Engineering Statistics
Professor Manjesh Hanawal
Industrial Engineering and Operations Research
Indian Institute of Technology, Bombay
Lecture 46
Unbiased Test, Uniformly Most Powerful Test, Neyman-Pearson Lemma, Interval Estimation

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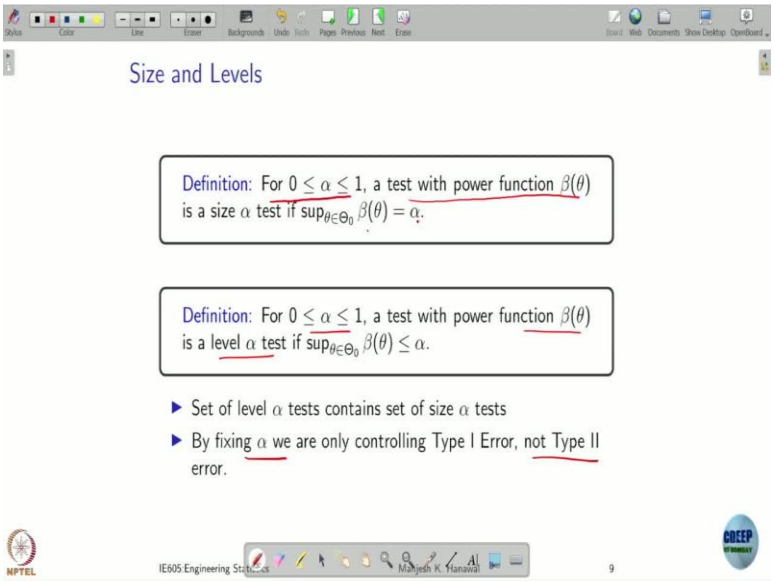

We are now going to introduce couple of definitions. Suppose some alpha is given to you, which is between 0, 1. Now, a test with power function b of theta, we are going to be call it as a size alpha test., if its maximum value over my null hypothesis is alpha, you understand what I am talking about. So, what is this? When I look beta of theta over theta, which is coming from null hypothesis, this is going to give me what? Type 1 error.

So, basically, when I say teste is size alpha test, that means it is type 1 error maximum value of type 1 error is alpha. A little generalization of this definition is, suppose I have alpha between 0, 1. And I have a test with power function beta of theta I am going to call this test level alpha test if its maximum value over my null hypothesis is less than or equal to alpha, basically, I am saying that type 1 error has to be less than or equals to alpha.

Now, if I give you a test, which is level alpha, is it also size alpha? Yes. We do not know. Like when I say level alpha is less than or equals to alpha, it is not exactly Alpha. But on the other way, if I do a test, which is size alpha, will it be level alpha test? That is true.

Now, here because these are related to the type 1 errors by fixing my alpha I am only controlling type 1 errors in this definitions for the tests there is nothing I am talking about type 2 errors in these definitions.

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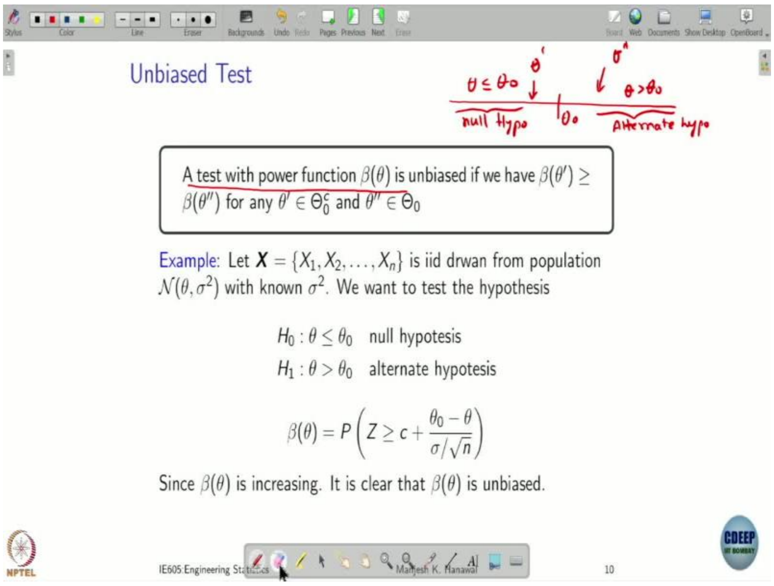

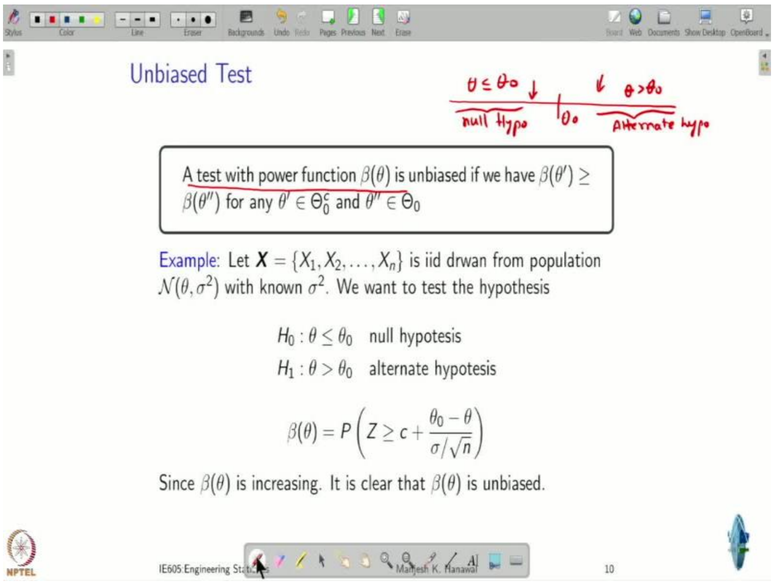

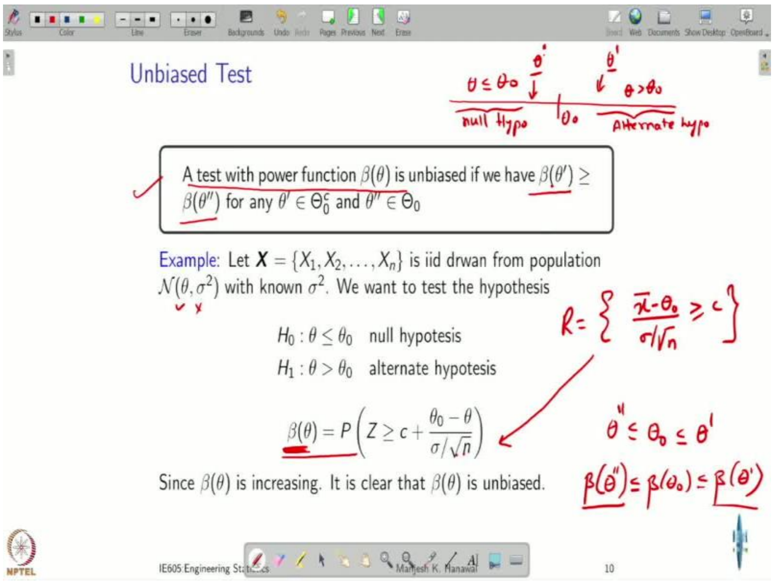

Is the definition of size and level of a test clear to you? Now, I am going to talk about one more test. We talked about unbiased estimators. Now, let us talk about unbiased tests. We are going to say that a test with power function b of theta is unbiased, so to understand it, let us talk about a picture I am taking simple real line. Let us say this is my theta not. This when theta is less than or equals to this theta not let us say this is my null hypothesis.

And when theta is greater than this theta not, let us call this a alternate hypothesis. Choose theta prime from this region and choose theta double prime from this region or alternate way I am missed I swapped it choose theta prime coming from an alternate hypothesis and choose theta double prime coming from a null hypothesis.

Now, if you have a test. Let us say if sample is coming from this theta prime that is alternate hypothesis. And it is our another possibility that sample is coming from this null hypothesis under which parameter that your rejection probability should be higher. The rejection probability should be higher when it is coming from alternate hypothesis you do not want to accept it belonging to the null hypothesis when the underlying parameter is already theta prime it should be higher.

On the contrary like if it is parameters from theta prime it being rejection should be smaller. So, that is exactly Unbiasedness properties trying to capture. It is saying that the probability of rejection of a sample that is coming from alternate hypothesis should be larger than probability of rejection of a sample that is coming from null hypothesis if that is the case that means basically what it is saying is you are tested such that it has more tendency to reject samples which are generated from the alternate hypothesis parameters that is what you expect your tests to be. If that is the case, then you are going to call it as unbiased test.

So, let us see whether the test we found for the Gaussian samples was unbiased or not. So, in the Gaussian samples, which are coming from some parameter theta sigma square where as usual we assumed theta is known and theta square is not known. For that your power function was this the LRT the constructed.

So, the LRT the LRT we construct that has a rejection region like this, what was that x bar minus theta divided by sigma by square root and being large that was theta naught being greater than or equals to 0. We had something like that, which get translated to this power function which we calculated in the couple of slides back.

So, this is a power function we have. Now, let us try to see that if I have a power function, I should be able to say whether it is a biased or unbiased test. Can you check whether this is an unbiased test? Apply this definition and check why is that1 minus 5, no I am I have to tell this in terms of my beta function power function. For my test here, I have given you my power function. And using this I need to tell whether it is a biased or unbiased tests.

So, that definition only requires you to use a power function nothing else.

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So, if theta increases here, beta theta is increasing. How does that conclude this is happening? This is not our text this is about the thetas parameters. Why is that? See, now, I have to check. So, I will take two values, let us say I take theta prime double prime from here, which is smaller than theta, theta not and I take theta prime which is larger than theta.

Since this b, now I know that theta prime is less than or equal to theta not and this is less than or equal to theta prime. Now, if beta is monotonically increasing, it must be the case that beta of theta double prime must be less than or equals to beta of theta less than a request a beta of agree if beta is monotonically increasing. That is it, we proved.

We have shown that beta of theta prime is greater than equals to beta of theta double from. So, the test we had for the question samples based on our likelihood ratio test is indeed unbiased. And notice that this is this works only for some C and n value or any C and n value.

It holds for any C and n value 2 all we want is monotonicity. Irrespective what is C and n this beta of theta function is monotonous.

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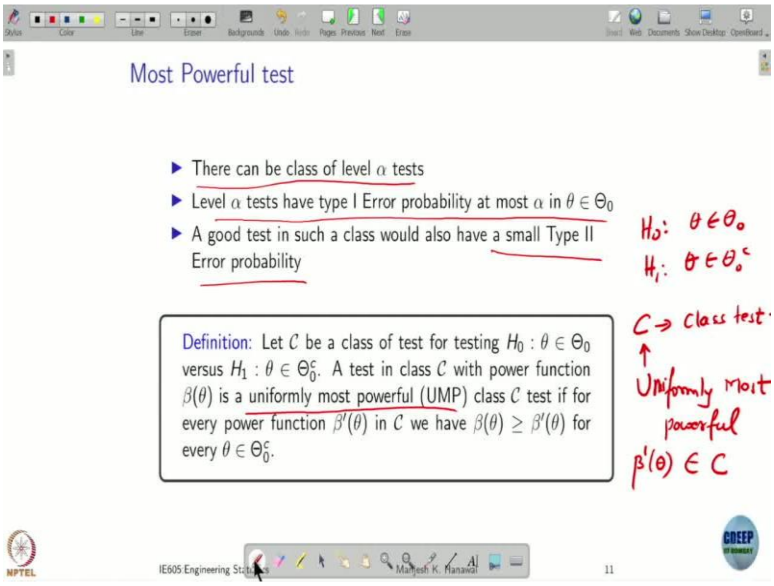

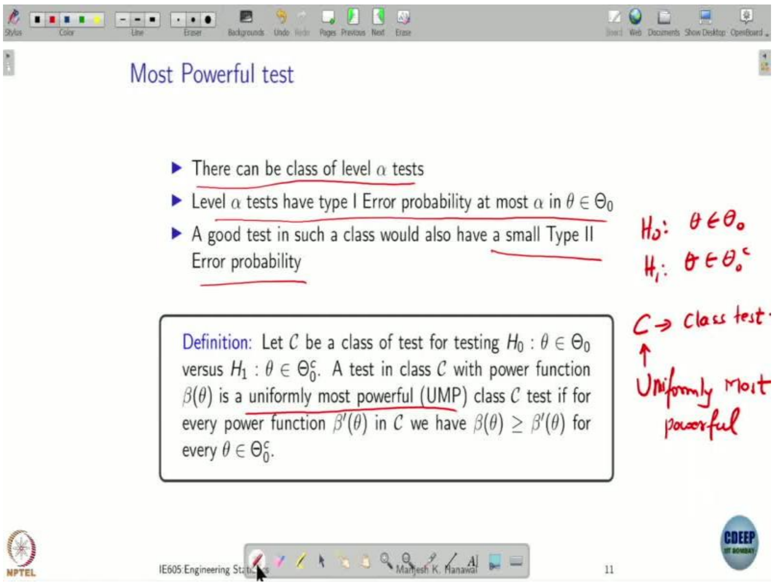

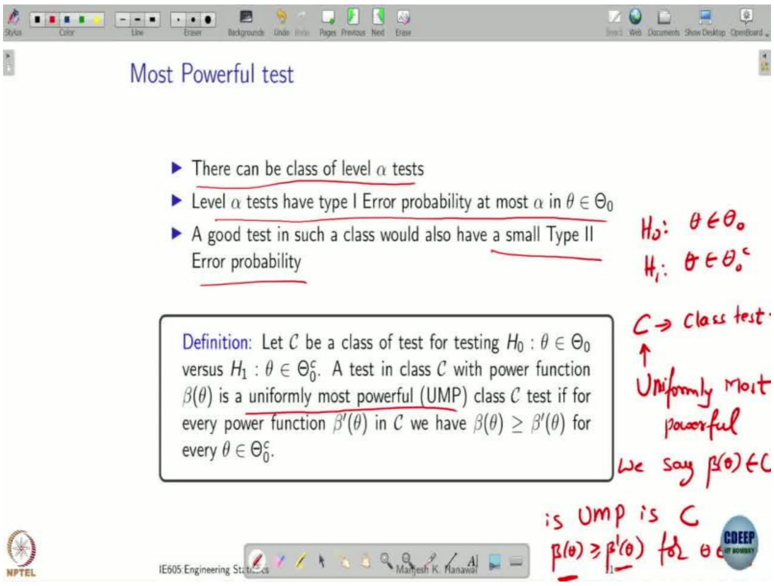

Now, this leads us to further look for the most powerful tests. So, again if you have to draw analogy with your estimators, how did we do estimators? First we started we are looking for estimator which were unbiased and among the unbiased estimator what we are trying to look the one with minimum variance we wanted someone with a minimum variance. Maybe that kind of analogy can also be here like of course, we want a test which is unbiased but now, the variance analogy is now captured through our type 2 error and type 1 error.

Now, let us say we have a class of level alpha tests I have many tests which are like level alpha, their type 1 error is alpha nothing more than that. So, we know that for level alpha type 1 error is at most theta. Now, first of all, if you have a level alpha test, I know that my type 1 error is not going to be more than alpha. But now, I want the level alpha test I want to consider our test to be good which has the smallest type 2 error among them. So, that leads to the definition of uniformly most powerful class which we define as follows.

Suppose, we have a test like this is we have a test where theta is I mean, we have this as usual null hypothesis and we have this alternate hypothesis let us say we have some C is a class of tests for the hypothesis and we are going to call this C to be uniformly most powerful if you take any beta that is you take any test in C, now every test we are associating them with that power function. You take any test with power function beta of prime of theta and C if we say wait a minute we have to replace this.

Now, we say some beta of theta belonging to C is uniformly more powerful test in C provide at if you take this beta of say a should be equals to beta prime of theta for theta belongs to theta C complement. What does that mean? This has a higher probability of rejection than any test in that class when my theta is coming from my alternate hypothesis. But it is saying that the whenever theta is coming from alternate hypothesis my probability of rejection is the highest in my class that guy is called uniformly the most powerful test in that class.

So, if you have a bunch of test and you give it a sample which is generated from an alternate hypothesis, if some tests rejects it, it did a good job because it is coming from your alternate hypothesis. But that UMP, if this guy has rejected that is also very likely very likely that it will also reject, because the probability of rejection is higher than probability of rejecting reject probability of projection of this guy, the test, we have selected.

In that way that is good in my set. And that is why I am going to call it as most uniformly most powerful, not only most powerful, but it is uniformly most powerful test.

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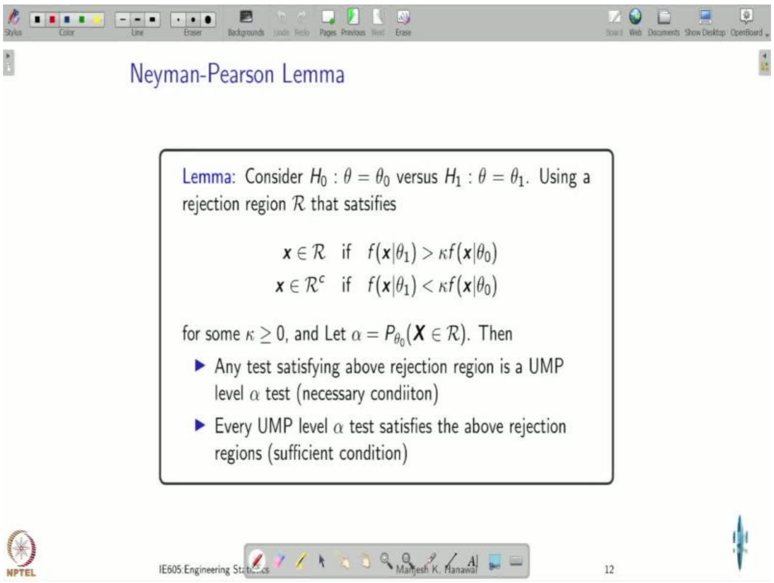

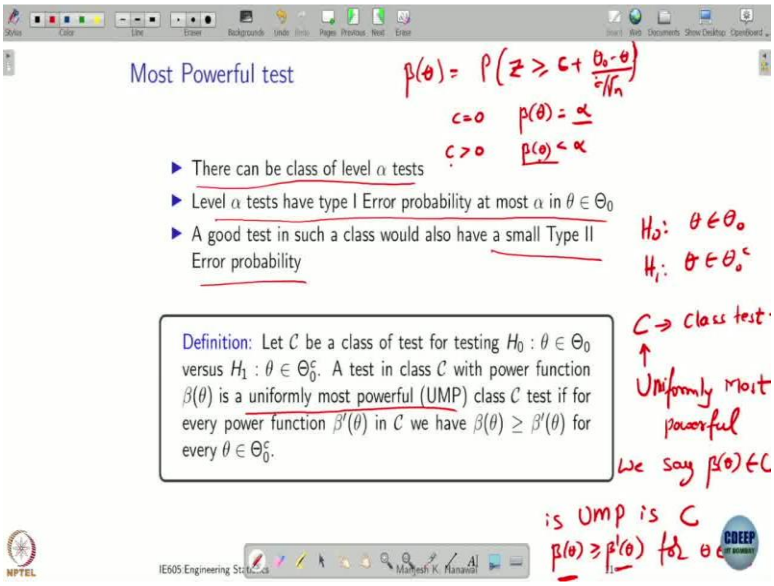

Now, the question comes fine, this is all good. I like unbiased tests. I like powerful tests. I was like, sorry, I like uniformly most powerful test by the way, did we decide we just decided discuss unbiased test, then we talked about most powerful test. And now is there in a way, what we are looking is a good test in my class of test. That is my most powerful test or other like uniformly most powerful test.

Now, the question is such a test always exist? See so, is it possible that you can come up with multiple tests level alpha tests? Just let us briefly discuss support, let us say I have this I have this greater than or equals to C plus, what was that theta minus theta sigma by square root n this is my I can choose from C and n and I will get some test. Let us say if you fix everything, let us now focus on C itself. Let us take C equals to 0. And whatever this guy gives me, wrote, let us call for C equals to 0, let us call this as some alpha itself, let us call alpha for c equals to 0.

Now, if I go into increase the C now I will make C greater than 0. What will happen to this probability it is going to be less than alpha now. So, what I can do is, by just choosing different, different C I can come up with so many tests. Like if my Cs are only allowed to be positive numbers, I know by setting 0 I am going to get some value type 1 error, let us call this alpha, and by choosing any C greater than 0, they are going to be all test are going to be less than alpha.

So, all of them are going to be in can I call them that belonging to a class with size level alpha by choosing different, different say, now I can construct a set class test, which are level alpha tests. Now, what I am looking at is among them, which is the one which is most powerful.

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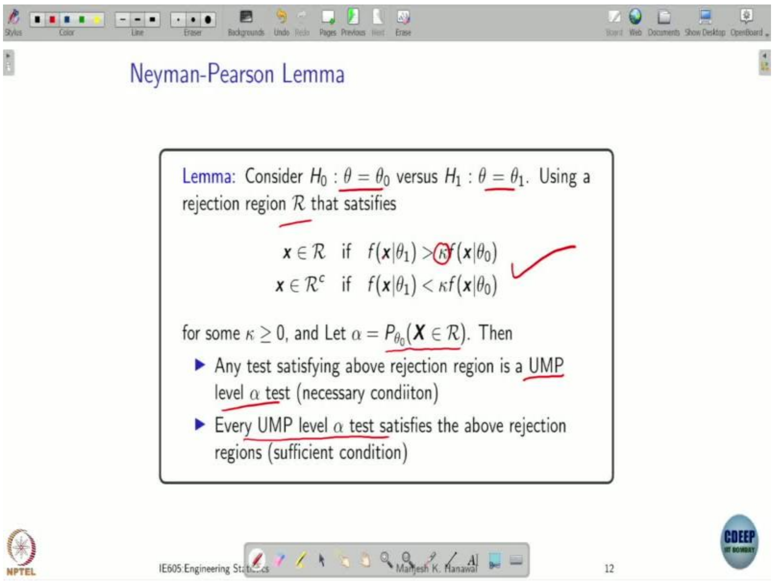

Now, this Neyman Pearson's Lemma is something which tells us when one can expect a most powerful test to exit exists and how does it look like. So, here is a simple version of the Neyman Pearson Lemma. Which only considers two simple hypotheses it only tries to distinguish whether my parameter theta 0 or theta 1 when it is theta 0, it is my null hypothesis when theta 1 is alternate hypothesis I just need to distinguish.

If your hypothesis test is there, and support let us say your rejection region is such that whenever you are sample x has a higher probability under theta 1. Then under parameter theta 0 I am talking probability but it is also density when it is a continuous case, then I am going to reject otherwise, I am going to accept. But there is some constant here, which is let say some constant.

So, you have defined you are rejection region now, in terms of the PDF or the probability mass function, whichever is applicable. And now let us, under this rejection region, I am going to define my alpha two be probability that my sample is going to be rejected and then my parameter theta 0 that is, it is my type 1 error. So, type 1 error I am going to call it as alpha.

Now, it says that any test which is with the above rejection region is going to be a uniformly most powerful test. If you are going to have a rejection region like this, then that is going to be uniformly more sorry most powerful test it is not just that it is uniformly most powerful level alpha test.

So, this is like a necessary condition. On the other hand, it says that you if you have any formerly most powerful level alpha tests or like it is saying that a very uniformly most powerful level alpha test what if you can come up with one it has to be like, look like this structure has to look like this.

So, it is basically saying how to construct that uniformly most power full test, and it is also telling you if somebody gives you a uniformly most powerful test how it will look like, and this is why this is your simple setting, but it is giving you a very nice characterization. So, here, this constant factor is missing, but if you can come up with a constant factor and have a rejection region like this, then it is going to be uniform, most powerful test, but under this simple case that you have just two parameters to test in your hypothesis testing setup.

So, any questions on this whatever we discussed so far about this most powerful test, and this unbiased? So, do you think anything else should be considered when we are looking into these tests? Because these are properties biasness is a property uniform most powerful being a property. Is there any other property that you think one should consider when we are defining such tests?

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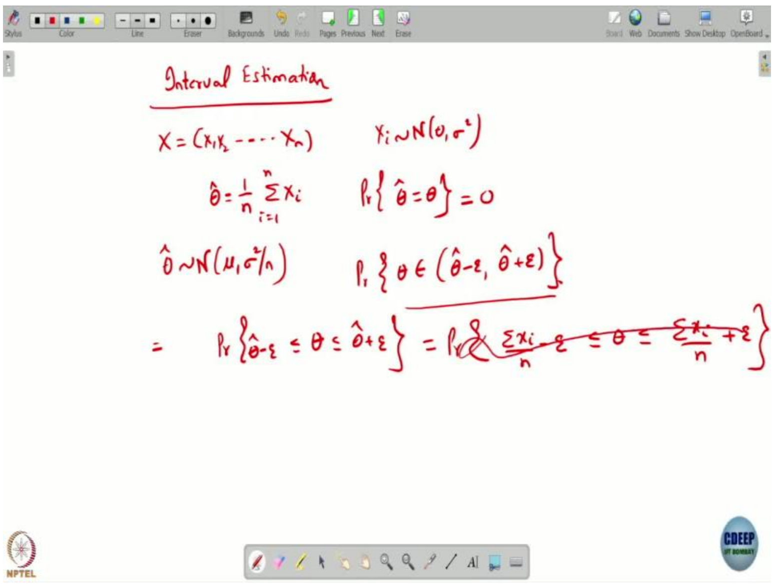

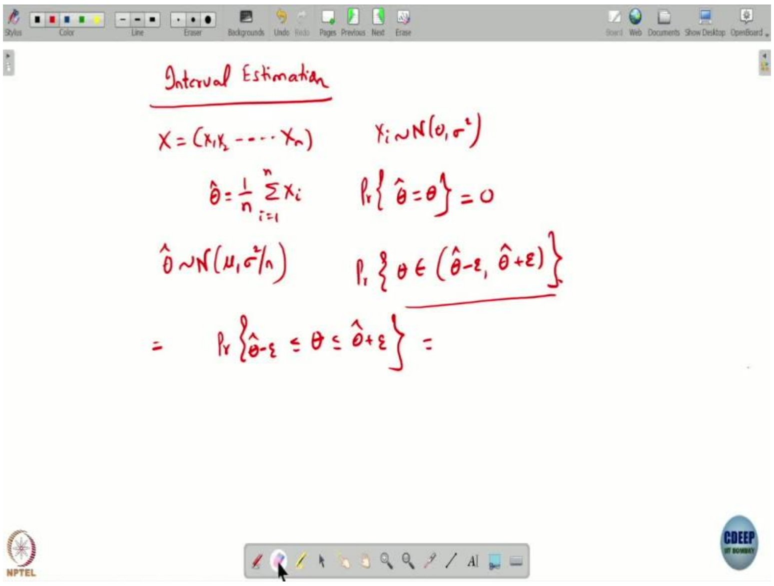

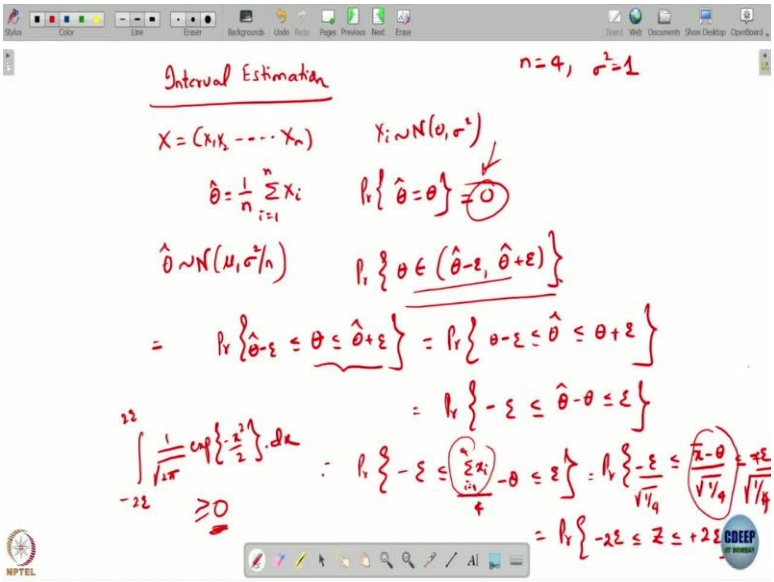

Now, just briefly discuss the next topic that we will cover in the next lecture called interval estimations. Now let us say you have these samples. Let us say they are all coming from Gaussian distribution with parameter theta and sigma square. So, what is the best estimator for theta? So, the best estimator for theta is sample. Now, what is the probability that theta hat is equals to theta that is your estimated value is exactly equal to the true value what is the probability.

What is this? Is it 0? Why is that? So, what is the distribution of theta hat? Normal with what parameters mu and sigma square by n. So, theta hat itself is a random variable which is continuous and continuous random variable taking any particular value, we know that is going to be 0. So, then you estimate your value theta to be theta hat a confidence that you have that is going to be the exact value is how much, you have 0 confidence in this because theta hat being exactly equals to being theta is 0.

On the other hand, if I say theta hat I will just say theta now belongs to theta hat let us say minus epsilon to theta hat plus epsilon. Can we calculate this probability? How? let us compute this. So, what we are basically asking is probability that theta is going to be less than or equals to theta hat plus epsilon and theta hat minus epsilon. And I know that this probability is simply summation of xi by n minus epsilon theta this is summation of xi by n plus epsilon or maybe I should have done something better instead of this I will get theta hat in between.

So, this is fine. I basically asking theta to be between theta hat minus epsilon and upper bounded by theta hat. Now, I want to write so, what is the random quantity in this theta hat is the random quantity. Is theta being random quantity? No, that is a fixed parameter. So, now let us try to write so, now I know that this theta hat from this quantity is going to be less than epsilon and this is going to be theta plus epsilon I can do this.

And let us do this probability that theta minus epsilon, what is this and I can further take this theta hat is summation xi by n or maybe before this I will do one more step, probability that minus epsilon greater than or equals to theta hat minus theta less than or equals to epsilon. For time being let us take only n equals to 4 sample and sigma square to be 1-unit variance. Now, minus epsilon x bar is basically xi I want to 4 divided by 4 minus theta minus epsilon. Now, what I will do is probability that minus epsilon that is this is simply x bar minus theta divided by I am going to do 1 by 4 minus epsilon 1 by 4 and plus 1 by square root of 4.

So, I have basically this quantity are written as x bar and everything else I have divided by square root of 1 by 4. So, what is square root of 1 by 4 here it is equivalent to sigma square by r like sigma by square root n which I want. Now, because of that what I can say about this that is now, I can write it as probability that minus epsilon this is going to be 2 epsilons this is now z this is going to be plus 2 epsilons. Now, that is a standard normal I should be able to compute this probability. So, this is basically minus 2 epsilons 2 epsilon 1 upon 2 pi exponential. This is a 0 thing minus x square by 2.

Now, will this be positive quantity this is going to be positive quantity. So, this is not going to be 0 like this. Earlier when you are asking exactly that to be theta plus 0, but now, instead of that you are asking whether it will be in this interval you are going to get some positive quantity. And now, we are going to study that instead of asking you exactly this value. We want your estimation to be lie in some interval and C that with that our confidence will be any better. Here our confidence was 0. But here we see that already our confidence can be positive. So, let us stop here. We will continue this interval estimations or confidence interval settings in the next class.