Mechanical Behaviour of Material - 1
Prof. Shashank Shekhar
Department of Materials science and Engineering
Indian Institute of Technology Kanpur

Module No # 02
Lecture No # 07
Viscoelasticity

Good morning students so today we will be talking about another topic viscoelasticity so we will start the plastic deformation after we have discussed this topic of viscoelasticity. So, in the previous lectures you saw that there is elasticity where you apply a stress and a certain strain or to be more precise you apply a load and a certain elongation is obtained in the material and it is not dependent on time.

But some materials and particularly polymers show a time dependent elastic deformation which is recoverable. And this phenomenon is called viscoelasticity so this is what we are going to talk about today. So like I mentioned let me write down some of the important features about viscoelasticity.

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→ time-dependent deformation that is recoverable
→ viscoelastic or anelastic deformation
→ When a load is applied, there is instantaneous elastic response, but the deformation increases with time.
→ different from creep which is time-dependent plastic deformation

It is time dependent deformation that is recoverable and this part is important the fact that it is recoverable this makes it different from creep phenomena we will come to it in a moment. So this phenomena where the time dependent deformation takes place it is called viscoelastic or anelastic deformation. So when a load is applied there is instantaneous elastic response but the deformation in these material keeps on increasing it does not stop with time.

And this continuous change in deformation with time is what is termed as the viscoelastic deformation there is instantaneous elastic response but the deformation increases with time. Now like I mentioned earlier that this phenomena is different from creep why because creep is a phenomenon which is plastic deformation which is time dependent plastic deformation. This viscoelastic or anelastic deformation is usually found in polymers.

Actually it; will you can also find it in metals and ceramics but then it is such so small in comparison to the overall strain that it is negligible. And the material where it is most obvious and most important is polymer that they are ignored. On the other hand when we are talking about polymers it is very significant

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→ Anelastic strain in metals and ceramics are usually so small that they are ignored

→ In polymers, strains are very significant

→ viscoelastic strains are undesirable

So in polymers usually this viscoelastic or anelastic deformation is non-desirable undesirable. You would see that over time the deformation takes place and the dimension changes and the material does not fit and these kinds of problems can occur. So for example you may see sagging of wooden beams denting off vinyl floors and loss of dimensional stability in some of the components. And these are all basically in examples of viscoelastic deformation which is undesirable.

And as I said mentioned in the example this does not happen only in the polymers it sometimes also happens in non-polymers and it is non-desirable over there too. But this is one of the primary reasons why polymers are not still used as such a structural material because this problem of viscoelasticity is what makes them undesirable. Now let us understand this viscoelastic train strains you know with the help of mathematical description.

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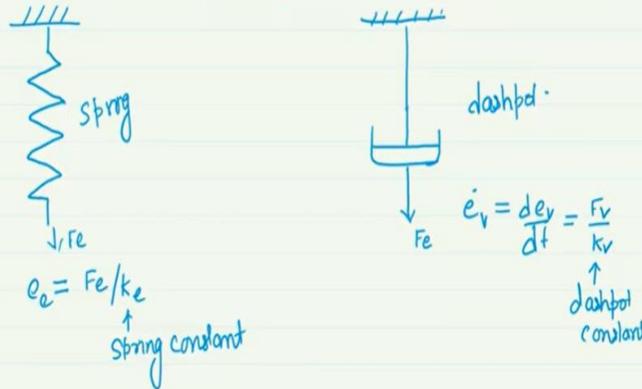
So overall the deformations that happen can we actually shown with the help of 2 different components. So one is a spring type component so you apply a force here F_e this is a spring and elastic strain F_e/k_e where k_e is the spring constant. The other component which defines and which becomes very important with respect to viscoelasticity is a dashpot. So it can be represented as if there is a viscous liquid here and there is a piston.

So you apply a force and you can imagine that the piston will not move instantaneously there will be time dependent phenomena. And here usually, it will be the relation is given like this $\dot{e}_v$ which is the rate of change of deformation, or elongation which is equal to de_v/dt this is equal to F_v/k_v where k_v is the dash part constant.

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And if your F_v is not a function of time then we can say that $e_v = (F_v \times t)/k_v$. Now these are 2 extreme examples and you can say 2 components which together show the overall behaviour of viscoelastic materials. So these are just elements and do not represent overall response of viscoelastic material to properly represent the response of a viscoelastic material.

We need to combine these 2 elements into some combinations as we will see and then that will represent in some of these models as we will see represent some of the viscoelastic behaviour of the materials. So some simple so we will be looking only at simple models and based on the simple models more complex ones we know the overall methodology we can build up more complex models depending upon the requirement. And in this particular course we will only look through some of the simple models.

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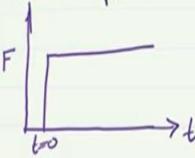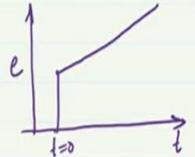
So first we will look at a series combinations this is our first simple model series combination which is also called as a Maxwell model. So this as the name suggests here we are assuming that there is a spring component and there is dashpot components which are put together in series. So here the constant is k_e here the constant is k_v here the force is being applied so f is constant in both of these.

So $F = F_1 = F_2$ and the overall elastic strain would be given by e_e the elastic the spring component plus the dashpot component. And this can be now this e_e from based on our previous equation can be written as $F/k_e + F\, t/(k_v)$ and therefore what we have is e equal to and this can be represented. So here we are assuming that a force was applied like this on the x-axis you have force on the y-axis of time and this is t = 0.

So you are applying so this is the condition for which we have calculated this. And in response the material the strain as you can see at t = 0 there is some constant value. So let us say this is t = 0 so there will be this constant value and after that it will increase linearly. So this is the response the extension this is how the extension in the material will take place. So here we are assuming that sudden force was applied.

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Sudden elongation 'e' at time t = 0.

$$F_0 = k_e e$$

this force causes dashpot to operate, gradually increasing strain e_v in.

$$F = k_e e_e = (e - e_v) k_e.$$
$$F = k_v \frac{de_v}{dt} = (e - e_v) \cdot k_e$$

Another condition where we can evaluate the response of this made area material is when where there is a sudden elongation e at time t = 0. Now when you apply a sudden in this condition if you apply a sudden elongation you know that this one cannot accommodate the strain instantaneously. So all that strain must be accommodated over here and therefore the force would also be accommodated entirely by the spring

And then gradually when over time this force there will be force acting on the dashboard and hence it will slowly take over or accommodate the force. So if we have the strain being completely accommodated by the spring then we know that F the initial

$$F = k_e e_e$$

where e is the total strain that was applied suddenly. And this strain remains constant but the force that is acting on the spring is now also acting on the dashboard and this force is will now make the dashboard active or will start to operate.

Gradually increasing the strain in that dashpot so initially there was no strain in the dashpot now the dashpot will start to take some of the strains meaning the strain in the spring will decrease. And therefore, the force in the spring will also decrease. And now that sum at some instantaneous time t when the force is decreasing for that instantaneous time let us say that e_e and e_v are the strain in the spring and the dashpot.

Then we know we can write some like this $k_e e_e$ where $e_e = e - e_v$. Hence

$$F = (e - e_v) k_e$$

F which is also acting on the dashpot where we can write $\frac{k_v de_v}{dt} = (e - e_v) k_e$

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$$\int \frac{de_v}{(e-e_v)} = \left(\frac{k_e}{k_v}\right) \cdot dt = \int \frac{dt}{\tau}$$

$$\ln\left(\frac{e-e_v}{e}\right) = -\frac{t}{\tau}$$

$$\frac{e-e_v}{e} = \exp(-t/\tau)$$

$$e-e_v = \frac{F}{k_e} \quad \& \quad F_0 = k_e e$$

$$\frac{F}{F_0} = \exp(-t/\tau)$$

$$\boxed{F = F_0 \exp(-t/\tau)}$$

Now this is this equation is in a differential form and we can integrate it .

$$\int \frac{de_v}{e - e_v} = \frac{k_e}{k_v} dt = \int \frac{dt}{\tau}$$
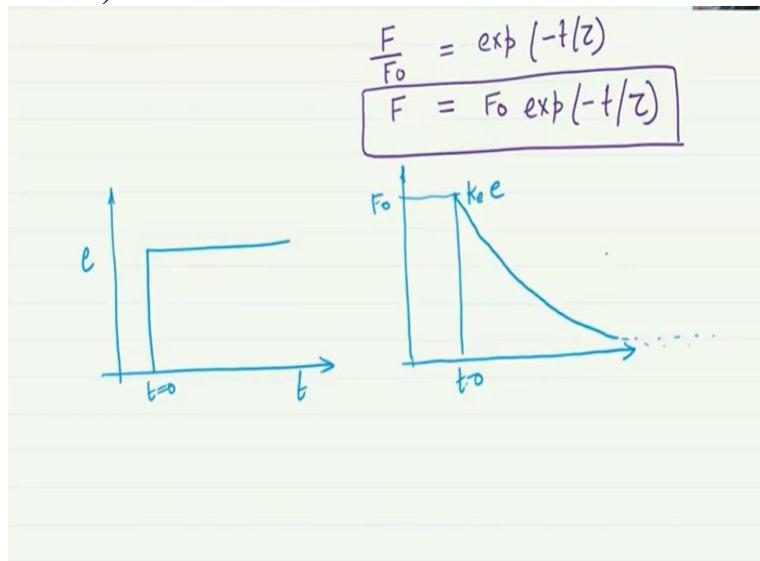
On solving the above equation

$$(e - e_v) = \exp(-\frac{t}{\tau})$$

On using above equation

$$F = F_0 \exp(-\frac{t}{\tau})$$

So for the condition where you know for this simple looking condition this configuration where we; have series combination of spring and dashpot. And we apply a sudden strain.
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So if we were to plot it would look something like this so if we are applying sudden strain so this is like this is t = 0. And as a response the force would change like this. So initially there is a force

$F_\circ$ which is the force that you see be generated because of the spring but then the dashpot starts to accommodate the strain and when the dashpot takes the strain then there is no force generated or resistance force generated by the dashboard and hence the overall force keeps decreasing.

And therefore eventually it will die off to 0 so this is equal to k e times e. And this gives us for this particular condition where we have series combination one when the sudden force was applied how the strain changes? And once in the second condition when the sudden strain was applied then how the force changes over time?

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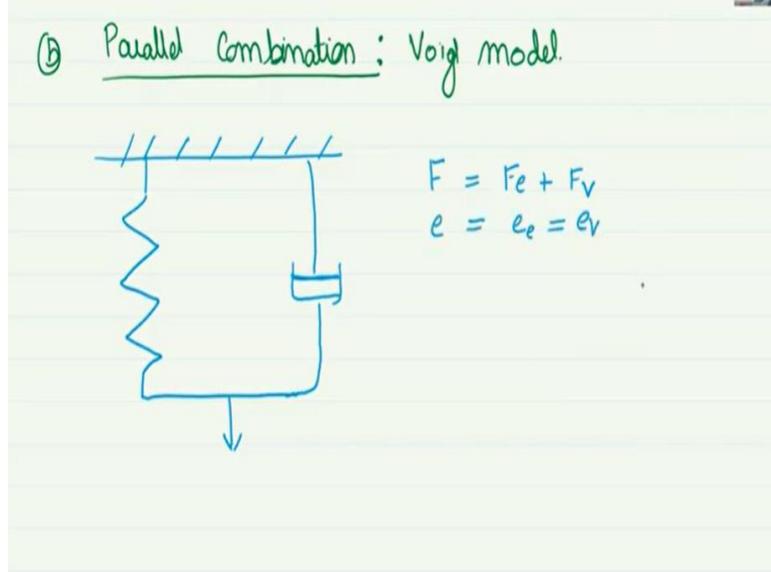
Now let us look at another simple combination which is a parallel combination which is also called Voigt model. Now as the name suggests what we have here is the 2 elements are combined in a parallel combination. So when you apply force, the force is being applied to both of them. And then so the force will be accommodated partly by this and partly by this. And therefore you can write this relation as $F_e + F_v$ but the strain in both of these 2 must be constant must be same and therefore $e = e_e = e_v$.

Now again we will do the similar kind of analysis first we will assume that a sudden force is applied.

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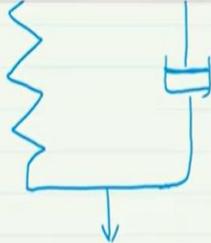
$$e = e_e = e_v$$

Sudden F is applied at time $t=0$

→ initially dashpot carries whole load

→ when deformation takes place, $\dot{e} = \dot{e}_e = \dot{e}_v$.

$$de = de_e = de_v$$

then spring operates

At time t = 0 so when you apply the force we know that the dashpot cannot take instantaneous strain. And which means if there is no strain here there will be no strain on the spring part and therefore there is no force in the spring. And all the force is being taken by all the force load is on the dashpot. So initially dashpot carries the whole load but over time when there is a constant force on this dashpot therefore it will deform and when this will deform it means there will also be deformation over here.

And so there will be some strain over here and therefore there will be equal strain in both the part there will whatever strain happens here that will be the amount of strain here and meaning that much force is being now accommodated over here. And if we write it in the differential form d e = de$_e$ it will come in handy and then spring operates.

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At $t = \infty$ $\qquad e_v(t=\infty) = \dfrac{F}{k_e}$

$$\frac{de_v}{dt} = \frac{F_v}{k_v} = \frac{F - F_e}{k_v} = \frac{F - ek_e}{k_v}$$

$$= \left(\frac{k_e}{k_v}\right)\left(\frac{F_e}{k_e} - e\right)$$

Now if you look at very long time at t = infinity what will happen? If you had applied a stress then eventually the deformation will take place here and this deformation will also imply that the spring will getting deformed and which means that the resistance force is increasing. So at some point

this resistance force will be balanced by the external force and that will mean that there is no more force being carried by the dashboard.

So, in the final equilibrium condition all the force would be carried by the spring and therefore, we can calculate the final amount of strain that would be applied. Let us call it e infinity which is at t = infinity and if the force applied was F. So the total amount of strain that is being applied that will be there in the system eventually will be equal to F/k_e. But we also know that $de_v/dt = F_v/k_v$ but $F_v = F - F_e$ so we can write it as $(F - F_e)/k_v$.

And if we write it with take it k_e/ k_v factor out what we have here is (F_e/k_e) - e
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$$= \left(\frac{k_e}{k_v}\right)\left(\frac{F_e}{k_e} - e\right)$$

$$\frac{de}{dt} = \frac{1}{\tau}(e_\infty - e)$$

$$\frac{de}{(e_\infty - e)} = \frac{dt}{\tau}$$

$$\ln\left(\frac{e_\infty - e}{e}\right) = -\frac{t}{\tau}$$

We can also write k_e /k_v as $1/\tau$ so this is $e_\infty - e$ is de/dt and therefore, this becomes $de/e_\infty - e = dt/\tau$. And then we when you integrate what we get is$\ln(e_\infty - e)/e = -\frac{t}{\tau}$ and now when we simplify it further what we get is?
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$$\frac{de}{(e_\infty - e)} = \frac{dt}{\tau}$$

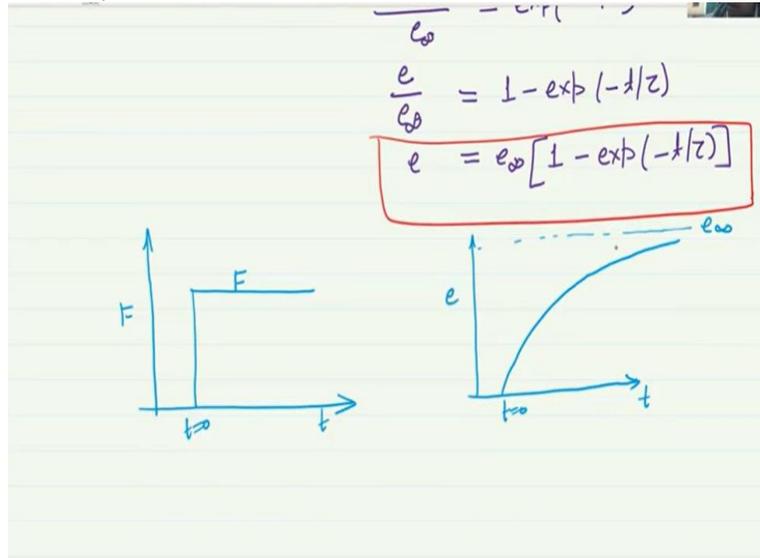$$\ln\left(\frac{e_\infty - e}{e_\infty}\right) = -\frac{t}{\tau}$$

$$\frac{e_\infty - e}{e_\infty} = \exp(-t/\tau)$$

$$\frac{e}{e_\infty} = 1 - \exp(-t/\tau)$$

$$e = e_\infty[1 - \exp(-t/\tau)]$$

$$e = e_\infty[1 - \exp(-\frac{t}{\tau})]$$

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Suddenly we applied certain force so this is t = 0 this is the force that we applied and the strain initially like we said that there will be 0 strain. But it will eventually rise to this value e infinity which is the final strain value. So it will look like this is the e infinity and this is t = 0. So and the y axis strain so for this condition where we have the 2 elements in parallel and if we apply sudden force then initially the strain is 0 and it eventually grows to infinity. So this is what we obtain when we look at the mathematical equations.

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Now the second condition is what if sudden extension is applied now here both spring and the spot are in parallel implies sudden extension of dashpot should also take place. There is no way that you can apply a sudden extension to this whole system and only the spring extends not the dashpot. But then we know that the dashpot cannot show or display behaviour where you can have sudden
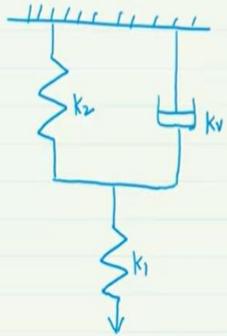
extension which means this type of problem cannot be solved considering this configuration therefore, this is impossible condition.

So first let me write but dashpot cannot have sudden extension and hence it implies in possible condition. So if you happen to model a particular material polymer material using this kind of configuration then you must also be aware that this kind of system will not accommodate any sudden extension. So this is what we understand from this mathematical formulation by solving this mathematical formulation.
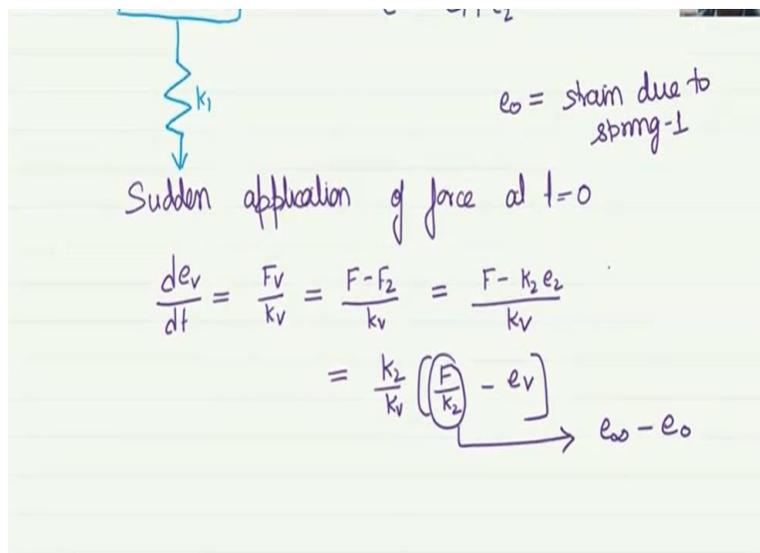
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Continuing with our simple model let us look at one more, simple model which is one level higher in terms of complexity than what we have already considered. So what we do here is combined parallel series configuration which is also called as Voigt Maxwell model. Now as the name suggests what we have here is a; configuration where you have both the things you have the spring and the dashpot in parallel and also in series.

So the string and the dashboard these 2 are in parallel and that dashboard and the spring these are in series. And we can call it k1 this has k2 and this has k_v so let us say this is where you are applying force and the basic equations that will govern this response F. So this F will be equal to the sum of this F which means $F = F_1 = F_2 + F_v$. So this is the force in the second spring and this one is this force in the dashboard and this is a force in the first string.

So this force is same as the total force that you will apply and $e_v = e_2$ because these 2 have to be same and total strain would be equal to e it should be $e_1 + e_2$ right so it will be equal to $e_1 + e_2$. Now let us solve this response for different conditions.

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k_1

$e_0 = \text{strain due to spring-1}$

Sudden application of force at $t=0$

$$\frac{de_v}{dt} = \frac{F_v}{k_v} = \frac{F-F_2}{k_v} = \frac{F-k_2 e_2}{k_v}$$

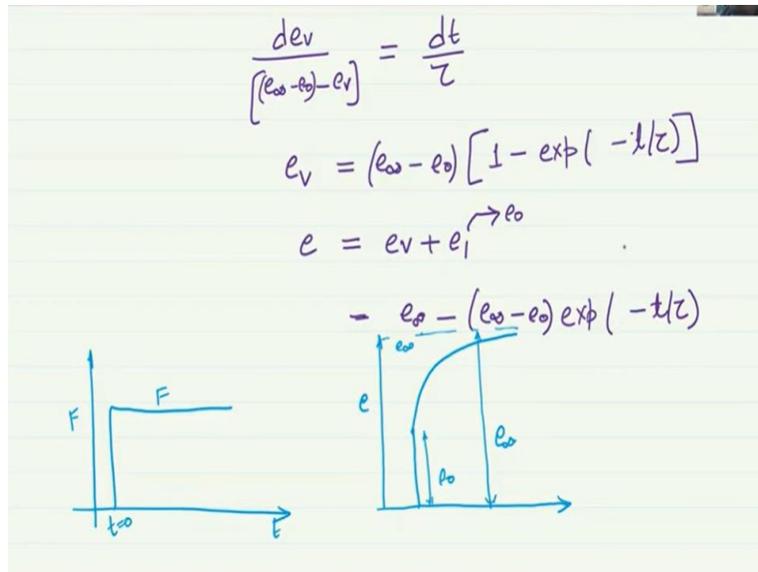$$= \frac{k_2}{k_v}\left(\frac{F}{k_2} - e_v\right) \longrightarrow e_\infty - e_0$$

So let us assume sudden application of force at time t equal to 0. now for this for the dashpot we know $de_v/dt = F_v/k_v$. But $F_v = F - F_2$ for that 2 in the parallel this is what we get. And $F_2 = (F - k_2 e_2)/k_v$ which is equal to if we take k_2 as common outside then what we get is $k_2/k_v [F/k_2 - e_v]$ .Now for F/k_2 we can write F/k_2 as the strain which is equal to e_∞ which will be the total strain minus the instantaneous strain.

So F/k_2 is a term which implies that total force is acting on this spring and what will be the elongation at that time. So when we are talking about the total force acting over here it means it is the condition where the total load is initially the load was being shared by these 2 but now we have the load being only shared by this part. And over here the strain would be equal to the total strain only when the total force is acting only on this and therefore there is no strain.

Therefore, this is F/k_2 now this one is equal to if we assume that the total strain is e_∞ then e_∞ minus the strain which was caused by this much by this spring which is let us call it e naught. So if the strain because of the spring constant for the first spring $= e_0$ therefore, F/k_2 can be written as $e_\infty - e_0$. And therefore this makes our task a lot more easier now we can again use the method that we applied here earlier we can write k_2/k_v as τ

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$$\frac{dev}{[(e_\infty - e_0) - e_v]} = \frac{dt}{\tau}$$

$$e_v = (e_\infty - e_0)\left[1 - \exp\left(-t/\tau\right)\right]$$

$$e = ev + e_1 \nearrow e_0$$

$$- e_\infty - (e_\infty - e_0)\exp\left(-t/\tau\right)$$

And therefore what we have here is d t/ d τ and now when we integrate it what we will have is so we have a variation for the strain in the dashpot and we know that total strain = e $_v$ + e $_1$which is same as e$_\circ$. Therefore when we expand it and then add it then it becomes $e_\infty - (e_\infty - e_\circ)\exp(-\frac{t}{\tau})$ And what this means is that we applied a sudden force like this so this is t = 0 this is time.

And in response to this the strain we get is like this where we have e naught because of the spring one and then exponentially reaching this value e_∞ which is primarily due to spring 2.So it is again if you look at it carefully it is just sum of the earlier 2 conclusions so this one is the series part and this is the parallel part and we are just getting a sum of these 2 when we apply a sudden force. Now let us look at the condition where we will be applying the sudden elongation.
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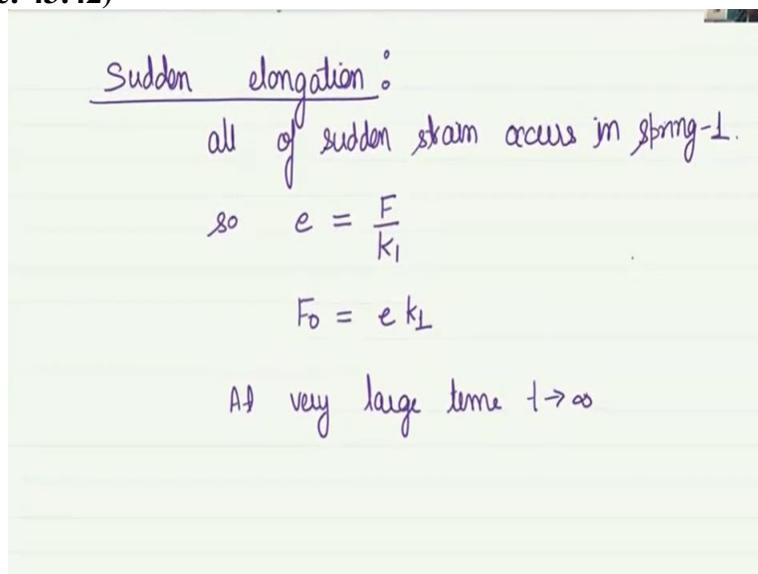
So let us look at the condition of sudden elongation and again, if were member from our earlier discussion discussions that this dashpot is not in a position to take sudden elongation. So it means when you apply a sudden elongation all that strain would be accommodated over here. And with

time then which will also mean that there is a force on the dashpot and therefore it will deform and therefore some amount of strain will get transferred to this.

And therefore the force will also get transferred to this and eventually it will take all loads that, has to exist. And again it would seem like it is just a combination of our earlier results so all of sudden strain occurs in spring one. So e which is the total strain that has been applied is equal to F/k_1 and at time $t = 0$ if we call the force as F_o then $F_o = e\,k_1$.

Now at time at very large time t tending to infinity what we will have we will be equilibrium condition where the strain will also be accommodated by spring 2 and the load will also be accommodated by spring 2. Which; will equal to the total force that we will be applying at that particular point of time.

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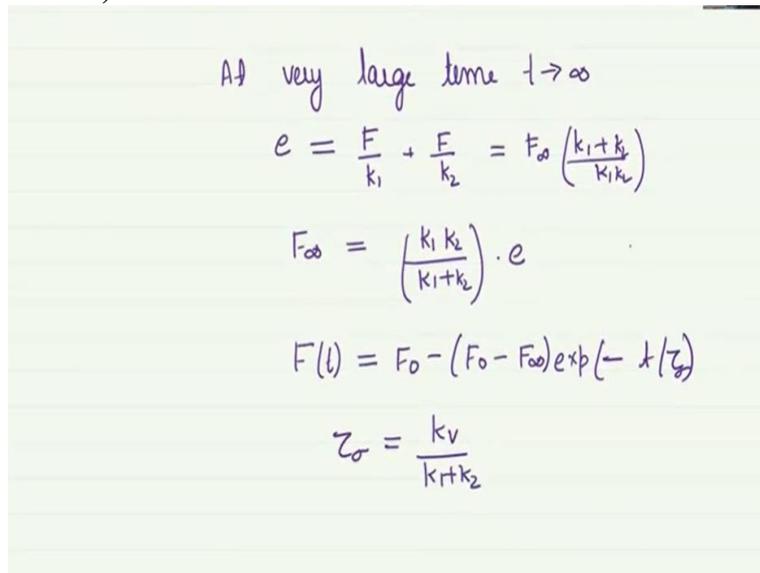
$$\text{At very large time } t \to \infty$$

$$e = \frac{F}{k_1} + \frac{F}{k_2} = F_\infty \left(\frac{k_1 + k_2}{k_1 k_2} \right)$$

$$F_\infty = \left(\frac{k_1 k_2}{k_1 + k_2} \right) \cdot e$$

$$F(t) = F_0 - (F_0 - F_\infty)\exp\left(- t/\tau_\sigma\right)$$

$$\tau_\sigma = \frac{k_v}{k_1 + k_2}$$

Therefore e would be $F/k_1 + F/k_2$. Or in other words if we call it

$$F_\infty = \frac{k_1 k_2}{k_1 + k_2} e$$

So this does not reduce to 0 that is one important thing that we need to realize over here. And there is a initial force which will be equal to $e\,k_1$ and eventually it reduces to this value F_∞

And how it reduces this one we will not derive I will just directly write the equation this will be $F(T) = F_o - (F_o - F_\infty)\exp(\text{-}t/\tau_\sigma)$. So τ_σ is basically the stress relaxation time and this stress relaxation time is given by $k_v/k_1 + k_2$. So the F force reduces from F_o to this particular value with time when we apply a sudden elongation. This is what we obtain from the mathematical description for this condition.

So we have not go into complex here but we see that the formulations are getting more and more, tough. But that is still tractable problem we can still solve it and we can get an explicit equation for the variation in strain or variation in force. Now to sum it up let us quickly solve one simple quick problem which will give us some idea about the relation between k_1 and k_2.

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So it is given that F_o, which is for this particular condition $\frac{F_o}{F_\infty} = 1.02$. And you are asked to find $\frac{\tau_e}{\tau_\sigma}$. Now if you go back and look at the relation you would see that $\frac{\tau_e}{\tau_\sigma} = (k_1 + k_2)/k_2$. And you would also find that $e = F_o / k_1$ so this is the initial force which is also and the total strain is also equal to because it is constant is equal to $(1/k_1+1/k_2)$

And therefore $\frac{F_o}{F_\infty} = \frac{(k_1+k_2)}{k_2}$ which means it is same as $\frac{\tau_e}{\tau_\sigma}$ which is equal to 1.02. So this is our understanding for visco elasticity we have looked at the mathematical formulation for visco elasticity and for different models different simplified conditions. And we also solved the variation or the response of the material in terms of force variation or the strain variation for 2 different conditions.

One whether; you are applying sudden force or other, where you are applying sudden; strain and we were able to obtain a relation for this. So with that we end this topic on visco-elasticity and next we will start discussing about plastic deformation thank you.