Engineering Statistics

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Week 01- Lecture 04

Total Probability law and Baye's theorem - I

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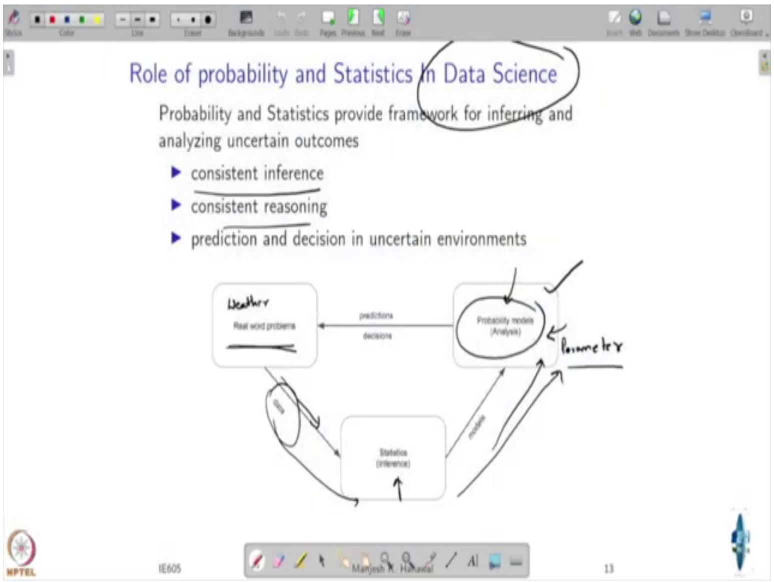

Last time we were talking about interplay between probability, statistics and data science. We said that real word is going to have some underlying model, probability model which we invite to capture, so that is where we try to come up with some probability models. And then from the real world we observe some data and we analyze that data and do some statistical analysis on that to get the parameters which will further go and help us improve our probability models.

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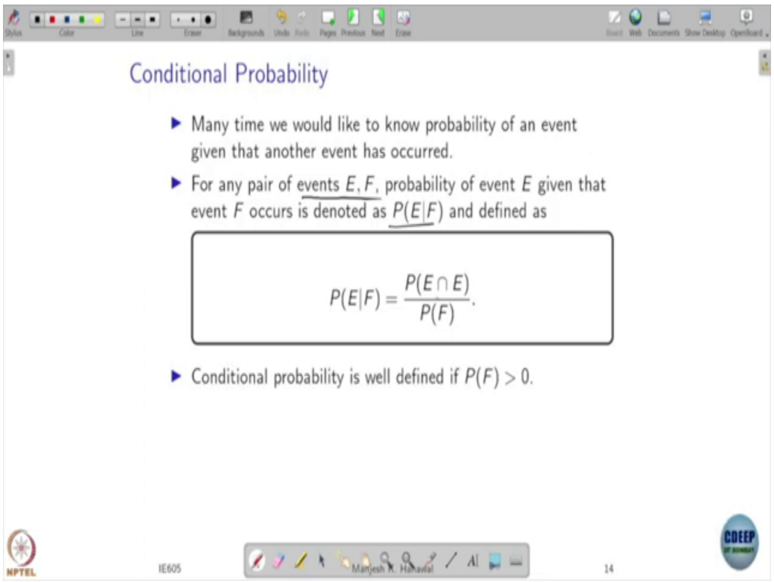

So, that was just the interplay but now let us get back to our study of the probability. So, I want to really go through quickly because some of this as I said Ms. Kavita is also covering and now I will not get into any example, so I will just go through these notions. Those of you have any issues, yeah just raise your hand we will discuss that.

Student: (())(1:15).

Professor: I should go beyond her.

Student: (())(1:20).

Professor: Then I will be just ending repeating the same thing.

Student: (())(1:31).

Professor: Okay, let us see, so maybe I will go now start maybe after 10-15 minutes I will ask you the pace and you can tell me. Okay, conditional probability. What is conditional probability like let us say you have two events and I want to know event happening, one particular event happening after you know that another event has already happened.

For example, if you take two events E and F and probability of event E given that F has occurred we are going to write it as P E given F and this event is defined in this way probability E given F is ratio of two quantities, the numerator is probability of the intersection between E and F divided by probability of F. And naturally this definition going to make sense then probability of F is greater than 0, otherwise this ratio is not defined.

And also, you do not want to condition on an event for which likelihood happening is 0 you want to condition on the things which have some positive probability of happening and based on that you want to see how the what is the probability of other events, so this makes sense.

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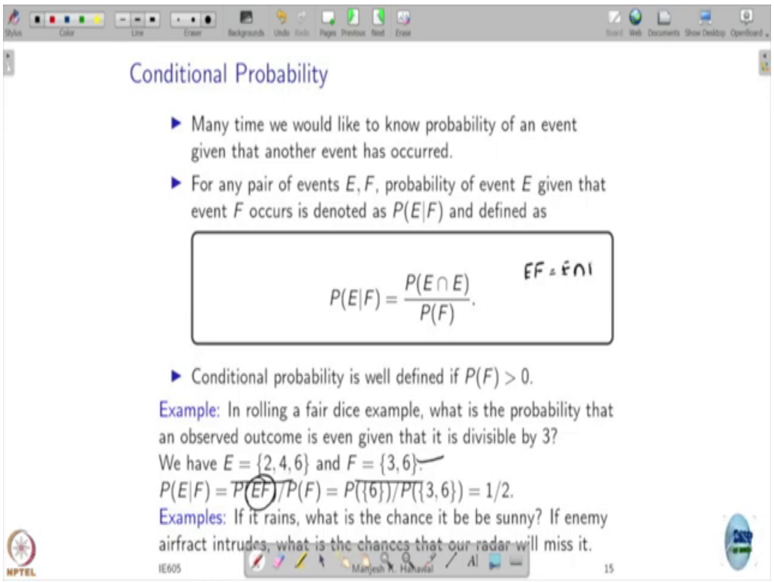

For example, if you are going to roll a dice, a fair dice and you are interested in two events, the first event is outcome is even, the second event is divisible by 3. Now, if you want to condition, want to understand okay event E, F has happened what is the probability that event E is going to happen?

Now, if you have been already told that say either 3 or F has happened now what is which one of them is divisible by 2 and which of them is even like the even, that event E will happen only if out of 3 and 6, 6 has happened so if 3 and 6 has happened you already have a you know what is the likelihood happening of E now, you know that 2 and 4 are not going to happen only possibility is 6 now your likelihood has changed about the whether you are going to see even number or not.

So, and now onward sometimes I will also use this notation instead of saying E ∩ F I may simply write it as E, F that already means that sorry what I am taking that it is going to be simply E and F that means already E ∩ F.

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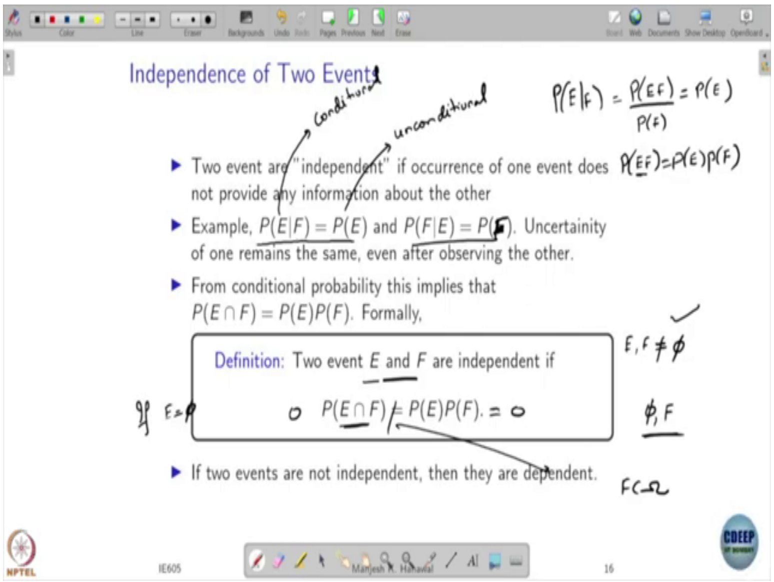

Now, with this let us we will get to the notion of independence of two events. We are going to say that two events are independent if occurring of one event does not provide any information about the other event. For example, I have let us say it is, if it so happens that probability of E given F is same as probability of E. So, you see that probability of, this is like an unconditional probability and whereas this is conditional.

So, the conditional probability is same as unconditional means happening of the event F is not providing improving anything about the probability of E and similarly if this is the case here also happening of E is not improving anything about my knowledge of F. So, whenever this happens in a way like observing this one event is not providing any information about the other maybe that means there is some kind of independence in this and that is where the independence comes into picture and what we are going to say formally is based on our conditional probability.

Student: (())(5:41).

Professor: That is right, that is the typo here. Now, if I go with my conditional probability P E given F this is equals to P(E ∩ F) divided by P(F) but if E and F are independent the left-hand side is simply equals to P of E so what this is going to say is now P of F is simply P of E into P of F. So, in a way like our intuition about independence is kind of implying that probability of E and F happening together is equals to the product of probability of E and F.

Now, that is what we are going to take as a definition of our independence, we are going to say that E and F are independent if the probability of their intersection is nothing but the probability of their, sorry is equals to product of the probability. Now, naturally if the two events does not satisfies this, we are going to say that to be dependent.

And one trivial thing we need to observe is whenever we are going to talk about E and F, obviously we need to assume that this E and F they are not equals to null set. If one of them is a null set what happens? So, this quantity is going to be null set if let us say is E is a null set, E intersection F is going to be null set, probability of a null set is 0.

Whereas on the other side also probability of null set is also 0, so this also becomes 0. If this is if let us say E is a null set but then let us say if I have E null set and some event F, does null set provides any information about F, event F happening or not? It may or not provide like I mean null set here that means if it is null set the complement that means omega has happened, F is a part of omega, F is a part of omega, so it may provide some information.

So, but according to this definition they are independent because both left-hand side and right-hand side is 0 so that is what when you are going to apply this definition we will make sure that both of them are not equals to a null set if it is null set it becomes a little bit vacuous definition.

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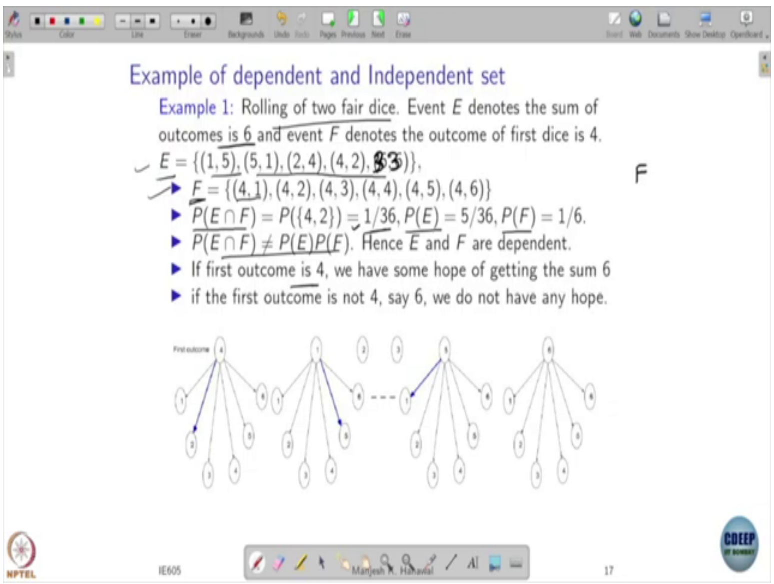

Now, for example I will just look into one example, other I will skip. Let us say we have two dice and we know that in this outcome there are 36 possibilities and now I am interested in an event when the outcome is 6. Now, let us say I am interested in two events, event E is like outcome is 6 so these are the possible outcomes, possibilities if it has to be 6, 1 5, 5 1, 2 4, 4 2 or 6, 6 can happen now something's wrong here.

Student: 3 3, 3 2.

Professor: And the second event F is outcome, the first outcome is 4, if the first outcome is 4, I mean 4 1, 4 2, 4 3 all of interest to me. Now, what I want to, now want to see, I want to understand whether these two events E and F are independent or not 0 So how to do this? First, I want to go and apply my definition, I will compute probability of this intersection. So, what is the probability of that intersection? The only thing that is common to E and F is 4 2 and what is the probability of 4 2 happening? 1/36, this is 1 out of 36 possibilities.

And what is the probability of E? E is there are about 5 possibilities E can happen in 5/36 and out, 5 out of 36. And what was P(F)? That this is 6 out of 36 probability that is 1/6. Now, if you look into that, 1/36 is not equal to the product of these two, so naturally as per our definition E and F are dependent.

Now, can you think about how, how does if the outcome is 4, how we can see that whether it will tell me something about whether my outcome is going to be 6. So, let us say F has happened, F has happened means my outcome, first outcome is 4. Now, does this provide me some information about whether my 6 can happen? There is a possibility like if 4 has happened, some if 4 has happened now that if two comes there is a chance that my, I can have some yeah the sum can be 6, so there is already some dependency here and that is a kind of getting implied here.

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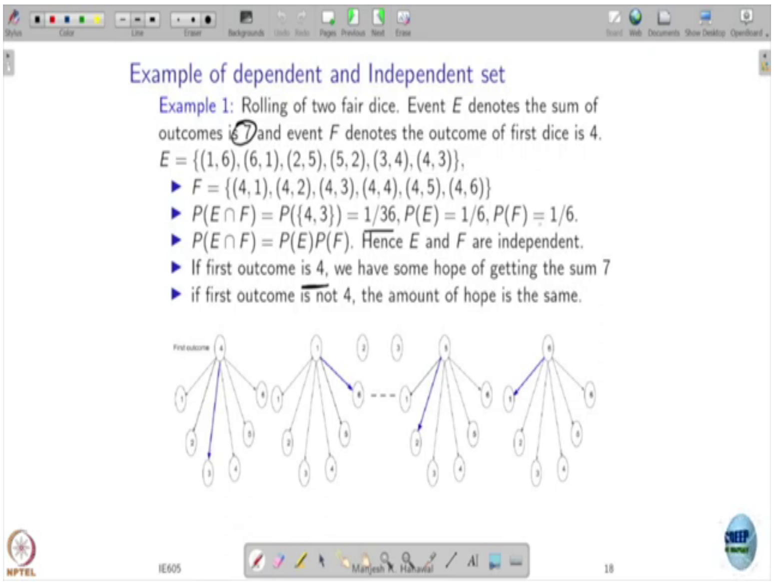

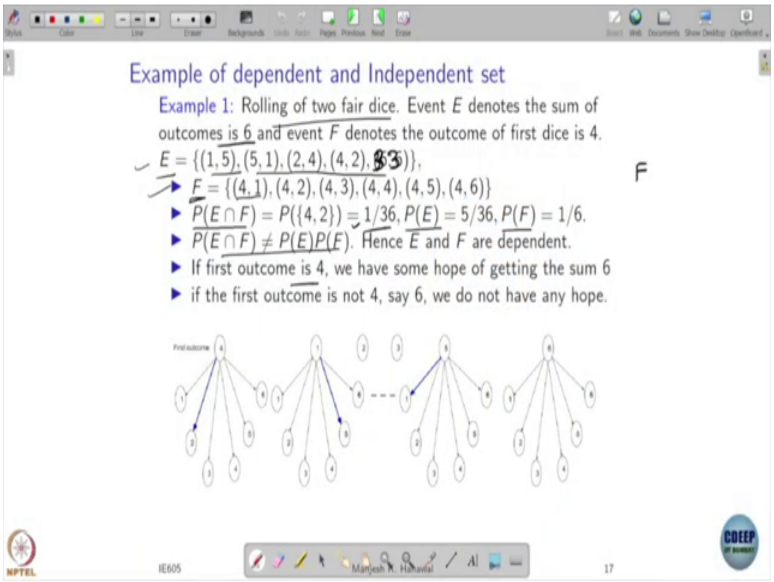

And similarly, I will not go into this example you can see that instead of 6 I am interested in sum being 7 and if let us say out, first outcome forms to be 4, is there a possibility that I can get 6? So, now my question is this like if my first outcome is 4 and I mean I am interested in sum to be 7. So, what happens in this case like will I get will if for first I have four then is there a chance that still I am going to get 7? So, these two events are dependent or independent, but what they are saying?

If you are going to look into this what is that probability of intersection this is 1 by 36 and P of E and P of F, 1 by 6, 1 by 6 that is also 1 by 36, so as per our definition what they are, independent but you are saying that is dependent. Why is it so? So, okay 4 has happened and now if 4 has happened you know that there is a chance that if 3 happens my sum can be 7. If

4 did not happen something else has happened, so 4 did not happen means it could be 1, 2, 3, 5 or 6.

And now does it still, if 4 does not happen can you still say that what is your likelihood of happening 7? It could be same, so event 4 happening or not it is not improving your knowledge about 7 happening. Now, can you just go back and apply the same analogy and see that why this is not true here? Like now we are concluding that these are independent, why because 4 happening or not they all has the same information about sum happening but why that is the case? Why?

Student: (())(13:19).

Professor: Do you all agree? If 4 did not happen let us say 6 happened can the sums be 6? No, it cannot be 6 something it has to be greater so that is a kind of intuition. So, often intuition may be not apparent whether looking into the two events whether they are going to be dependent or independent but we will go with our definition, definition is easy to verify just compute their intersection and see the probability and see that that probability is equals to the product of the each of the events, fine.

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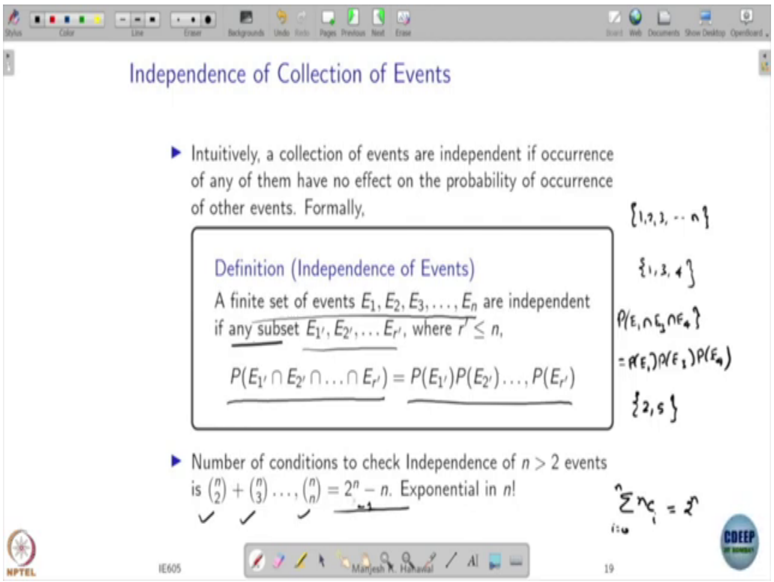

Now, in collect so often you may have to deal with more than two events maybe you will have to deal with a sequence of events and want to see whether they are independent or not. Now, how you are going to check that? We are going to extend the definition we had for two for the multiple event case and what we are now going to do is if you have this finite set of events E1, E2, E3 up to all the way up to En, now we are going to say that they are independent if you take any subset of this n events and if you look into their joint probability that probability should be equals to the product of their individual probability.

What we are saying is suppose let us say you have this 1, 2, 3, up to n and let us take, I took some numbers 1, 3 and 4, this is the one subset of this. Now, I looked into probability of E1 intersection of E3 intersection E4, this should be equals to probability of E1 into probability of E3 and probability of E4, this should happen.

This has now happened on one set, one subset of 1, 2, 3, 4, n. But what we are asking is for any subset, other subset could be let us say other subset could be 2, 5, like that if you take any subset this should happen. Now, you now look into that at least when we want to talk about independence I need at least two sets.

So, when I have only two sets I just need to see that whether take them and see their intersection but now when I have more than two there are so many subsets possible and that is what we are trying to do there. So, you have to select 2 2 at a time how many possibilities n choose 2 are there, you have to take 3 3 how many percent possibilities n choose 3 are there and at the end all of them.

So, this will lead to total sum of 2^n and I think there is also minus 1 missing here. So, all of you know what is this sum to n choose 0 plus n choose 1 plus and choose 3 like that, n choose I if I start from I equals to 0 to n this is equals to 2^n. So, if you just add this I am subtracting n n minus 1 because there is no n choose 0 and n choose 1 term in this.

And now you see that this is growing like exponentially in n, so if you have more events to check for independence you have to do a lot of that combinatorial things are just exploding, but because of this often most of the times we will be not interested in independence of the all events but some subset, some kind of weaker notion called as pairwise independence.

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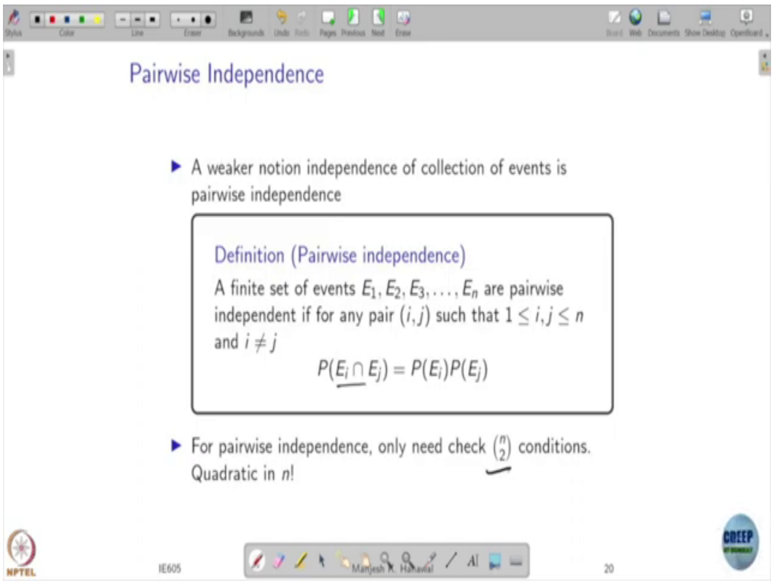

So, in this what we are going to do is we take the set of events and look into just to take 2 2 at a time I am only interested in 2 2 and see that their intersection there, the probability of their intersection is nothing but the product of their probability if this happens then I am going to say they are pairwise independence and as you see that in this case I only need to check and choose two conditions, and it is going to be quadratic in n you can check and often it so happens that obviously it is independent it is implies that pairwise independence but pairwise independence need not simplify that the set of those events are independent.