Modeling of Tundish Steelmaking Process in Continuous Casting
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Lecture – 20
Navier – Stokes Equations for Newtonian Fluids

Welcome to the lecture on Navier-Stokes Equations for Newtonian Fluids. So, we had discussed about the conservation equations, we discussed about the equation for the momentum, continuity, and also about the energy. So, especially in if you look at the momentum conservation equation or the energy equation we had the x momentum or y momentum or z momentum equations, then you had the energy equation. In all these equations what you see that you come across the viscous stresses term.

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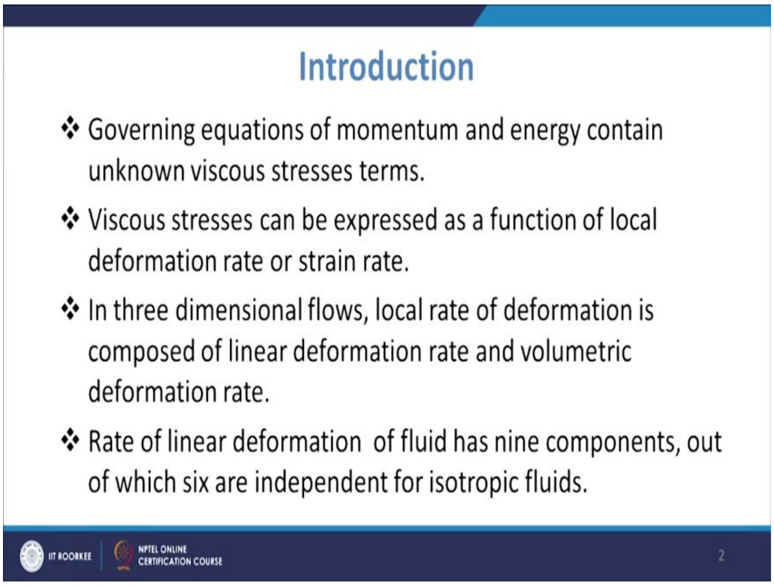

So, the governing equations of momentum and energy they contain unknown viscous stresses terms that is: τ_{xx}, τ_{xy}, you know, τ_{xz}, or xy or so. So now, these need to be in expressed in a measurable manner. And they are basically expressed as the function of local deformation rate or the strain rate. So, in that case the equations will have something by which you can so you can do the experiment you can measure those strain rates and further you can calculate the values. So, those stresses are basically expressed in terms of these strain rates.

So, mostly what we do is, normally when we talked about the three dimensional flows in that the local rate of deformation is composed of linear deformation rate as well as the volumetric deformation rate. And, if you talk about the linear deformation rate of the fluid it has nine components, out of which six are independent for the isotropic fluids. So, they are normally denoted by S_{ij}. So, that is the I mean linear deformation terms. And their suffix ij will have similar meaning the way we tried to even define for the stress terms. So, in that only we are also defining them.

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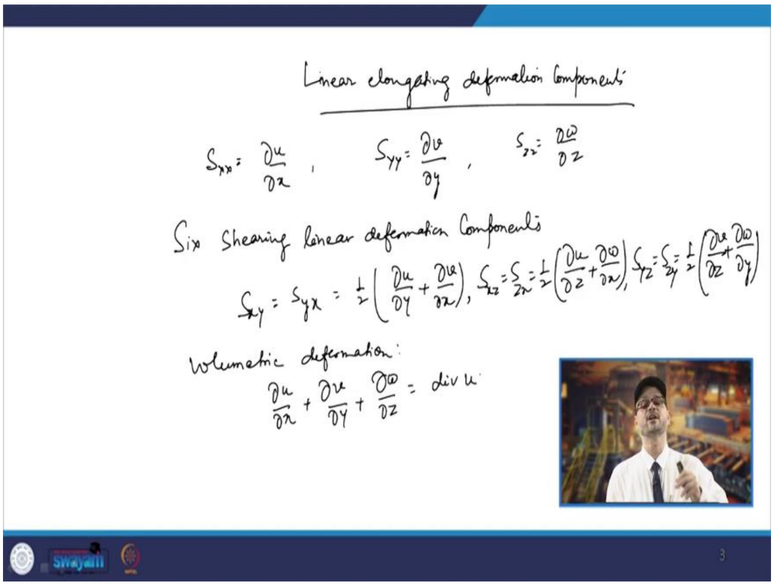

So, if you talk about the you know linear terms. So, you will have the linear; so you have linear elongating deformation components. So, they are the three; so those three are basically the S_{xx}, $S_{yy}$and S_{zz}. So, their values are $\frac{\partial u}{\partial x}$ S_{xx}, is, then S_{yy} is $\frac{\partial v}{\partial y}$, and S_{zz} will be $\frac{\partial w}{\partial z}$.

Now you have apart from that, so as you had we saw that we have a nine component, so you have six shearing also components; so, shearing linear deformations components. So, they will be you know when. So, in the S_{ij}, when $i = j$, so that will be linear elongated deformation component and then you have shearing linear deformation components.

So, in that you will have the you know S_{xy} or S_{yx} you can have and they are normally defined by $\frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$. Similarly you have S_{xz}, or S_{zx}, they will be $\frac{1}{2}(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z})$.. And accordingly S_{yz} or S_{zy} they will be $\frac{1}{2}(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y})$... So, this way you have the nine you know strain rate linear deformation rate components. So then, these are deformation components.

Then we have the you know volumetric deformation. And volumetric deformation will be defined by; so that will be $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$. So, that will be $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$. So, that way that is also can be that can be represented by $div\, u$. So, what is seen that these are the strain components, this will be required in our coming studies.

So now, if we talk about the Newtonian fluid; now in the Newtonian fluid the viscous stresses are proportional to the rates of deformation. So, as we have understood earlier also that they will be related to that rates of the formation. And the in the three-dimensional form of the Newton's law of viscosity for compressive flows you will have a two constants of proportionality comes.

You know, we talked about the Newton's law of viscosity and we saw that they are the proportionality term is coming that is coefficient of viscosity. Now in this case, when we talk about the compressive flows; compressible flows in those case we deal with two constants of proportionality and that is mu and lambda.

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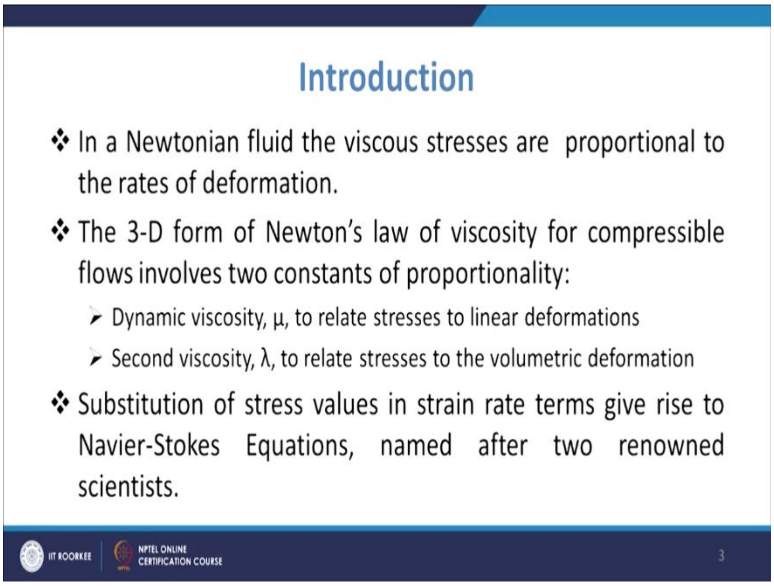

So, mu is known as the dynamic viscosity which will where it will be relating the stress to the linear deformation that is what we have seen in the case of Newton's law of viscosity earlier. But, in the case of compressible flow you have a second term, that is second viscosity λ. So, this is used for relating the stresses to the volumetric deformation. So, this which is happening in the case of compressible flows.

So, you know now this stress values; substitution of these stress values in the strain rate terms give rise to the Navier-Stoke equations and how that is you know, that we will see. So, what we have seen that since there are you know these two you know λ as well as μ. These are the constant of proportionality is used.

And, if we define the viscous stress components; so, you have the nine viscous stress component τ_{xx} τ_{xy} τ_{xx} τ_{xz} τ_{zx} . So, altogether you have the nine viscous stress component which are encountered in the momentum as well as the energy equations.

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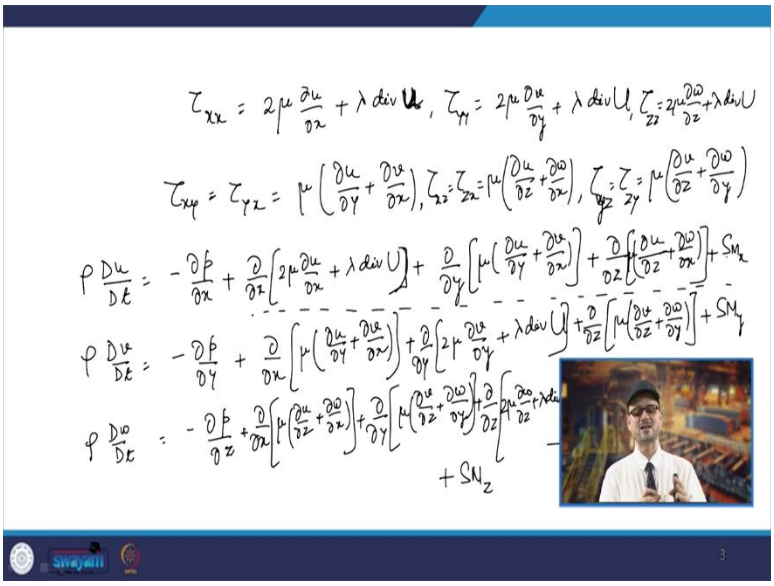

Now, these components of the viscous stresses they can be expressed you know. So, they are expressed like in the term of the λ and μ and also the deformation component. So, what you see is the τ_{xx} it is defined as $2\mu\frac{\partial u}{\partial x} + \lambda div.U$. So, that is you know $\lambda div.U$, so that is for the τ_{xx} . Similarly τ_{yy} will be $2\mu\frac{\partial v}{\partial y} + \lambda div.U$. And τ_{zz} will be again $2\mu\frac{\partial w}{\partial z} + \lambda div.U$. So, that way we have these three stress component.

And then out of that you have you know out of the nine, six are independent. So, basically τ_{xy} will be taken as a τ_{yx} . And this will be $\mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$. Similarly τ_{zx} equal to τ_{zx} , so that will be $\mu(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z})$and τ_{zy} or τ_{yz} it will be $\mu(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z})$. So, that way you have these you know viscous stresses which are there. So, they can be expressed in terms of these deformation components.

And you know these components; so these components values when they, so they these are you know expressed in terms of strain rate. And then when they are written for the these momentum or the energy equation in terms of these strain rate terms. So, that was done by independently by two scientists Navier as well as Strokes. So, that is why this equation which we obtain is known as the Navier Stokes equations.

So, you know the term that is λ so not much was known about λ earlier, but its effect is said to be very very small as compared to μ. And it is normally said to be about $-\frac{2}{3}\mu$. So, λ is said to be that is you know said by, that is basically suggested by sltzing. So, sltzing was also a researcher and on his name there is a book also. So, he has suggested the value of lambda as $-\frac{2}{3}\mu$.

Now for the incompressible flows we know that $div\ U = 0$. So, that way you have you know accordingly you can have many meanings for the incompressible fluids and then you will have the terminologies. So, once you give substitute these values into the momentum equations x momentum z y or z momentum, so they will be having the different form. So, when you substitute, so what happens? So, if suppose you take the x momentum. So, it becomes $\rho \frac{Du}{Dt}$. Now that will be, so you had the first term that is $-\frac{\partial p}{\partial x}$ so that was as usual.

Now, we have the it will be $\frac{\partial}{\partial x}[2\mu \frac{\partial u}{\partial x} + \lambda div. U]$. So, this will be the first term. Similarly you will have $\frac{\partial}{\partial y}$ of you know τ_{xy} . So, that will be $\mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$. And similarly the third term will be $\frac{\partial}{\partial z}$. So, it will be $\mu(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z})$. So, this will be the these are the three terms. And then you had the source term that was S_{Mx}. So, this was the momentum my question and this is what you get for the you know x momentum equation.

Similarly for the y momentum equation you get $\rho \frac{Dv}{Dt}$, so that will be minus of $-\frac{\partial p}{\partial y}$. And then you have the terms will be coming like that so you will have $\frac{\partial u}{\partial x}$ of now you have the terms like mu of d be $\mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$.. Similarly, in this term you will have $\frac{\partial}{\partial y}$ of this term will come $2\mu\frac{\partial v}{\partial y} + \lambda div.U$.

So, that term will be there in the y term y component. And then you have the this third term will be $\frac{\partial}{\partial z}$. of be $\mu(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z})$... So, then and then you will have the source term that is S_{My}. So, this is the equation for the y momentum. And when you substitute in the z momentum equation you will have the equation rho of $\rho\frac{Dw}{Dt}$.

So, this will be equal to $-\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}[\mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$. Similarly, $\frac{\partial}{\partial y}$ of $\mu(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z})$. And then you will have the $\frac{\partial}{\partial z}$ times you will have $2\mu\frac{\partial w}{\partial z} + \lambda div.U$. And then you will have the source term in the for the z $\frac{\partial}{\partial z}$ component.

So, this is this is the these are the three equations which are the Navier Stoke equations, because you have expressed these viscous stresses term in the term in the form of strain rate you know terms. Now what we see if you look at these term. So, if you take any of these values. So, we can express them.

You know now if you take these many terms; if you take this term. So, you can express that as $\frac{\partial}{\partial x}(\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}\mu(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}) + \frac{\partial}{\partial z}(\mu\frac{\partial u}{\partial z})$,. So, that will be one term. And there will be another term that will be separate and plus S_{Mx}. So, that we are basically writing as the source term. So, if you take this part and we write further.

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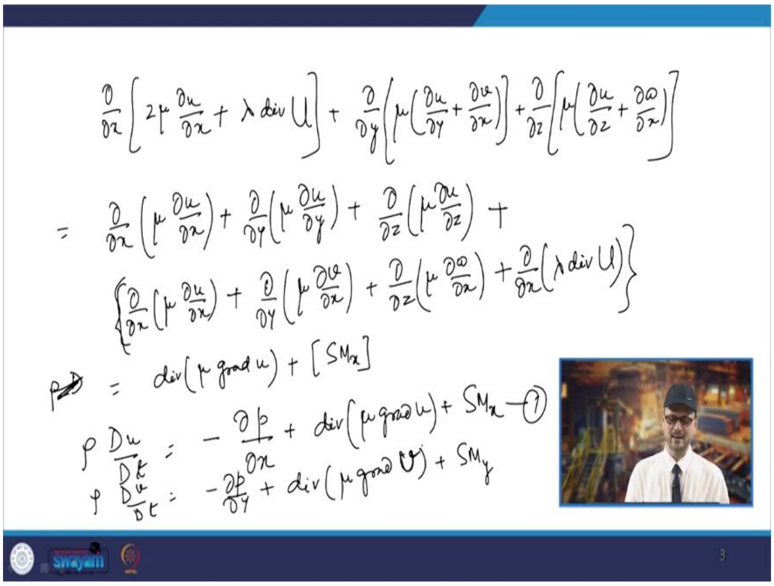

So, we are writing basically $\frac{\partial}{\partial x}[2\mu\frac{\partial u}{\partial x}+\lambda div. U]+\frac{\partial}{\partial y}\mu(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})+\frac{\partial}{\partial z}[\mu(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z})]$. . .

So, if you try to write further this equation that you can write as first you take one of these term together. So, it will be $\frac{\partial}{\partial x}(\mu\frac{\partial u}{\partial x})+\frac{\partial}{\partial y}\mu(\frac{\partial u}{\partial y})+\frac{\partial}{\partial z}[\mu(\frac{\partial u}{\partial x})]+\frac{\partial}{\partial x}(\mu\frac{\partial u}{\partial x})+\frac{\partial}{\partial y}\mu(\frac{\partial v}{\partial x})+\frac{\partial}{\partial z}[\mu(\frac{\partial w}{\partial x})]+\frac{\partial}{\partial x}(\lambda div. U)$..

So, this term will be you know $div\ (\mu\ grad\ u)$. So, that will be this term. And then a rest term you can take in the bracket as S_{Mx}.

So, that way you have taken this bracketed term. So, this basically this bracketed term is taken into this bracket. So, that is the equation for the you know Navier Stokes equation in the x direction. Similarly, so then in that case you can write. So, it you are writing; so that will be ρ of Du. So, this is basically $\rho\frac{Du}{Dt}$. So, that will be basically div of; so you will have term left is dou p by dou x it was in the earlier one so this is $-\frac{\partial p}{\partial x}$ was left. So, that is that terms comes here. And then you have $div\ (\mu\ grad\ u)$. And then you haveS_{Mx}.

So, this is the you know Navier Stoke equation for the x momentum you know component. So, you have you will have three equations for the x, y and z momentum. So, as that is for the x momentum. And if you go for the y one it will be $\rho\frac{Dv}{Dt}$ so that will be $-\frac{\partial p}{\partial y}+div\ (\mu\ grad\ v)+S_{My}$. So, you know this is u and this is v.

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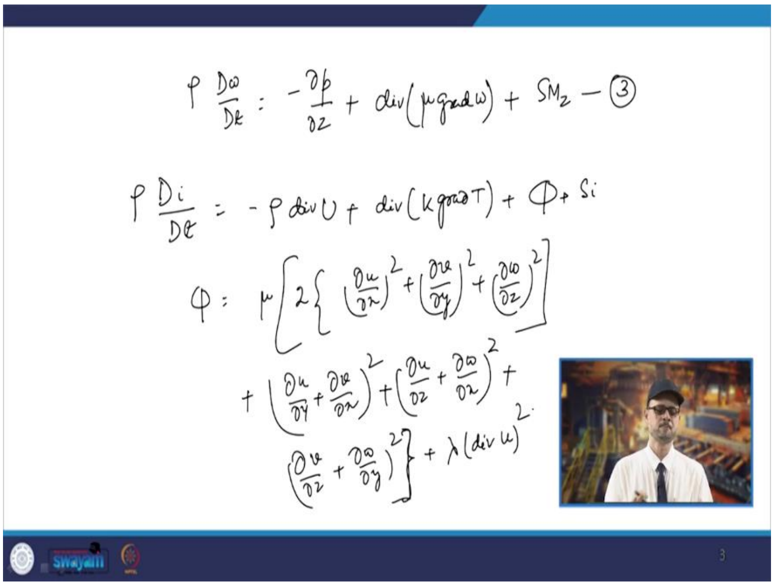

And the third term will be for the z. $\rho \frac{Dw}{Dt} + div\ (\mu\ grad\ w) + S_{Mz}$. So, you have S_{My}., S_{Mx}., and S_{Mz}.. These three equations you know they are the Navier Stoke equations and normally they are used for you know while solving. And where typically when we use them for the you know using the finite volume method.

Now, if you go for the you know energy equation also. So, in the last class we had discussed about the energy equations. And, for the energy equation which we derived we had you know that time it will e will be for the internal energy component i plus you have the kinetic energy that is $\frac{1}{2}(u^2 + v^2 + w^2)$.

So accordingly, you will have the you know the terminologies coming for that, because the $\frac{1}{2}(u^2 + v^2 + w^2)$ that part for that you will have the u square term and otherwise you have the for the i term coming up. So, you can use this I mean you can write the energy equation also.

So, for that for the internal energy $\rho \frac{Di}{Dt}$; so, that will also come as $-\rho div(U) + div(k\ gradT) + \varphi + S_i$. So, φ is basically you are getting this you know by the this is known as the dissipation function. So, this phi which is dissipation function it is basically$\mu[2\{(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial y})^2 + (\frac{\partial w}{\partial z})^2\}] + (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})^2 + (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})^2 + (\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})^2$. And then you are putting into the bracket.

. So, this is basically these are the terms that will come under bracket plus $\lambda(divu)^2$; that is square. So, that was basically that you can get when we are you know we are converting this e term into the two terms i plus the the other term, that is for the $\frac{1}{2}(u^2 + v^2 + w^2)$.

So, actually that this might be not clear to you, maybe I can that you can understand what we did in our you know while we got for expression for the $\rho \frac{DE}{Dt}$.

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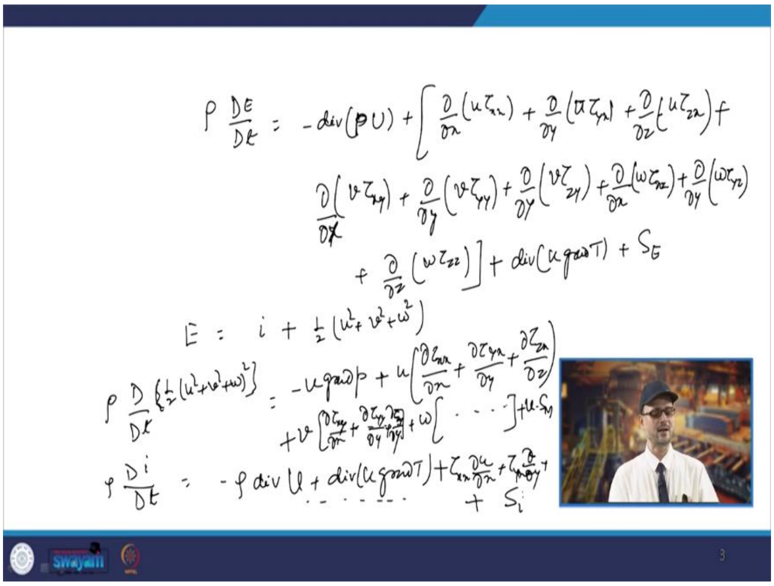

So, that was you know that was $-div(\rho U$. And then you had the terms like $[\frac{\partial}{\partial x}(u\tau_{xx}) + \frac{\partial}{\partial y}(u\tau_{xy}) + \frac{\partial}{\partial z}(u\tau_{xz}) + \frac{\partial}{\partial x}(v\tau_{yx}) + \frac{\partial}{\partial y}(v\tau_{yy}) + \frac{\partial}{\partial z}(v\tau_{yz}) + \frac{\partial}{\partial x}(w\tau_{zx}) + \frac{\partial}{\partial y}(w\tau_{zy}) + \frac{\partial}{\partial z}(w\tau_{zz})$.

So, this way you had the d $\rho \frac{DE}{Dt}$ and then you had the term $div.(\, k\ grad\ T)$ and you had the term S_E. So, this was the energy equation which we derived.

Now in that basically what we take E, this E will be internal energy i term then and that will be for the temperature. And then you have the kinetic energy term that is half of u square plus v square plus w square. So, what we do is normally when we are separating, so you will have the terms for the kinetic energy. And for that what we see that in the in that case you have the multiplied terms of u times you know those those terms. So, those terms will be with these $\frac{1}{2}(u^2 + v^2 + w^2)$.

So, if you segregate it will be for the one will be rho times d $\frac{D}{Dt}(\frac{1}{2}(u^2+v^2+w^2)$.. And one will be $\rho\frac{Di}{Dt}$. Now if you see in in this term. So, one will be for i which will be for the internal energy and that will be for, so that we can take for the temperature term. And so that though these terms will go with that term.

But, for this you know for this part four of the $\frac{1}{2}(u^2+v^2+w^2)$.. part that is kinetic energy part that will be obtained by multiplying the u component with the x momentum. So, that can be understood that this will be basically $u\frac{\partial}{\partial x}\tau_{xx}$. Similarly $u\frac{\partial}{\partial y}\tau_{xy}$.

So, all these terminologies will be coming for this expression and others will come with with this. So in that, and this will also be you know divided in two terms so it will be $-u\ grad\ p$. And, you know accordingly then you will have the terms like you know; so this is basically p u. And, you have $u(\frac{\partial}{\partial x}\tau_{xx+}\frac{\partial}{\partial y}\tau_{yx+}\frac{\partial}{\partial z}\tau_{zx})$.

So, this will be further you will have v times that things will come here; $v(\frac{\partial}{\partial x}\tau_{yx+}\frac{\partial}{\partial y}\tau_{yx+}\frac{\partial}{\partial z}\tau_{zy})$.. So, similarly plus w times you will have these components basically, they will be coming together. So, all these terms will come here. And then you have the u times the source term.

So, these terms like you have if you take v times, so $v(\frac{\partial}{\partial x}\tau_{yx+}\frac{\partial}{\partial y}\tau_{yx+}\frac{\partial}{\partial z}\tau_{zy})$... So, similarly for w also these terms will come. And if you take, so that those components will be for the for that component for the$\frac{D(}{Dt}\frac{1}{2}(u^2+v^2+w^2)$. kinetic energy part.

And, for the internal energy part you will have other terms coming. And for that you will have $a\ -\rho\ div(u)+div(ugrad\ T)$; this term will come with the internal energy part. And then you will have the $\tau_{xx}\ \frac{\partial u}{\partial x}$. Then you have a $\tau_{yx}\ \frac{\partial u}{\partial y}$.

So, this way you have all these terms multiplied by the these x stress term multiplied by these $\frac{\partial u}{\partial y}$ and . So, that way these terminologies will come alongside here and then you will have $S_{\ i}$. So, this way you will have. So, that is why we have written the expression for $\frac{Di}{Dt}$ and you have $\rho div(u\ gradT)$. So, that for that just understanding for the you have here for the $div($ $gradT$ t and this term was here. So, if you put that in in this term you will be getting you know this term, and this φ was will be coming like this.

So, this is about the Navier Stoke equation which will be solved. Normally, we solve with the proper boundary conditions, and then you have solution methodology which is adopted. And then we get the proper value of output parameters. That we will see in our coming lectures.

Thank you very much.