Evolutionary Dynamics

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Week 9

Lecture 44

Thank you. Hi, welcome back, everyone. In this video, we will work on the traveling wave model of adaptation that we looked at the end of the last video, and we will derive the speed at which that wave is moving toward the right. This video is perhaps—no, not perhaps—this is definitely the most mathematical topic we are going to discuss in this course. So, we will take it slow, but even if you don't get the precise math of it, what would be nice is if you paid attention to just understand the logic of what is being talked about.

And as far as that, I'm able to communicate the logic and geometry associated with what is happening in the population. I think that's a good step toward understanding evolutionary processes. So, let's go back to the traveling wave and set up this model. At a given instant in time, the population structure looks like this. Some structure.

It really doesn't matter what the precise structure is. Now, in this context, this is fitness, and this is the number of individuals. The total population size is N, and we are looking at this traveling wave where, in tau time, The lagging edge is rendered extinct by selection. So, a few things happen in this tau time.

And what we are interested in is the speed of this wave at which it's moving to right. which is also the speed of adaptation, speed of how mean fitness is changing with time and so on and so forth. So in tau time, a few things happen. First, lagging edge goes extinct. Lagging edge goes extinct.

So this goes extinct in tau time. Of course, this is also decreasing, but it doesn't go extinct in tau time. Second, new leading edge is established. What that means is that although individuals of this genotype increase in frequency because their fitness is much higher as compared to the population mean, but in this tau time, beneficial mutations kept on happening and in tau time, one of those beneficial mutations was able to establish itself in the population and this becomes the new leading edge. And so these two changes take place at the two extreme ends of this distribution.

In between the frequency of other genotypes. So this is happening at extreme ends. of the distribution in between frequency of other genotype changes, genotypes change to give the impression as if that in this tau time, the entire distribution moved rightwards by S naught amount. They change and it's as if that the entire distribution moved one step to the right.

And this one step in our framework is S naught unit because we are saying all beneficial mutations confer this S naught benefit. So, now this we saw at the end of last lecture. This meant that in this tau time, These three combined mean that the mean population fitness increased by S naught in tau time. As a result of that, we can say that the speed or the rate at which mean fitness is increasing.

is simply equal to S naught divided by tau. S naught is the amount by which the mean fitness increased and tau is the time in which it happened. So, just as you have speed of a car as distance upon time, the speed of increase of mean fitness is simply by how much did the fitness increase and divided by how much time it took to increase that fitness. So all of this is going on in this context. Now, S naught we know because we know the nature of beneficial mutations.

We know that most S naught values are around 0.03, 0.04. So we have a good estimate of S naught from experimental data. It's this tau that we have to worry about. So our exercise in this video is going to be that how do we find this value of tau so that we can estimate the speed at which this wave is moving.

And to find tau, we will take two different approaches. First is that this is the population mean and we say that the fitness of this population mean is 1. this as of this moment because this becomes leading edge after tau time as of this moment this is leading edge And this leading edge is K beneficial mutations ahead of the population mean. Although I have drawn as if this is 1, 2, 3, 4.

Although I have drawn as if the leading edge is 5 beneficial mutations ahead of the population mean. I have no basis of doing that. I have no idea what is the value of K. So, I am going to say that the leading edge is K beneficial mutations ahead of the population mean what might be that value of K is something that I do not know yet. So, I am going to look at this travelling wave in two different ways.

and we are going to solve for both tau and K. So, at the end of this exercise, I will have an expression for K and tau and get an estimate of how this travelling wave, what is the width of this travelling wave and what is the speed of this travelling wave at which it is moving to the right. That is the goal that we have for this video. So, how do we do this? We are going to look at two different aspects of the travelling wave. Let me just draw this again.

Let us say this is population mean. This is leading edge. I am going to look at this travelling wave in two different ways and I will come up with two different equations and I need two equations because I am solving for not one but actually two variables k and tau. Although I am only interested in tau to be able to solve the speed problem but as I will see I can't solve tau without knowing the value of k and hence I have two unknown variables tau and k. And hence to get these two unknown variables, I can't get that from one equation.

I need two equations to be able to solve for two unknowns. And that is why we are going to be doing two different equations. So the first question we ask is how long will it be before before leading edge is no longer leading edge. And the answer to this is in one sense simple because we have said that the traveling wave moves one unit to the right every tau time.

So, leading edge will no longer be the leading edge after tau time right because after tau time we have been saying that this continues to increase in frequency, but in tau time it will pick up one beneficial mutation which is able to survive drift and get established in the population and hence the current leading edge will no longer be the leading edge after tau time. So, this is tau, but in this exercise we are going to see if we can derive an expression for this tau time. So we are going to derive an expression for tau. Okay, so the first question that in the leading edge, how many individuals are there?

of this moment how many individuals are there are there in leading edge now and the answer is 1 upon KS Because remember, the advantage that leading edge enjoys over the population mean, this fitness is 1, this fitness is 1 plus K times S0. So, the advantage that it enjoys over the population mean is K times S0. And we had discussed that if the advantage that you enjoy over the population mean is S0, once you have 1 upon S0 individuals, then you have escaped the clutches of drift and your established and selection will drive you home to fixation. However, in this case, the advantage is KS0.

And since we are saying that our leading edge is fully established, that means its numbers must be 1 by KS0. So that we can say confidently that it has escaped drift and this is my leading edge. So the number of individuals is KS naught as once it gets established. Next question, what is its advantage in fitness over population mean? This one is simple.

Population mean is 1. Leading edge's fitness is 1 plus K times S naught. So, its advantage over the mean is simply K times S naught. So, this is just equal to K times S naught. This is important because the rate of increase, the rate at which this frequency is going to increase, this rate is proportional to its advantage over population mean.

If the advantage is huge, then its frequency is going to increase faster. If the advantage is only just a little bit over the population mean, its frequencies are going to increase much slower. So, what that means is that its numbers are going to increase as 1 upon KS naught. This is the initial number of cells that are present. So, let us call this N naught into e to the power KS naught.

T. This is the numbers as they will increase here. However, mutations are also happening at the rate mu. But these mutations are not able to survive drift and this is going to be mu B. But these mutations are not going to be able to all survive drift. Most of them will be lost because of drift. So what is the probability that these beneficial mutations are able to survive drift?

If a beneficial mutation has a selection coefficient of S0, the probability that it survives drift is also S0. In this case, when this mutation occurs, That becomes the new leading edge. Its advantage over the population is also Ks naught because the population mean has shifted by that time to this point. So this is also Ks naught.

So mutations occurring at the leading edge are Ks naught above. So the probability that it will survive drift is also Ks naught. So this is multiplied by the probability of survival of drift. And I am going to integrate this from 0 to tau dt. And this expression gives me how many beneficial mutations occur and survive drift in the leading edge.

Let me go over this again. This is the number of individuals in the leading edge. Let me mark the current leading edge as red. So this is the current leading edge. The number of individuals in the current leading edge is one upon Ks naught because these are individuals that have already survived drift and established themselves as the leading edge.

This number is going to increase exponentially as Ks naught into T because Ks naught is the advantage that this leading edge has over the population mean. Ks naught times T. But I am not interested in the rate at which this frequency is increasing. I am interested in the mutations that are occurring in this leading edge. Hence, I have to multiply with mu B. But most of the mutations that occur will not survive because of the action of drift.

What is the fraction of these beneficial mutations that will survive? That is simply equal to K times S0. Hence, I multiply it with this to indicate what fraction of mutations will survive. And then I integrate this over 0 to tau. Tau is the time over which the new leading edge is established and I integrate it from 0 to tau.

And I am interested in solving for tau because when this new leading edge is established, so let me use another color. When this new leading edge is established, the number of mutations that have established is equal to 1. Because when one of these mutations establish when one of these mutations establishes itself then my current leading edge is no longer leading edge and hence I will equate it to 1. That is the time tau that I am interested in when the current leading edge is no longer leading edge.

So when I solve for tau this equation, this Ks naught cancels. Let me write this again. So I have 1 equals 1 upon Ks naught, 0 to tau, e to the power Ks naught into t, mu B, Ks naught dt. This is just copied from the last slide. This Ks naught cancels.

$$1 = \int_0^{\tau} \frac{1}{Ks}\, \mu_b\, Ks\, dt$$

$$1 = \int_0^{\tau} \mu_b\; dt$$

I integrate it, I get 1 equals 1 upon Ks naught mu b comes out e to the power Ks naught t. I apply the limits 0 to tau and I get Ks naught by mu b is equal to Ks naught tau minus e to the power Ks naught into 0 which is just 1. And if k is sufficiently large, this is just can be approximated at e to the power k s naught into tau. So, I get this equation that k times s naught times mu b is approximately equal to e to the power k s naught into tau. And as you can see from this equation that I cannot solve for tau. Remember, this entire discussion is so that we can get an estimate of tau so that I can get the speed of the wave that is traveling towards the right.

But as you can see, I can solve for tau here, but I also need k. Without k, I cannot do it. So, this is equation 1. Now, we follow the other approach. Where we ask we need another equation because we have two equations and we need we have two variables hence we need two equations. In the other equation we ask the following question.

In how much time. will leading edge become population mean. So leading edge, this is the leading edge, which is Ks naught. So in tau time, as this is moving towards the right, its numbers are increasing. In tau time, it will reach this number.

In 2 tau times, it will reach this number, and so on and so on and so on, right? So, for the first—so the way we will approach this is that for the first tau time, first tau time, what is the advantage that the leading edge has over the population mean? And the answer to that is k times s naught because this is 1, this is 1 plus K s naught. So, the advantage of the leading edge over the mean is 1 times K s naught. So, for the first tau time, its growth in numbers will take place as 1 upon K s naught.

This is from the last equation: e to the power K s naught into tau. After this tau time, what has happened is that numbers—this wave has changed, and this number has gone to this. There is a new leading edge. This number has gone to this. This number has gone to this.

This has gone to this. This has gone to this. Then, the old leading edge has now decreased to this. This one has decreased to this, this, this, and this has been extinct. So, everything has moved to the right, and now we are looking at the right—this red-colored wave.

So, in the next instant of time we for the next tau time what is the advantage that remember we are asking that how long will it take this to become highest in frequency it will reach this number. So, for the next tau time, what is the advantage that this has over the population mean which has moved here? And the answer now is K minus 1 times S naught. It is no longer K times S naught because K S naught is the advantage that leading edge enjoys over the population mean. This genotype is no longer leading edge now.

Hence, its numbers will grow as its starting numbers were e to the power k s naught e to the power k s naught tau, which was in the previous time. Now, for the next tau time, it will grow as e to the power k minus 1 times s naught tau. And so on and so forth, it will keep on happening. So eventually, by the time this wave is moving to the right and the numbers associated with this genotype become so high, the expression will be Nt times is equal to 1 by ks0 into e to the power ks0 tau times e to the power k minus 1 s0 tau. going all the way to e to the power s naught tau.

$$N(t) = \frac{1}{Ks} e^{Ks\tau} e^{(K-1)s\tau} \dots e^{s\tau}$$

$$N(t) = \frac{1}{Ks} e^{K\frac{(K+1)}{2}s\tau}$$

$$N \sim \frac{1}{Ks} e^{K\frac{(K+1)}{2}s\tau}$$

And we can sum all these exponents and we get this as 1 by K times s naught into e to the power k into k plus 1 by 2 s naught into tau. So now the question is that what is what is this left hand side? How many cells are there in this state when this fitness level is the most abundantly represented in the population? And if the total population size is n, what experimental data has shown that this can be successfully approximated as simply n by 2? About half the cells are in the travelling wave model that I have.

If this is the distribution I get, about half the cells are present here. And if I plug that here, this is my second equation where the only unknowns I have are K and tau. This, coupled with the first equation—where again the only unknowns are K and tau—I can solve these two equations for the values of K and tau. And this sort of forms what we will do in the exercises for this week. We will have several values of n, s naught, and in this case, mu b, which will give us a sense of how wide these distributions are and what the typical values of K and tau are. That will give us an idea of how fast these waves are moving to the right.

So this was slightly mathematically involved. I hope you got a sense of this wave moving to the right and how we have these two approaches to analyze the wave. In the first one, we asked how much time it would take for the leading edge to become the population mean. And in the first one we discussed, how long it would take for the leading edge to no longer be the leading edge. At the end of this, we have two equations and two unknowns, which we'll solve together.

So that concludes this mathematical part that I wanted to discuss. Next up, we have one more topic in one lecture. After that, we enter the final phase of this course, which is purely experiments. We'll continue these discussions in the next video. Thank you.