal{F}\left(\frac{\rho g D^2}{\sigma}\right)$$

Another example, if we take a small pipette and we let a small drop of liquid pendant from this pipette from the circular orifice of this pipette you would see the liquid drop something like what shown in this figure and this would oscillate in surface tension.

The time period of oscillation has to be found out. It can be argued that the frequency oscillation should be a function of the surface tension of the liquid, the density of the liquid, the diameter D of the orifice, and of course, the acceleration due to gravity, because it oscillates under gravity force. So, n is equal to 4 in this problem, 4 independent parameters. We choose the MLT system again quite convenient. We understand the dimension very intuitively and we write the dimensions of all the quantities dependent as well as independent F, σ, ρ, D and g. You can verify for any one of them.

Now, we choose a basic set. A basic set would consist of three quantities. Let us choose σ, ρ, D from among the list of independent parameters. We choose 3 as the basic group. So, that we write the functional relationship of F and the remaining independent parameters in terms of these dimensions of σ, ρ, D. σ, ρ, D is yellowed is the basic group. So, now we find out by using the same technique as we used before.

For example, frequency f has a dimension of T raised to minus 1. We can write $T^{-1} = \left(MT^{-2}\right)^{\alpha}\left(ML^{-3}\right)^{\beta}(L)^{\gamma}$. From this, we can find out the value of alpha as one half, value of beta as minus one half and value of gamma as minus 3 by 2. So, f is sigma raised to power half, rho raised to power minus one half and D raised to power minus 3 by 2 in dimensions. Similarly, we do for the remaining independent parameter g. This would be sigma divided by rho divided by D squared.

So, that the dimensionless equation will be formed out of two dimensionless parameters, the dependent dimensionless group $\dfrac{f}{\sqrt{\sigma/\rho D^3}}$. The independent group of parameters is $\left(\dfrac{\rho g D^2}{\sigma}\right)$ and therefore, the functional relationship is simply this: $\dfrac{f}{\sqrt{\sigma/\rho D^3}} = F\left(\dfrac{\rho g D^2}{\sigma}\right)$. I think you should be convinced by now that it is quite easy to find out the groups that define the functional relationship. Let us do one example in a little bit of more details. Let us consider deformation of an elastic ball striking a wall or a floor.

Deformation of an elastic ball striking a wall

We wish to investigate the deformation that occurs in elastic balls when they impact on a wall. We might be interested, for example, in finding out what determines the diameter d of the circular imprint left on the wall after a freshly dyed ball has rebounded from it.

$$d = \mathcal{F}\begin{pmatrix} \text{diameter } D, \\ \text{velocity } V \text{ just prior to contact,} \\ \text{elasticity } E, \\ \text{and Poisson's ratio } \nu, \text{ and} \\ \text{mass of the ball, } m. \end{pmatrix}$$
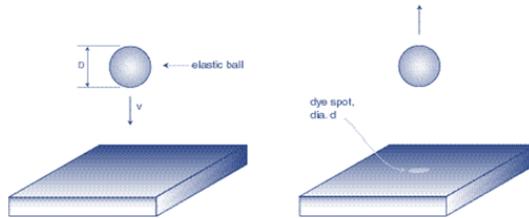
Thus, $d = \mathcal{F}(D, V, E, \nu, m)$

We wish to investigate the deformation that occurs in elastic balls with the impact on a wall. We might be interested, for example, in finding out what determines the diameter D of the circular imprint left on the wall after a freshly dyed ball has rebounded from it. So, this ball is traveling towards the wall with the velocity V, elastic ball of diameter capital D and after hitting the wall and rebounding, it leaves a dye spot or diameter lowercase d as shown on the right. Clearly the diameter of the dye spot, lowercase d, would be a function of the diameter D of the ball, the velocity V just prior to the contact with the wall, the elasticity E of the ball material, the Poisson's ratio ν and the mass of the ball m. Since we have taken the velocity V just prior to the impact the gravity effect is not significant.

Thus lowercase d is a function of D, V, E, ν, the Poisson's ratio, and m, the mass of the ball. We organize our analysis in this table. The basic group we write as V, m and D, the velocity, mass and the diameter capital D of the ball. Velocity has dimension $L\,T^{-1}$, mass is obviously M, and diameter is L. The other quantities E and ν in the independent parameters, E has a dimension $M\,L^{-1}\,T^{-2}$.

ν, the Poisson ratio is dimensionless, and d is the dependent parameter, the diameter of imprint left on the wall is length L. So, for these three quantities E, ν and d we write in the forms of $V^a M^b D^c$, and find out the value a, b and c, so that the dimensions match. For E the dimensions would be V^2 M D^{-3}. So, that ED^3 /MV^2 would be a non dimensional pi number, non dimensional independent pi number. Π_2, the second pi group refers to ν which is dimensionless.

Deformation of an elastic ball striking a wall

	Variables					
	Basic			Others		
	V	M	D	E	ν	d
Dimensions	LT^{-1}	M	L	$ML^{-1}T^{-2}$	-	L
Non-dimensional	V^a	M^b	D^c			
Value of exponents						
Π_1	2	1	-3	$\dfrac{ED^3}{MV^2}$		
Π_2	0	0	0		ν	
Π_0	0	0	1			d/D

$$\frac{d}{D} = \mathcal{F}\left(\frac{ED^3}{MV^2}, \nu\right)$$

$$\frac{d}{D} = \mathcal{F}\left(\frac{E}{\rho V^2}, \nu\right)$$

So, this itself is a non dimensional pi number ν. No calculation, no analysis needed, and the dependent parameter, non dimensionalized, is denoted Π_0, and clearly, it would have a dimension of V^0 M^0 D^1. So, that the non dimensional dependent parameter is $\frac{d}{D}$. And so the functional relationship would be something like this: $\frac{d}{D} = F\left(\frac{ED^3}{MV^2}, \nu\right)$. Mass divided by the diameter cubed would be like density because mass is density times volume, and volume is like D^3. So, D^3/M is like $1/\rho$. So, $\frac{d}{D}$ is a function of $\frac{E}{\rho V^2}$, where ρ is the density of the material. V is the velocity just before the impact. Sonin in the text that has been noted as your textbook, gives you data on the various experiments done with elastic ball striking a wall.

It uses three materials, alumina, aluminum and rubber. The values E and ν is given each experiments were done with three velocities and the value of the imprint lowercase $\frac{d}{D}$ was obtained. These was plotted with these symbols given on the right in this figure. You see the all the points collapse in one curve. In fact this quite independent of the Poisson ratio which is denoted by gamma here rather than nu it is denoted by gamma.

We plotted $\frac{d}{D}$ versus $\frac{E}{\rho V^2}$, and there is only one line independent of the material, independent of ν, independent of D, independent of E, independent of ρ, and this confirms that the analysis that we did is all right. With this single curve we can predict the imprint diameter for any future experiment with whatever material, whatever size, and whatever velocity we impact a wall with. The imprint diameter can be predicted.

There is one another example where the power of dimensional analysis is beautifully on display. The increase in pressure delta p in a centrifugal

Dimensional Analysis of Centrifugal Pumps

We shall work in MLT system.

We next write the dimensions of all the relevant variables in the chosen system. Determine the minimum number r of the dimensions required to express these variables. For most problems of fluid mechanics, this number equals 3.

Variable	M	L	T
$\dot{Q}$ (independent)	0	3	−1
D (independent)	0	1	0
ρ (independent)	1	−3	0
N (independent)	0	0	−1
Δp (dependent)	1	−1	−3

$r = 3$

pump is believed to depend upon the diameter D of its impeller, the density ρ of the fluid pump, the rpm N at which the impeller is rotating and on Q rate or $\dot{Q}$, the rate of the volume of the liquid being pumped.

Thus, Δp the increase in pressure is a function of D, ρ, N, $\dot{Q}$. The value of n here is 4. We will work in MLT system. As usual we next write the dimensions of all the relevant variables in the chosen system. We determine minimum number r of the dimension required to express these variables.

For most problems in fluid mechanics this number equals 3. MLT is a system that we use. So, three variables are used. So, the four independent parameters and the dependent parameterΔp.

The dimensions we have written here in MLT system. The value of r or k is 3 here. So, the number of resultant pi numbers would be 2, one dependent and one independent. The total number of non-dimensional groups of parameter which can be formed by combining the n physical parameters and variables of a problem equal to n minus r where r is a number of primary dimension required to express the dimensional formula of the n physical quantities the Buckingham pi theorem which we had stated earlier. Therefore, in this example we should have two non-dimensional numbers, non-dimensional pi's, one dependent and one independent.

For our example let us choose D, ρ and N as the basic group. This is conventional for centrifugal pumps. We deal in diameter of centrifugal pump, density of the liquid pumped and RPM that we operate on. It is now possible to form n minus r independent pi's out of the variable for i from r plus 1 to n, that is, from 4 to 5, two parameters.

And we do the analysis as before. The basic group D, ρ and N has a dimension of L, $M L^{-3}$, and T^{-1}, respectively. The other groups are Δp, the dependent parameter which is a

dimensional pressure: $M\,L^{-1}\,T^{-2}$: force divided by area, force $M\,L\,T^{-2}$, divided by area, L^2. So force divided by area would be $M\,L^{-1}\,T^{-2}$ and $\dot{Q}$ the volume flow rate would be volume divided by time: $L^3\,T^{-1}$. So we can form the non-dimensional parameters D^a, ρ^b, N^c. The values a, b and c are determined for the other variables, other parameters.

Dimensional Analysis of Centrifugal Pumps

(e) It is now possible to form $(n-r)$ independent π's out of the variables
$$\pi_i = X_i X_1^{a_1} X_2^{a_2} X_3^{a_3} \ldots X_r^{a_r} \quad \text{for i from } r+1 \text{ to } n.$$

	Variables				
	Basic group			Others	
	D	ρ	N	Δp	$\dot{Q}$
Dimensions	L	ML^{-3}	T^{-1}	$ML^{-1}T^{-2}$	L^3T^{-1}
Non-dimensional parameters	D^a	ρ^b	N^c		
	Values of exponent				
Π_2	-3	0	-1		$\dot{Q}/ND^3$
Π_1	-2	-1	-2	$\dfrac{\Delta p}{\rho N^2 D^2}$	

For $\dot{Q}$, we get value of a as -3 and value of c is -1. So that $\dot{Q}/ND^3$ is a dependent pi number that is formed. This involves a bit of algebra but it is quite simple. For the dependent parameter Δp, the dimension $M\,L^{-1}\,T^{-2}$ are matched with L^a, $M\,L^{-3}$ raised to power b, T^{-1} minus 1 raised to power c. And by matching the dimension we get the value of a as -2, value of b as -1, and value of c at -2, so that the non-dimensional pi number, dependent pi number, comes down to $\frac{\Delta p}{\rho N^2 D^2}$. I have marked with a rectangle the independent parameter and the other one Π_2 is the dependent parameter, marked in red. $\frac{\Delta p}{\rho N^2 D^2}$, the pressure developed by the pump is usually expressed in terms of the head H of the fluid, such that Δp is $\rho\,g\,H$. So Δp is replaced by ρgH, where H is the head developed by the pump. Usually we do not express pressure developed by the pump, we express a head developed by the pump.

Using this a non-dimensional dependent parameter becomes like $\frac{gH}{N^2 D^2}$. So for Δp I substitute ρgH, and the dependent parameters becomes $\frac{gH}{N^2 D^2}$. And that is a function $\dot{Q}/ND^3$. The parameter $\frac{gH}{N^2 D^2}$ is termed as head coefficient in pump engineering, and the parameter $\dot{Q}/ND^3$ is termed as the flow coefficient C_Q such that C_H now is a function of C_Q. The dependent non-dimensional parameter is function of only one independent parameter and of course the geometry of the pump which we have not written but is understood all through.

So all pumps will have to be geometrically similar. This curve shows the cumulative data for various pumps operating at various speeds and with various flow coefficients. Notice that in

this area for low flow coefficients, for low discharge rates, for relatively low discharge rate all the curves for different RPMs collapse into one curve. The deviation is only on higher flow rates. What does it indicate? It indicates that our prediction for the performance of pumps is valid for the domain that we considered. The deviation is at higher flow rates then the flow velocities would be larger and larger flow velocity would lead to cavitation.

Since we had not considered the effect of cavitation in our formulation, obviously, this was to be expected. The deviation from a single curve from the predicted curve is only where the cavitation is suspected, and we do not operate our pumps in this region because of the damage that the cavitation and the resultant collapse of vapor bubbles results in the impeller.

A problem involving gravity

Two-body orbits

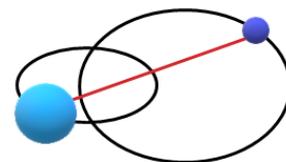$$F = G\frac{m_1 m_2}{r^2}$$

$$\tau = \mathcal{F}(m_1, m_2, r, G)$$

Basis set: m_1, r, G

	M	L	T
τ			-1
m_1	1		
m_2	1		
r		1	
G	-1	3	-2

$$\tau\left(\frac{m_1 G}{r^3}\right)^{1/2} ; \frac{m_2}{m_1}$$

	m_1	r	G
τ	1/2	-1/2	3/2
m_2	1		

$$\frac{\tau^2 m_1 G}{r^3} = \mathcal{F}\left(\frac{m_2}{m_1}\right) \qquad \tau^2 = \frac{r^3}{m_1 G}\mathcal{F}\left(\frac{m_2}{m_1}\right)$$

Let us do one more example. In this example we take two body orbits. Two bodies are orbiting under the influence of the gravitation force between them. The force of gravity is given by G times m_1 m_2 divided by r^2, where r is the distance between the two, m_1 and m_2 are the masses of the two body and G is the universal gravitational constant. So time period τ of the circular orbits would be a function of m_1, m_2, r and G. We construct the table showing the dimensions in MLT system.

τ, the independent parameter is written in red. m_1, m_2 r and G. The dimensions are easy to verify. The dimension of G, the universal gravitational constant, is obtained from this equation of the gravitational force. We use m_1, r and G as the basic set, and as we do this the dimension of τ comes out to be m_1 `raised to power one half, r raised to power −1/2, and G raised to power −3/2. And of course the dimension of` m_2 would be simply m_1.

$\frac{m_2}{m_1}$ is dimensionless. So that the two parameters, the dependent pi number would be

$\tau\left(\frac{m_1 G}{r^3}\right)^{1/2}$ and $\frac{m_2}{m_1}$. The first one would be the function of the second one. We can manipulate

this to write τ^2, the time period, would be like $\frac{r^3}{m_1 G} F\left(\frac{m_2}{m_1}\right)$. Santiago has plotted the orbital time period or the square of the orbital time period τ^2 against the cube of the semi-major axis in astronomical units for the solar system. The third one from the left is our earth as all these are the planets of the solar system.

Planetary motion around the Sun

$$\tau^2 = \frac{r^3}{m_1 G} \mathcal{F}\left(\frac{m_2}{m_1}\right)$$

$$\tau = 2\pi \frac{r^{3/2}}{\sqrt{m_1 G}}$$
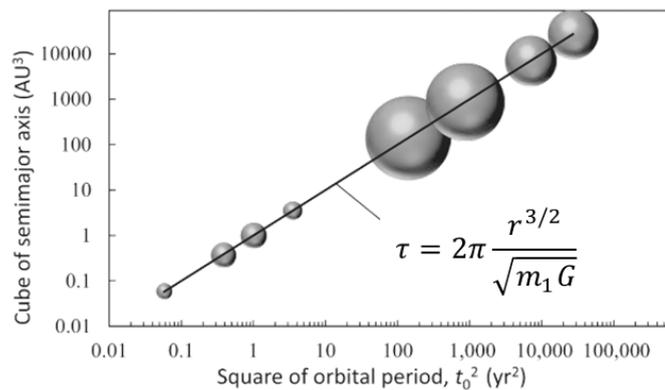
[Santiago, J G, *First Course in Dimensional Analysis,* MIT Press, 2019

They are on one line as was expected using the mass of the system. Let us do a little mathematics. τ^2 is this since it should not matter as what we call body one and body two. So that by symmetry $\frac{r^3}{m_1 G}$ function of $\frac{m_2}{m_1}$ should be like $\frac{r^3}{m_1 G}$ of $\frac{m_1}{m_2}$, reversing the role of m1 and m2. And if we do this, we get function of $\frac{m_2}{m_1}$ should be equal to $\frac{m_1}{m_2}$ into function $\frac{m_1}{m_2}$.

That is, f(x) is like 1/x, $f\left(\frac{1}{x}\right)$. One solution of this functional equation is if $f(x) = \frac{C}{1+x} \cdot \frac{C}{1+x}$ satisfies this functional equation. And so that τ^2, the time period of the planetary motion can be written as some constant $\frac{cr^3}{G(m_1 + m_2)}$. This is the Kepler's third law of planetary motion, obtained without much algebra, without understanding the physics completely, without solving complex equations which Kepler did.

So, we are able to obtain solutions. This is the power of dimension analysis.

Thank you.