Engineering Statistics
Professor Manjesh Hanawal
Industrial Engineering and Operations Research
Indian Institute of Technology, Bombay
Lecture 02
Consequences of Axioms

Any questions on sample space, events or probability or what we call the mutually exclusive, anything like these are the basic operations, you should be clear about how to do them. Now, let us move on. So, in axioms of this probability, just assume these three basic things that probability of the event should be non-negative probability or the entire sample space should be 1 and there should be if I take mutually exclusive event, finite number of mutually exclusive event and if I take the probability that should be nothing but probability of this individual event, and this can be extended also to uncountably many. We will come back to that but for the time being, let us focus on the finite additivity.

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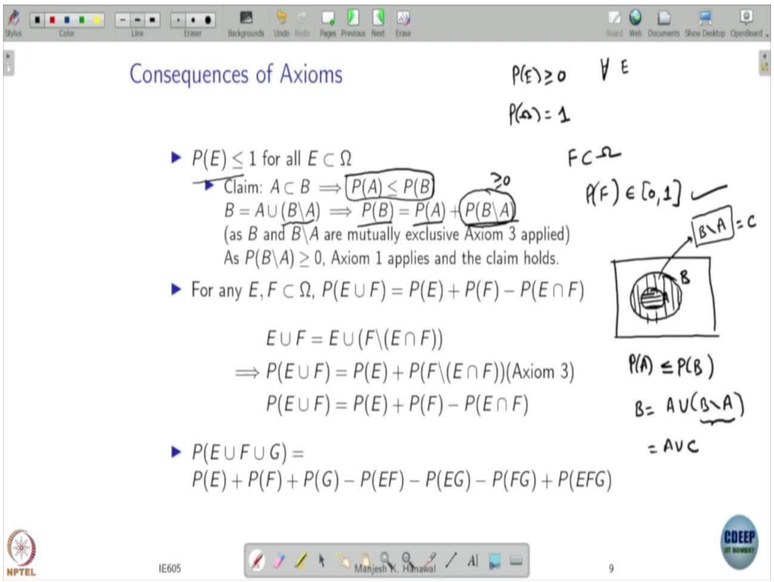

Consequences of Axioms

- $P(E) \leq 1$ for all $E \subset \Omega$
  - Claim: $A \subset B \implies P(A) \leq P(B)$
    $B = A \cup (B \backslash A) \implies P(B) = P(A) + P(B \backslash A)$
    (as B and $B \backslash A$ are mutually exclusive Axiom 3 applied)
    As $P(B \backslash A) \geq 0$, Axiom 1 applies and the claim holds.
- For any $E, F \subset \Omega$, $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$$E \cup F = E \cup (F \backslash (E \cap F))$$
$$\implies P(E \cup F) = P(E) + P(F \backslash (E \cap F)) \text{(Axiom 3)}$$
$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

- $P(E \cup F \cup G) =$
  $P(E) + P(F) + P(G) - P(EF) - P(EG) - P(FG) + P(EFG)$

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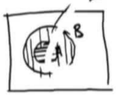
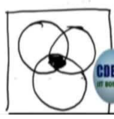
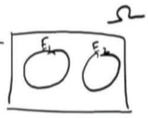




Probability of Events

In a random experiment we want to know/assign 'likelihood' of each event. This is done by defining probabilities. Intuitively, probability should satisfy some basic properties given by following axioms:

- Non-negativity: $P(E) \geq 0$ for all $E \subset \Omega$
- Normalization: $P(\Omega) = 1$
- (Finite) additivity For mutually exclusive events E_1 and E_2, $P(E_1 \cup E_2) = P(E_1) + P(E_2)$. (to be extended)



These three axioms itself they say a lot of other, they imply a lot of other properties, the first property is going to be so, since I said P of E is equals to greater than 0 and P of omega is 1 what to expect? Can any, let us take another F, which is a subset of omega can F, P of F, what do you expect this to be? In terms of the value in what value it should be?

It has to be expected to be 0 to 1. But I did not assume this. I did not assume this here. But these three assumptions itself imply that this is indeed true. And you can work out that, I will just so only based on our non-negativity, normalization and finite additivity these three properties, you can argue that probability of any event is going to be less than or equal to 1.

So, for that, we just let us quickly go through that. Let us take let us first take any two events one is B. And I have another event A. Here is event A is a subset of B, it is. Now, if this is the case, naturally, I expect probability of A to be less than or equal to probability of B. This is intuitive. But I did not… in the axiom I did not assume this. But what we are now going to assume that the three axioms also imply this. So how is that?

Now, what we can do is, this event B, I can write it in two parts. Let me call this, the one with horizontal lines. So, the one in the vertical lines, that portion, can I write it as B minus A. So, this is what this one in the vertical I am going to call it as B minus A. That is, I remove all the elements of A from B and whatever remains, I am going to denote it as B minus A.

So now, B can be written as union of these two things, A union B subtract A. Let us call this simply C that is A union C. So, this portion is what I am calling it as C. So, is A and C mutually exclusive? Yes, there is no overlap between them. Now, but then, from my now can I say that probably of B is nothing but probability of A plus probability of B minus A, why is that?

Student: The third property.

Professor: The third property mutually exclusive property that I have applied. And if that is the case, I am done. I am saying that this I have already proved why is that? This quantity here is another event, B minus A is also a subset of omega. Now, we know that from the first axiom, this has to be greater than or equals to 0, if this has to be greater than or equal to 0, then it must be the case that $P(B) \geq P(A)$. So, fine.

Now, the axiom said, how the probability of each event should look, now does this using that. So, if you again, look into this carefully I can treat probability as a function on my events, like I said, f is subset, all subsets. And now P is a function on this, which is going to give a value of [0, 1] between them.

So, probability is actually function on the events, for every possible event, it is assigning a value in the value in the interval [0, 1]. But now, if you have events, you may take union of events, let us say if you have two events, and you have union of events, E ∪ F. And you want to see that how this probability of the union is related to the probability of the individual events P and F, so what we are trying to do here is? We have one event, and another event and let us call this as well left as A and this has B. And now intersection is the entire thing.

Now, probability of E ∪ F, one. So, what when you want to compute probability of E ∪ F for what you are going to do is, you want to take this match. And also, the likelihood of this match. So, when you did this, you double counted this region here. So, this naturally gives me an intuition that the union of when I take probability of the union of this, we should be equal to probability of E probability of F minus this portion which I double counted, this is a relation. And in fact, this relation is actually implied by our three axioms.

It is not that I am making this heuristic argument here like the probability of E union F is equals to this intuitively, it should be like this, but actually, this is what our three axioms also said. And these are the steps based on which we can use axioms and find it, here I will not go into the details, but just using these axioms 1, 2, 3 you will be able to derive this and you can extend this to more than two subsets when you have let us say three subsets.

Let say when you have three subsets and they can be arbitrary there can be overlap in any way, I mean, these are three circles, they may not overlap at all or maybe only two overlap and they and the third may not overlap with other two any possibilities there, but let us consider this is a case where all of them are overlapping with each other, where this is the common region among all.

You can again verify that this probability of this intersect union of these three can be written as probability of these three each of these circles from that you are going to take off, that is common between A and F and that is common between E and G, and that is common between F and G, but when you are taking out you would have taken common between E, F, G one extra time. So, you need to bring that back and you will get this relation. And again, this is actually implied by our three axioms like it: to prove this, you will, you will first prove this property and then extend it to the three subset case.

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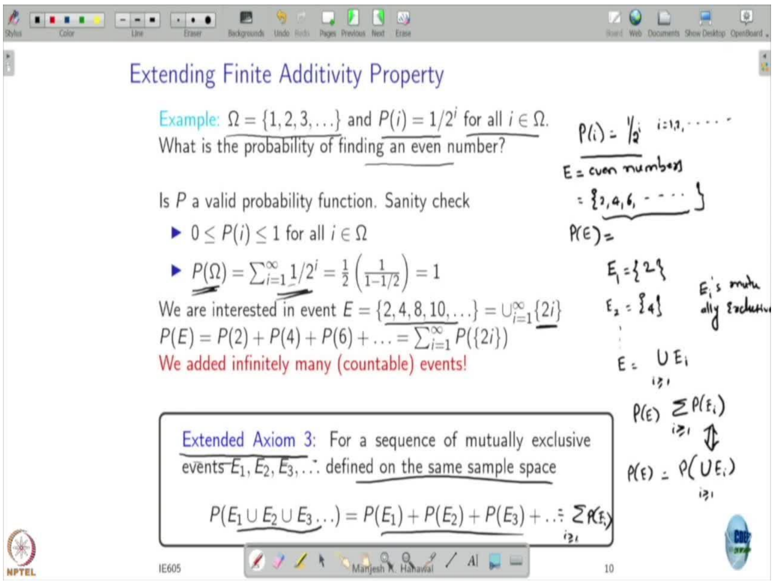

Now, I was talking about finite additivity. Why we need to worry about the finite additivity case, like when I talked about these three axioms, the last axioms, I restricted to finite additivity. But, in practice, I need to extend this to include countable events also. And now, to look into that, let us see what you want to do . Let us first see where I may need count ability additivity.

Let us take omega, my sample space, where omega is any possible integer 1, 2, 3, like that. And I am assigning the probability that ith number happening is $(1/2)^i$. And this is true for all i ∈ omega. Now, I am interested in finding what is the probability that an even number happens.

So, notice that here I have given you probabilities of each of the outcomes. Let us say what I am saying is P(i) = $(1/2)^i$, for i = 1, 2, 3 like this. Now, what I am asking is I am interested in event E, which is even numbers which are nothing but 2, 4, 6 like this. And now I want to find the probability of E, how will I find the probability of E?

Probability of E can be thought of as probability of 2 happening 4 happening 6 happening, but event 2 happening and 4 happening, if I am going to treat them as separately event 2 happening and event 4 happening, are they mutually exclusive? They are. Then I can treat each break as mutually exclusive event.

And then I can use those probabilities to add up to get this but now, how many times I need to add let us go back to this. So, first of all, whenever I say something is a probability, you need to verify that like, that axioms at least it follows the three axioms, the axioms are saying that it should be for every $i \geq 0$, which is absolutely obvious here.

And if I am going to take a P(omega), so the second axiom said this should be equals to 1 and is it 1 in this case. Let omega is nothing but 1, 2, 3, 4 all the way up to infinity. You just had all these probabilities you will get 1 here. And now I am interested in this event. And this event could be thought of 2, 4. I can write it as union of events let me write it as event 1. Let me write it as how to write, event I am going to write it as event 1 is to event 2 is 4 like this I will write, now E is nothing but union of these events.

And now I know that E i's are mutually exclusive. So E i's are mutually exclusive. Now, let us say if mutually exclusive, and then what I need is probability of E, I know that when if this union is over finite not a finite union, this is a countably many outcomes are there. So, if I had to do that I need to do this, I need to have to do this or like P of E is nothing but P of union of E i. And if this P of this has to be equal to this, there I need to add in countably many, so, I need to allow myself flexibility not just add finite mutually exclusive event even add countably many mutually exclusive event and that is where I need to extend my X third axiom.

And now, we are going to say that if you have a sequence of mutually exclusive events, E 1, E1, E2 like this, then they are all defined on the same span sample space, then the probability of the union of those mutually exclusive events is nothing but sum of all of that. So, that is nothing but summation of P(E), but this is how we are extending the finite additivity of the mutually exclusive event to the countably many case. But now, the question is this clear to all of you? But now, finite countable, finite additivity and uncountable additivity is fine. But what about the uncountable case?