

**Mechanical Behaviour of Materials - 1**  
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**Module - 1**  
**Lecture - 4**  
**Principle Stress**

Good morning, students. So, we will continue our discussion today on elasticity. Today we will look into one important concept in elastic behaviour, which is principal stresses.

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The slide is titled "Principal Stresses" and contains the following content:

- Ch-1: Hosford
- Ch-2: Dieter

→ Find a plane where no shear stresses are acting. → Principal stresses.

→ Principal stresses are maximum/minimum stresses and the planes are called principal planes

→ Principal stresses are important in determining onset of yielding

$$\sigma_p^3 - I_1 \sigma_p^2 - I_2 \sigma_p - I_3 = 0$$

So, the reference from where you can read more about it are chapter 1 in the book by Hosford and chapter 2 in the book by Dieter. We started with normal stress and looked at planes where both normal and shear stresses were acting. So, if you remember our earlier drawing, we were looking; so, some weight that was put over this and we were looking at stresses under normal conditions.

And then we also looked at a particular plane which was like this. So, we found that on these oblique planes, we have both normal stresses and shear stresses happening. We can also work backwards in the sense that we can start with a condition where both normal stresses and shear stresses are working and find a plane where no shear stresses are acting and only normal stresses are acting. So, these kinds of stresses are called principal stresses.

We can find a plane where; these are what are termed principal stresses. So, the normal stresses under that condition are called; we will understand this with the help of some examples; but before that, what is the formal definition? So, the formal definition is that principle stresses are maximum, minimum stresses and the; so, if you look at a particular system; so, for example, let us say we are looking at this system.

So, in this, if you look at various planes, there will be a different amount of normal stresses. And certain plane will have the maximum normal stress. So, that value of normal stress will be called

principal stresses. And in that particular plane, you will have no shear stresses, and those planes would be called principal planes; on planes, and the planes are called principal planes.

This concept is very useful because, first it helps to simplify the overall thing. You can look at the element which where only principal normal stresses are acting and hence a lot of important parameters can be obtained. And also this principal stresses would also be useful, as you would see, when we are trying to find the onset of yielding. So, in usually the equation that you will obtain will involve all the 9 stresses.

But then, the same equation can also apply to the condition, to the element which is rotated in a way where only principal stress is acting. And thus, there are only 3 elements acting on it. And hence, you can get a much simpler equation for finding the onset of yielding. How do we find the value of the principal stresses? So, there is an equation. We will not derive that equation.

We will look at that equation and you would see that there are some parameters which are called invariants. And those invariants are related to the values of stress and strain. And these invariants, as the name suggests, values would remain invariable no matter what particular orientation of element you are selecting. So, for finding the principal stress for any particular given element inside a given component, so, you will know all the values of stresses, shear stresses and normal stresses. And from that, you will calculate the invariants. And once you have the invariants, you will use this equation to calculate the principal stresses.

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Handwritten notes on a slide showing the cubic equation for principal stresses and the definitions of stress invariants  $I_1$ ,  $I_2$ , and  $I_3$ .

onset of yielding  
 $\sigma_p^3 - I_1 \sigma_p^2 - I_2 \sigma_p - I_3 = 0$   
 Roots of this eqn  $\rightarrow$  Principal stresses

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$I_2 = \sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 - \sigma_{xx} \sigma_{yy} - \sigma_{yy} \sigma_{zz} - \sigma_{zz} \sigma_{xx}$$

$$I_3 = \sigma_{xx} \sigma_{yy} \sigma_{zz} + 2 \sigma_{xy} \sigma_{yz} \sigma_{zx} - \sigma_{xx} \sigma_{yz}^2 - \sigma_{yy} \sigma_{zx}^2 - \sigma_{zz} \sigma_{xy}^2$$

So, as you can see, this is the equation. So,  $\sigma_p$  is the variable which is, the root of this equation would give you the 3 principal stresses. As you can realise, there will be normal stresses along the 3 directions, so, 3 roots would be found; and therefore, we have a cubic equation. So, roots of this equation gives principal stresses. And what are these invariants? So, these invariants, as I have already mentioned, their values will remain invariable or constant no matter how you orient the element.

So, you have the one particular point inside the component where you want to calculate the stress with the state of stress. So, for that particular point, you can have various orientations. And no

matter how you change the orientation, these values or even  $I_2$  and  $I_3$ , they will remain constant. So,

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$I_2 = \sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2 - \sigma_{xx}\sigma_{yy} - \sigma_{yy}\sigma_{zz} - \sigma_{zz}\sigma_{xx}$$

$$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{xy}\sigma_{yz}\sigma_{zx} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{zx}^2 - \sigma_{zz}\sigma_{xy}^2$$

And for  $I_3$  we have; so, these are the values of the invariants. And as you can see, these values of invariants are described in terms of  $xx$ ,  $yy$ ,  $xy$  meaning all the normal and shear stresses are concerned, but then, as we said, it is invariant, it means that these would be valid even if you are talking or taking the element. Now, orientation, here you have only normal stresses, meaning principal stresses.

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$$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{xy}\sigma_{yz}\sigma_{zx} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{zx}^2 - \sigma_{zz}\sigma_{xy}^2$$

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)$$

$$I_3 = \sigma_1\sigma_2\sigma_3$$

And therefore, in that condition, we will write; so, the principal stresses we are defining or writing, nomenclature is to use the subscript 1, 2, 3. So,  $\sigma_1$  defines principal stress 1;  $I_2$  defines principal stress 2;  $I_3$  defines principal stress 3. And therefore, these values of  $I_1$ ,  $I_2$ ,  $I_3$  will translate to; so, you can go back and look at over here,  $\sigma_{xx}$ ,  $\sigma_{yy}$ ,  $\sigma_{zz}$  which would be basically, there will be  $\sigma_{11}$ ,  $\sigma_{22}$ ,  $\sigma_{33}$ .

So, this becomes  $\sigma_1$  or  $\sigma_{11}$  is same as  $\sigma_1$ ;  $\sigma_2 + \sigma_3$ .  $I_2$ , now, here you can go back and see that there will be no; in the principle plane orientation, there will be no shear stresses. So, these three would be 0. And what you will have is only this one and this one, and this one and this one, and this one and this one. So, this becomes 1, this becomes 2; 2 and 3, and 3 and 1. And therefore,

$$\text{this becomes } I_2 = -(\sigma_1\sigma_2 + \sigma_3\sigma_2 + \sigma_1\sigma_3)$$

And similarly, if you look at  $I_3$ , all the, if you get rid of all the shear stress component, then all these terms go away and you have only the first element, which means it will become

$$I_3 = \sigma_1\sigma_2\sigma_3$$

Now, let us try to calculate the values of  $\sigma_1$ ,  $I_2$ ,  $I_3$  for the 2 conditions, for the example that we had discussed earlier. So, let me draw that condition again, that we had earlier.

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$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)$$

$$I_3 = \sigma_1\sigma_2\sigma_3$$

Applying to earlier example

$$I_1 = \sigma_{yy}$$

$$I_1 = \sigma_{x'x'} + \sigma_{y'y'} + \sigma_{z'z'}$$

$$\sigma_{y'y'} = \sigma_{yy} \cos^2 \theta$$

$$\sigma_{x'x'} = \sigma_{yy} \sin^2 \theta$$

$$\sigma_{x'y'} = \sigma_{yy} \cos \theta \sin \theta$$

So, here, given that the stress remains constant, so, this whole thing can be taken as single point, because the stress are not changing. If the stress were changing, then one point would be considered different from the other point. And with that respect, we can assume this is one point or one for stress, one point represents the whole element. But this is one condition; the other condition where we had the oblique plane; so, the element can be assumed to be something like this in this case.

And this becomes

$$\sigma_{y'y'} = \sigma_{yy} (\cos \theta)^2$$

$$\sigma_{x'x'} = \sigma_{yy} (\sin \theta)^2$$

$$\sigma_{y'x'} = \sigma_{yy} \sin \theta \cos \theta$$

So, as you can see, if theta is 0, then  $\sigma_{y'y'}$  will be equal to sigma yy, which is what you would expect. And  $\sigma_{x'x'}$  is normal to this; so, when theta is equal to 0, this  $\sigma_{x'x'}$  will become; this is sin square theta.

So, there is a square term because there is also the effect of area. And the other one is the direction of the force. So, we are taking the component of the force as well as the component of the area; and therefore we have the square term. If you go back to the notes, you would see, this is what type relation was coming out to. Now, we have these two conditions. We will find the invariant in both the conditions.

And as I said, we are expecting that since it is an invariant, therefore the value should remain same here and over here, in the left and the right conditions.

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So, for the left  $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$  ; and therefore, the only element that we have here is sigma yy. For  $I_2$ , if you go back and look at the values, all the values are 0, and therefore it turns out to be equal to 0. For  $I_3$ , again all the components for  $I_3$  turns out to be 0, and therefore  $I_3$  is equal to 0. So, this is the value of  $I_1, I_2, I_3$  that we obtained for this left condition.

Now, we must get the same values for this oblique condition because it is representing the same element, only the difference is in orientation. And as we have said that invariants should remain unchanged when we change the orientation for the same point. Now, coming to  $I_1$ ; so,

$$I_1 = \sigma_{x'x'} + \sigma_{y'y'} + \sigma_{z'z'}$$

On substituting the values of stresses in above equation it comes out to be

$$I_1 = \sigma_y$$

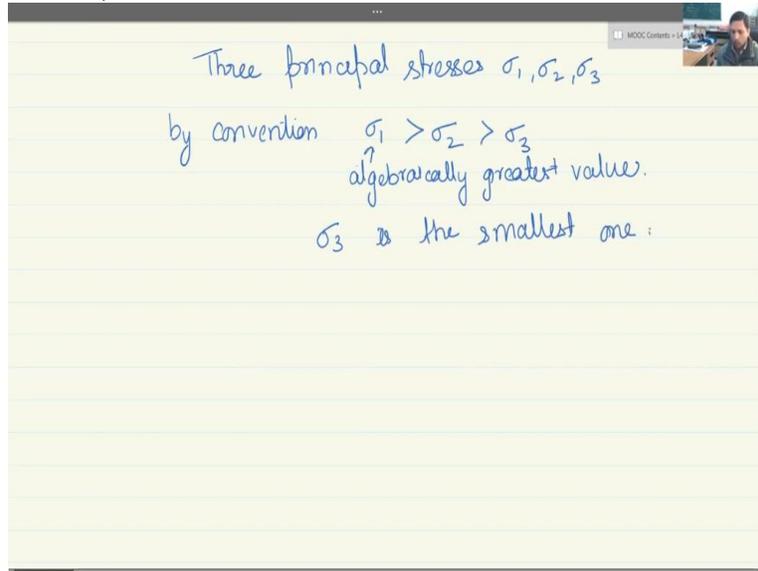
Likewise substituting the value of stresses in expression related to  $I_2$ , it comes out to be 0.

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And therefore,  $I_2$  is equal to 0 which is same as on the left hand side or the first condition.

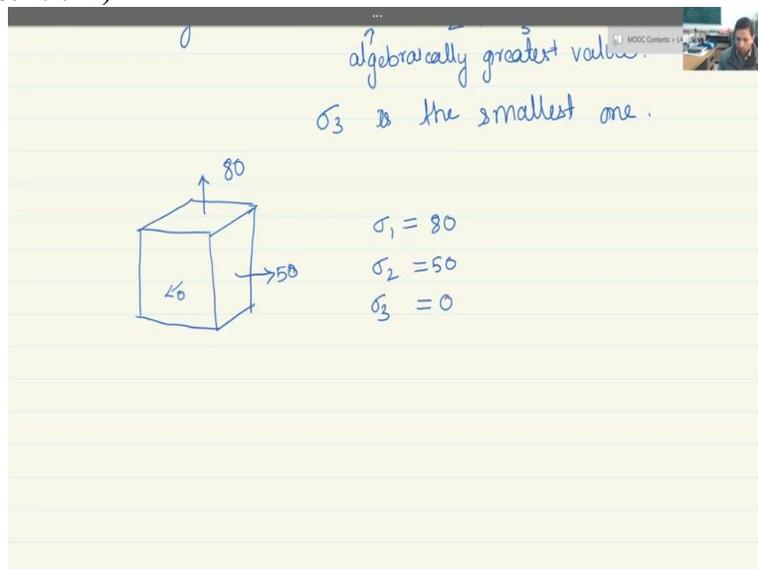
Now, coming to  $I_3$ ; for the  $I_3$ , you would see that these are the elements  $xx, yy, zz$ . So,  $zz$  is 0; so, this term goes to 0; all the other are anyway 0. So,  $I_3$  is also 0. What do we see that, yes, indeed invariants are same in both the cases. And this is not some, you can say, coincidence, but as you would see, this is the way it will be for all the conditions. So, now moving on, we will look a little bit more about the principal stresses.

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And as I mentioned earlier that when we are talking about the principal stresses, they will be, they are normal stresses when there are no shear stresses, and there will still be 3 numbers for the 3 different planes. So, we will have; this is usual nomenclature,  $\sigma_1, \sigma_2, \sigma_3$ ; and by convention,  $\sigma_1 > \sigma_2 > \sigma_3$ . So,  $\sigma_1$  is algebraically the largest principal stress, meaning, including the positive and negative sign, whichever comes out to be the highest number, that would be defined as  $\sigma_1$ .  $\sigma_3$  would be the smallest one. So, now, let us look at some simple examples.

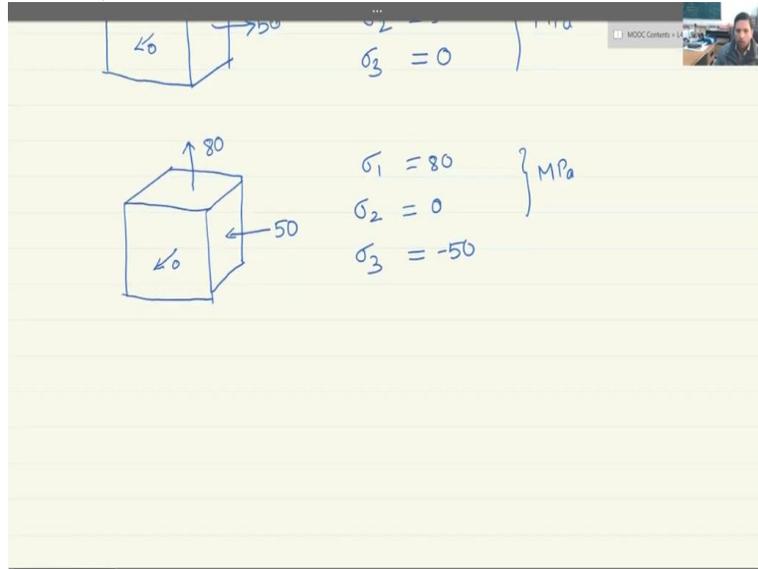
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So, let us say you have 80 MPa is the normal stress and there are no shear stresses; and 50 MPa is the normal stress in this direction. So, what will be the value of  $\sigma_1$ ? Which is the  $\sigma_1$  value? So, yes,

this is the  $\sigma_1$  value. So,  $\sigma_1$  is equal to 50. What is  $\sigma_2$ ?  $\sigma_2$  is 50. And what about  $\sigma_3$ ? Is there any  $\sigma_3$ ? Yes, there is indeed, because this is, the value here is 0. So, the stress acting in this direction is 0; so, that is the value of  $\sigma_3$ , equal to 0.

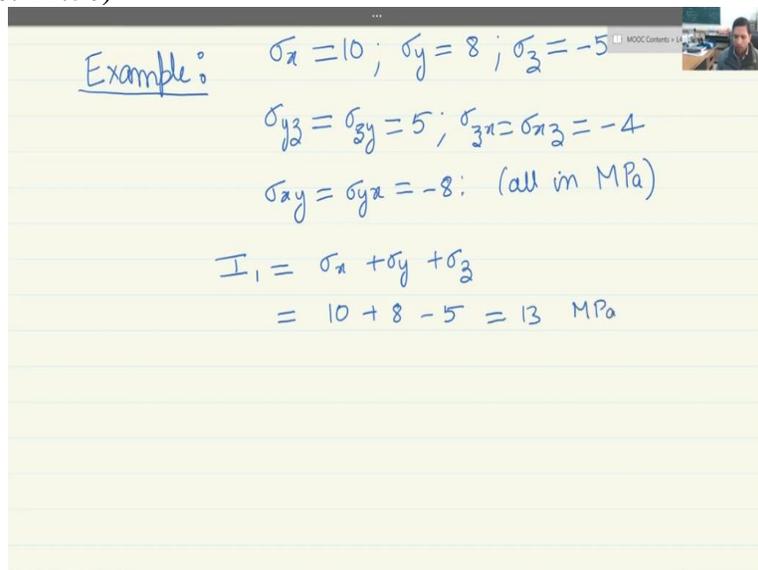
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Now, let us look at another example. Here, 80 is still acting on the top face, but 50 is now acting in the compressive direction. So, what is the value of sigma 1? So, algebraically, still this is the highest value;  $\sigma_1$  is equal to 80, and there is a unit of megapascal which is there for all of these.  $\sigma_2$ ; now, you may be tempted here to say that -50 is  $\sigma_2$ , but that is not the case. Here also you have 0.

And amongst -50 and 0, 0 is higher; so,  $\sigma_2$  will be 0. And sigma 3 will be equal to -50. So, that is how the  $\sigma_1$ ,  $\sigma_2$  and  $\sigma_3$  are defined. Now, moving on, let us look at one example.

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So, let us say you are given that  $\sigma_x$  is equal to 10; and again I am assuming that the units are in MPa.  $\sigma_y$  is equal to 8;  $\sigma_z$  is equal to -5;  $\sigma_{yz}$  is equal to  $\sigma_{zy}$  is equal to 5; and  $\sigma_{zx}$  is equal to  $\sigma_{xz}$  is

equal to -4; and  $\sigma_{xy} = \sigma_{yx} = -8$ . So, you are given all of these. Now, you have to find the values of  $I_1$ ,  $I_2$  and  $I_3$  for this. So, again we will apply.

Here, let me just write the title, applying to earlier example.  $I_1$  we know is equal to  $\sigma_x + \sigma_y + \sigma_z$ . And since we will be able to find  $I_1$ ,  $I_2$ ,  $I_3$ , we should also see what will be the values of principal stresses that we can obtain from these given conditions. So, here these are given. So, this is  $10 + 8 - 5 = 13$  MPa. Actually I would not write the units because there will be some square terms; so, there it will become mega Pascal square. We will assume that the units are consistent with each other.

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$$I_2 = \sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2 - \sigma_{xx}\sigma_{yy} - \sigma_{yy}\sigma_{zz} - \sigma_{zz}\sigma_{xx}$$

$$= (-8)^2 + 5^2 + (-4)^2 - 10 \times 8 - (8 \times -5) - (-5 \times 10)$$

$$= 115$$

$$I_2 = \sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2 - \sigma_{xx}\sigma_{yy} - \sigma_{yy}\sigma_{zz} - \sigma_{zz}\sigma_{xx}$$

So, putting in the values, this will become 115. So, that is the value of  $I_2$ .

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$$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{xy}\sigma_{yz}\sigma_{zx} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{zx}^2 - \sigma_{zz}\sigma_{xy}^2$$

$$= 10 \cdot 8 \cdot (-5) + 2(-4)(5)(-8) - 10(5)^2 - 8(-4)^2 + (-5)(-8)^2$$

$$= -138$$

Now, calculating the value of  $I_3$ ,

$$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{xy}\sigma_{yz}\sigma_{zx} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{zx}^2 - \sigma_{zz}\sigma_{xy}^2$$

And when you solve it, you would see that this comes out to -138. So, now, we have the value of  $I_1$  is 13;  $I_2$  is 115;  $I_3$  is -138.

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The image shows a handwritten derivation on a yellow background. The first part calculates the third invariant  $I_3$  using the formula  $I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{xy}\sigma_{yz}\sigma_{zx} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{zx}^2 - \sigma_{zz}\sigma_{xy}^2$ . The values substituted are  $\sigma_{xx}=10$ ,  $\sigma_{yy}=8$ ,  $\sigma_{zz}=-5$ ,  $\sigma_{xy}=-4$ ,  $\sigma_{yz}=-8$ , and  $\sigma_{zx}=5$ . The calculation proceeds as follows:  $I_3 = 10 \cdot 8 \cdot (-5) + 2(-4)(5)(-8) - 10(5)^2 - 8(-4)^2 + (-5)(-8)^2 = -138$ . The second part shows the cubic equation  $\sigma_p^3 - 13\sigma_p^2 - 115\sigma_p + 138 = 0$  and lists its three roots:  $\sigma_p = 1.079$  ( $\sigma_2$ ),  $18.72$  ( $\sigma_1$ ), and  $-6.82$  ( $\sigma_3$ ).

And therefore, we can put it in the equation form for finding the principal stresses,

$$\sigma_p^3 - 13\sigma_p^2 - 115\sigma_p + 138 = 0$$

And this is a cubic equation which you can solve using any of the solvers here like Mathematica. And what you would get is that, sigma p is equal to 1.079; that is one value; other value is 18.72; and still other value is 6.82.

So, again we can see that these are 3 values of, as you would expect, because there are 3 principal stresses. So, this one is sigma 1, this one is sigma 2 and this is sigma 3. So, the concept of principal stresses should now be amply clear to you. Well, once we get the invariants from a given condition, we can find the principal stresses, meaning an orientation. So far, we have not tried to look at the angle theta, but this is not something we are trying to obtain yet.

And you can easily obtain the theta, but we will look at it a little later. What we have found out so far are only the values of  $\sigma_1, \sigma_2, \sigma_3$  meaning, let us say there was this orientation and then you move to treat it in some other condition. So, here, earlier one, you had normal as well as shear stresses acting, but now you have only normal stresses acting. So, we have found the values of those  $\sigma_1, \sigma_2, \sigma_3$  but still not given you the equation for finding the theta; we will come to that.

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18.72  $\sigma_1$   
-6.82  $\sigma_3$

When shear stress on one plane is zero.

- > Normal stress in z-direction is a principal stress
- > Other two principal stresses lie in the plane
- > simple transformation rules

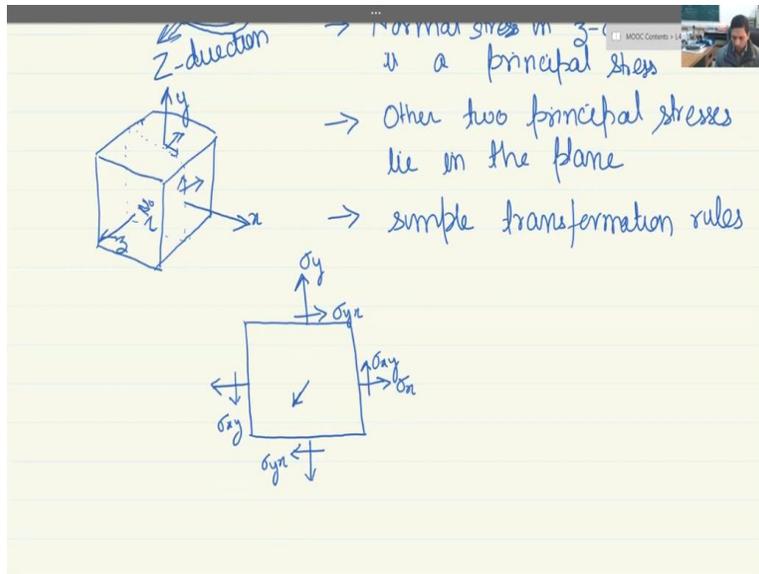
So, for that, before we are able to do that, let us again talk about transformation. We talked about transformation earlier, which was a very general equation; now we will look at a transformation for a specific case. So, in engineering, you would see that the components or the stress condition, loading conditions for a material are usually either plane stress or plane strain condition; and therefore, when we are talking about transformation, our task becomes much easier.

So, let us say we have a component which is like this. As you can see, it is a thin element and therefore there will be no; and let us assume that there are no shear stresses on the z side, z component. So, basically, there may be normal stress in the z direction, but there are no principal stresses. So, the conditions that you would usually see are, when shear stress on one plane is 0, normal stress; so, this becomes the principal stress in that direction, because there are no shear stresses.

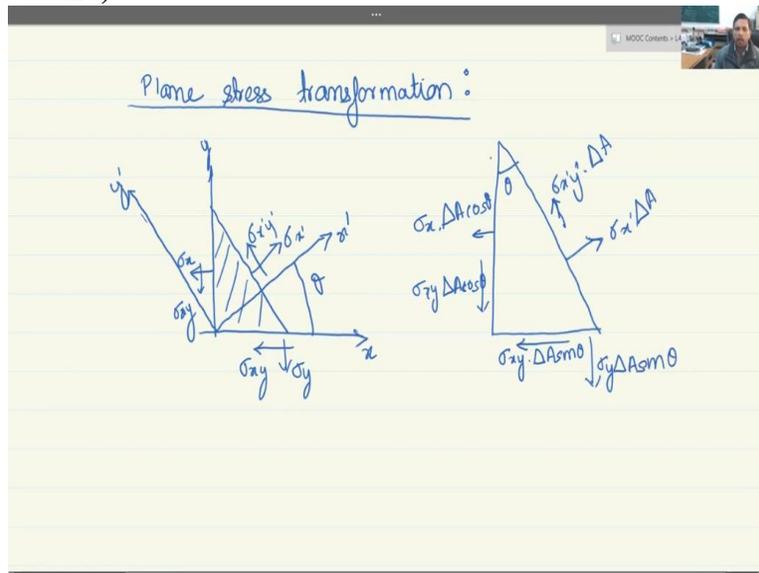
We are not talking about the parallel directions along these, but along this particular direction. For this particular direction, we have only normal stress acting; and therefore, this becomes the principal stress. So, let us say this is z direction. So, normal stress in the z direction is a principle stress. Other two principle stresses lie in this plane. So, this is a plane, this whole plane; so, other two stresses lie in a; so, for this particular situation which is very common we encounter in engineering, we can find a simple transformation rules.

So, to describe it in terms of element, what we have here is, here you may have shear stresses, here you may have shear stresses, but not over here. So, there are no outcome over here. That is what we have usually and in this particular case, if you want to transform;

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So, if you look at one cross section, then you will have  $\sigma_y$ ,  $\sigma_{yx}$ ,  $\sigma_x$ . So, this is a plane stress condition and most of the time when you will, because your third direction you already have a principal stress, so, the rotation that you would need is only in this plane, in the plane that is shown over here and that kind of rotation is much simpler to describe. So, plane stress transformation. (Refer Slide Time: 33:26)



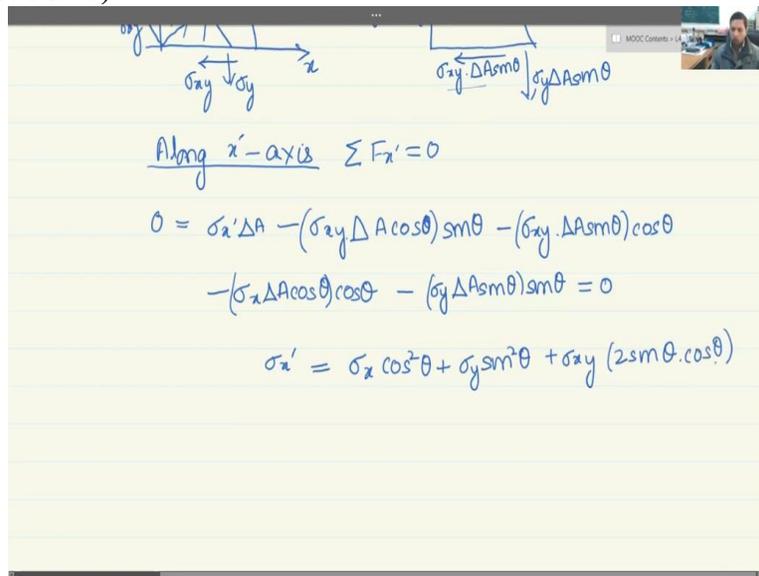
So, you are given something like this; it is your original x-axis; it is your original y-axis. And now you want to rotate and find stresses in another axis which has x prime and y prime and this is rotated by an angle theta. So, this is theta then. So, basically this is normal parallel to the y direction. In the original xy, you had, this is also sigma y and this as  $\sigma_{xy}$ ; this as  $\sigma_x$ ; and this as  $\sigma_{xy}$ .

And if the area that is on which this is acting; so, let us say this is, this area is  $\Delta A$ . First let us look at sigma x prime y prime. And if we call this area as  $\Delta A$ , then the force acting here is  $\sigma_{x'y'} \Delta A$ . And similarly, if the stress acting here is  $\sigma_{x'}$ , then the force acting here is  $\sigma_{x'} \Delta A$ . On the other hand, here, the stress that is acting is  $\sigma_x$ .

And since this is  $\Delta A$  and this is theta, so, therefore this one becomes the area into, to calculate the force, the area, this one is  $\Delta A \cos \theta$ ; and therefore, sigma x times delta A cos theta is the force acting in this direction. And similarly, sigma xy is the shear stress and area is  $\Delta A \cos \theta$ . Here, this is sigma xy and area here is  $\Delta A \sin \theta$ . So, this is a triangular element and if it is not rotating, it remains stationary under equilibrium, then these forces must balance out.

And it is on this principle that we will be able to find a general relation for a plane stress transformation. So, let us look at the stresses along x-axis, along x prime axis actually. So, the overall body is not moving, therefore  $\sum F_x = 0$ . And therefore, we know that the total sum should come out to 0. And therefore, we can write it as 0 equal to  $\sigma_{x'} \Delta A$ ; that is the force acting along the x prime direction; minus, this is the force which is acting like this; and we will have to find a component along x prime.

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So, overall this becomes

$$\sigma_{x'} \Delta A - (\sigma_{xy} \Delta A \cos \theta) \sin \theta - (\sigma_{xy} \Delta A \sin \theta) \cos \theta - (\sigma_x \Delta A \cos \theta) \cos \theta - (\sigma_y \Delta A \sin \theta) \sin \theta = 0$$

Again taking a component and when you equate it, you would see that

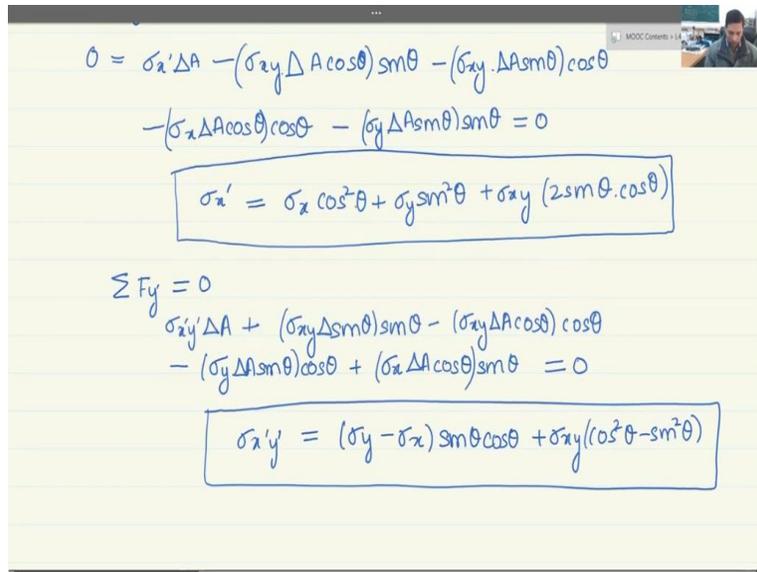
$$\sigma_{x'} = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + \sigma_{xy} (2 \sin \theta \cos \theta)$$

So, this is the equation that we obtain when we are balancing the forces along x prime direction. Now, we can balance the forces along y prime direction.

And again, since it is in equilibrium, we know that it should turn out to be equal to 0. So, again, let us look at the components. So, first, this is the y prime component, which is this one. And then, we will have to take the components of each of these by taking either sin theta or cos theta; so,

$\sigma_{xy}$ .

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$$0 = \sigma_x' \Delta A - (\sigma_{xy} \Delta A \cos \theta) \sin \theta - (\sigma_{xy} \Delta A \sin \theta) \cos \theta - (\sigma_x \Delta A \cos \theta) \cos \theta - (\sigma_y \Delta A \sin \theta) \sin \theta = 0$$

$$\sigma_x' = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + \sigma_{xy} (2 \sin \theta \cos \theta)$$

$$\sum F_y = 0$$

$$\sigma_x' y' \Delta A + (\sigma_{xy} \Delta A \sin \theta) \sin \theta - (\sigma_{xy} \Delta A \cos \theta) \cos \theta - (\sigma_y \Delta A \sin \theta) \cos \theta + (\sigma_x \Delta A \cos \theta) \sin \theta = 0$$

$$\sigma_x' y' = (\sigma_y - \sigma_x) \sin \theta \cos \theta + \sigma_{xy} (\cos^2 \theta - \sin^2 \theta)$$

So, these are the various components we have taken along the y-axis and we know this will be 0. And when we simplify or take away all the 0 elements, then what we see is that,

$$\sigma_{x'y'} = (\sigma_y - \sigma_x) \sin \theta \cos \theta + \sigma_{xy} (\cos^2 \theta - \sin^2 \theta)$$

And this is something that you should try on your own and you would see that it is, that to ensure that this is what it comes out to; and this will also give you a practice and to how to derive this.

So, what we have here, an equation for sigma x prime and sigma x prime y prime. So, this will give you rotation. So, you have stresses, let us say in the xy coordinate. Then, this equation lets you get the normal stresses along x prime y prime direction and also shear stresses along x prime y prime direction. So, this is the plane stress transformation. Now, this plane stress transformation can be made a little bit more, you can say simpler or to look if we apply some of these trigonometric equations.

So, but before that let me just highlight the 2 equations that we have here. And remember that we have obtained the equation for sigma x prime. Now, sigma y prime will only be the same thing, but theta will be theta plus 90 degrees. So, then you can get sigma y prime. And therefore, sigma x prime and sigma y prime are not very different.

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$$\sigma_{x'y'} = (\sigma_y - \sigma_x) \sin\theta \cos\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta)$$

$$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$2\sin\theta \cos\theta = \sin 2\theta$$

$$\cos^2\theta - \sin^2\theta = \cos 2\theta$$

Now, if we apply some of these trigonometric equations which are the ones that I am talking about. So, these are the equations that I am talking about: cos square theta is equal to cos 2 theta plus 1 by 2. And similarly, sin square theta is equal to 1 minus cos 2 theta by 2. So, basically what we are doing is, this equation gets transformed from theta to 2 theta format. And there is a reason; you would see why 2 theta becomes a characteristic value compared to theta.

And another equation that may be handy is 2 sin theta cos theta is equal to sin 2 theta. And similarly, cos square theta minus sin square theta is equal to cos 2 theta. So, now, once you apply these equations over there, what you would get are the following equations.

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$$2\sin\theta \cos\theta = \sin 2\theta$$

$$\cos^2\theta - \sin^2\theta = \cos 2\theta$$

$$\sigma_{x'} = \left[\frac{\sigma_x + \sigma_y}{2}\right] + \left[\frac{\sigma_x - \sigma_y}{2}\right] \cos 2\theta + \sigma_{xy} \sin 2\theta$$

$$\sigma_{y'} = \left[\frac{\sigma_x + \sigma_y}{2}\right] - \left[\frac{\sigma_x - \sigma_y}{2}\right] \cos 2\theta - \sigma_{xy} \sin 2\theta$$

$$\sigma_{x'y'} = -\left[\frac{\sigma_x - \sigma_y}{2}\right] \sin 2\theta + \sigma_{xy} \cos 2\theta$$

So, this will become your go to equation whenever you want to do any transformation;

$$\sigma_{x'} = \left[\frac{\sigma_x + \sigma_y}{2}\right] + \left[\frac{\sigma_x - \sigma_y}{2}\right] \cos 2\theta + \sigma_{xy} \sin 2\theta$$

And like I said, sigma x prime and sigma y prime are same thing, only that they are rotated by 90 degrees as will also be obvious from this. So, the first term does not have any theta, so it comes out to be

$$\sigma_{y'} = \left[ \frac{\sigma_x + \sigma_y}{2} \right] - \left[ \frac{\sigma_x - \sigma_y}{2} \right] \cos 2\theta - \sigma_{xy} \sin 2\theta$$

So, as you can see, when theta is rotated by 90 degrees 2 theta, it becomes 180 degrees. So, it is still cos 2 theta, but there is a minus sign. And over here we will have again minus sign. So, this gives us sigma y prime and  $\sigma_{x'y'}$  is equal to

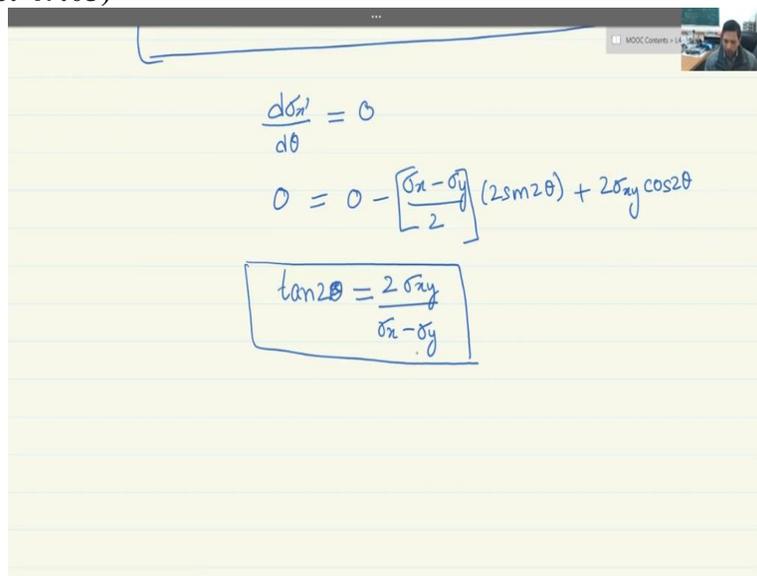
$$\sigma_{x'y'} = - \left[ \frac{\sigma_x - \sigma_y}{2} \right] \sin 2\theta + \sigma_{xy} \cos 2\theta$$

So, this is an equation that will become very important for you whenever we are talking about transformation, any kind of transformation.

So, this is given in the original frame sigma x, sigma y and sigma xy. And now, you can get  $\sigma_{x'}$  which is oriented at an angle theta and  $\sigma_{y'}$  which is at 90 degrees to  $\sigma_{x'}$  and  $\sigma_{x'y'}$  which is the shear stress for the xy orientation. And these equations will come in very handy. Now, once you have this equation, one important question that may arise to you is, how to obtain principal stress from this?

Because we have these stresses, so, should not there be any way where we can get principal stresses from this? And the answer is yes. Why? Because we know that principal stresses are where the stress value is highest. That is one characteristics. The other characteristics is that on that particular plane there are no shear stresses.

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So, going with the first condition, we know that

$$\frac{\partial \sigma_{x'}}{\partial \theta} = 0$$

So, if we differentiate the first equation with respect to theta, we should; and assuming that it is giving us the maximum, or we can check for that,  $d^2 \sigma_{x'} / d\theta^2$  is less than 0; then

we would know that, if it is less than 0, then we know it is we are getting maximum. So, let us differentiate this and what you would get is

$$0 = 0 - \left[ \frac{\sigma_x - \sigma_y}{2} \right] 2 \sin 2\theta + 2\sigma_{xy} \cos 2\theta$$

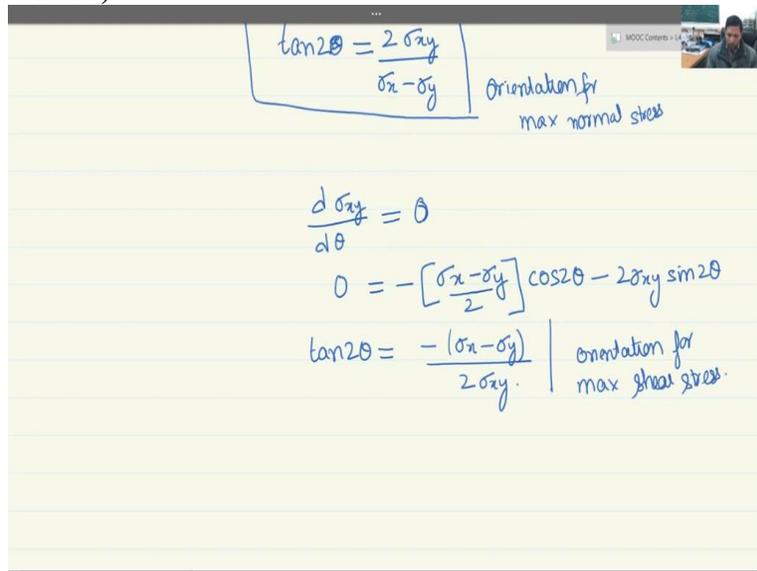
Or, in other words, we have

$$\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_x - \sigma_y}$$

so, what is it that we are getting from this equation? This is by equation that I mentioned earlier that would give you the orientation for the principal stresses. So, this  $2\theta$ ,  $\tan 2\theta$ ; if you get  $2\theta$  equal to  $\tan^{-1}$  of this, will give you the  $\theta$  at which you would get maximum normal stress, and the shear stresses would be 0.

On the other hand, here we can also; from this also you can get the orientation where you can get the maximum shear stress, but let us do it independently; and for that, we know that the maximum shear stress does, the plane where you will have the maximum shear stress, then, for that, you will have  $\frac{\partial \sigma_{xy}}{\partial \theta} = 0$ .

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Handwritten notes on a yellow background showing the derivation of the orientation for maximum normal and shear stress. The first part shows the equation for maximum normal stress:  $\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_x - \sigma_y}$  with the note "orientation for max normal stress". The second part shows the derivation for maximum shear stress:  $\frac{d\sigma_{xy}}{d\theta} = 0$ , leading to  $0 = -\left[\frac{\sigma_x - \sigma_y}{2}\right] \cos 2\theta - 2\sigma_{xy} \sin 2\theta$ , which simplifies to  $\tan 2\theta = \frac{-(\sigma_x - \sigma_y)}{2\sigma_{xy}}$  with the note "orientation for max shear stress".

And again when you differentiate, what we get is the equation like this. And in this case, the  $\theta$  where the shear stress would be maximum, what you would get? So, this is the relation for the condition where you will get maximum shear stress. Now, what do we see the orientation for? So, this is the shear orientation for maximum normal stress and this is giving you the orientation for; from these two relation, you can clearly see that the orientation for maximum normal stress and orientation for maximum shear stress would be away by 90 degrees.

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$\tan 2\theta = \frac{-\sigma_x - \sigma_y}{2\sigma_{xy}}$  | orientation for max shear stress



- \* Max normal stress and max shear stress away by  $2\theta = 90^\circ \Rightarrow \theta = 45^\circ$
- \*  $\sigma_x' + \sigma_y' = \sigma_x + \sigma_y$
- \* max shear stress occurs at halfway between max and min normal stresses.

So, these are some of the important things that we can notice from here. Apart from that, few more things that we can notice, and those are like  $\sigma_x' + \sigma_y' = \sigma_x + \sigma_y$ . So, that is, the sum of the normal stresses on two perpendicular planes is an invariant quantity. And that we already know because  $\sigma_z$  is a constant and it is independent of orientation or angle; so, for that particular plane orientation.

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- \*  $\sigma_x' + \sigma_y' = \sigma_x + \sigma_y$
- \* max shear stress occurs at halfway between max and min normal stresses.
- \* The variation of normal and shear stress occurs in the form of a sine wave, with a period of  $\theta = 180^\circ$ .

These relations are true only for 2D state of stress.

And another one is something we have already discussed, maximum normal stress and maximum shear stress orientation are away by  $\theta$  equal to 45 degrees. And maximum shear stress occurs at an angle halfway between maximum and minimum normal stress; halfway between max and min normal stresses. So, there is one thing that I made a mistake here.

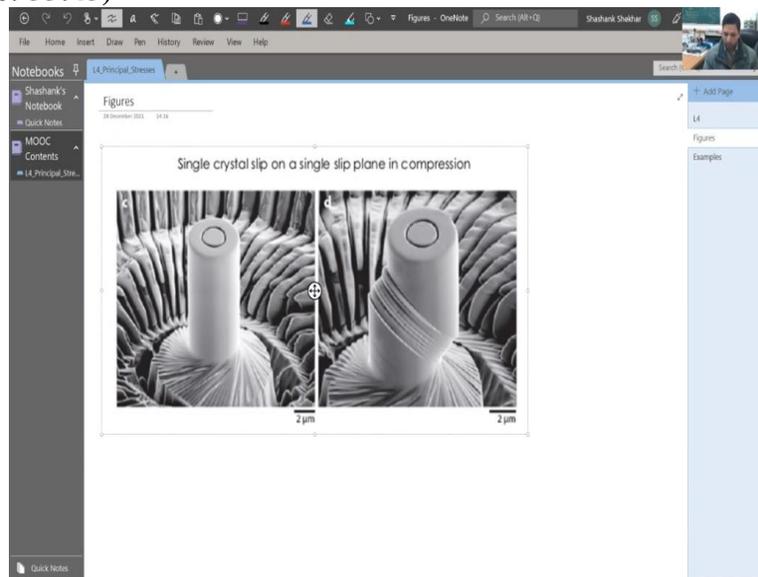
So, maximum normal stress and maximum shear stress are away by  $2\theta$  equal to 90, not  $\theta$  equal to 45 degrees, because we have measured these in or counted these in terms of  $2\theta$ , which would mean that  $\theta$ , but that they are away by  $\theta$  equal to 45 degrees. So, in general, that would mean

that if you have some cubic structure like this and you are applying only normal stress here, then, at this particular orientation, you will have only shear stress and no normal stress.

So, this is 45 degrees away and when you go back between from here to here, you will have the maximum normal stress; at this angle you have maximum shear stress. And if you come here, you can see, this is 0 normal stress. So, between maximum and minimum normal stress, you see that there is a maximum shear stress condition. So, the points that have been mentioned can be understood with respect to this particular graph.

The variation of normal stresses and shear stresses occur in the form of sin wave with a period of theta equal to 180 degrees. If you were to plot normal stress versus theta and shear stress versus theta, you would see that it is a sinusoidal function with a period of theta equal to 180 degrees. So, after 180 degrees, again it will come back to its original position and follow the same path.

So, these relationships that we have described here are true only for 2D state of stress. So, we have today talked about one important concept about stresses, which is principal stresses, and we also looked at plane stress transformation. And before we go, I would like to show you important figure. **(Refer Slide Time: 55:45)**



So, here, what you are looking at is a small pillar, which is also called as micro pillar that you are able to make in FIB, focused ion beam electron microscope. So, this is of the order of, you can see, the scale is given here, 2 micrometre. So, the diameter is something of the order of 5 micrometre and the length maybe of the order of 20 or 40 micrometre at the most. And what we do usually, why researchers make this kind of pillar is to characterize properties, particularly mechanical properties of one particular single crystal of a material and that too in a particular orientation.

So, this could be a single crystal and then one can compress it. So, you can see that after compression, what they are getting; since it is a single crystal, they are getting shearing; so, they are all parallel shears. So, the shearing has actually taken place just as we would imagine. So, on one certain particular plane, the shear stresses are very high and therefore the material collapses along those plates.

Now, the question for you to think for the time being is, what are these angles at? How are they oriented with respect to the normal direction? How can we find out at what particular orientation they would fail? So, with that, we will close this today's session with this thought for you to work upon, and we will come back. And next time what we will be looking at is again another important aspect which is Mohr circle. And you would see, whatever we have learnt today, you would be able to imagine and present in a much better way using this, the concept of Mohr circle. So, thank you.