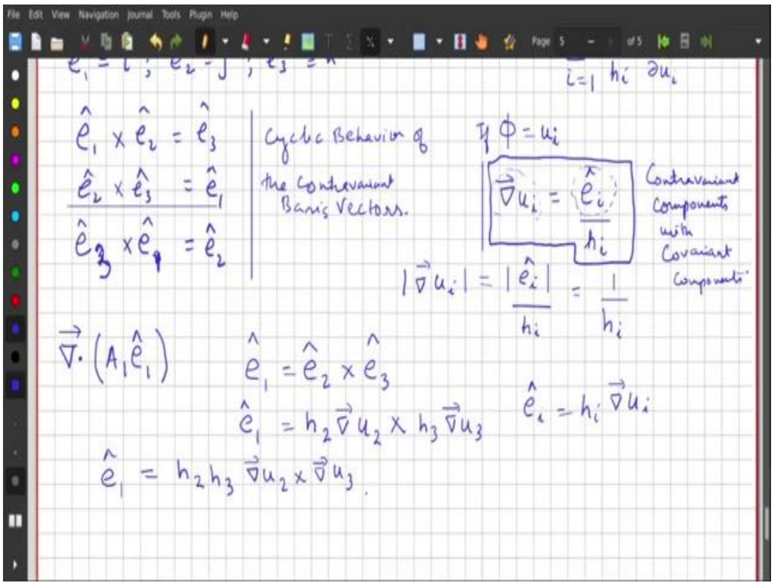
Statistical Thermodynamics for Engineers
Professor. Saptarshi Basu
Indian Institute of Science, Bengaluru
Lecture 23
Supplementary Video 8 Coordinate System 3

Welcome everyone to another segment of introductory lectures where we were discussing about coordinate systems in general.

(Refer Slide Time: 0:11)



And in the last lecture, we saw the cyclic behavior. So, we saw the cyclic behavior of the basis vectors, cyclic behavior of the contravariant and basis vectors. So, we will continue from here in this lecture and develop the general ideas of writing the various vector operators like the grad, the divergence and the Laplacian in a general fashion in any coordinate system. So, we are developing the idea from a generic co-ordinate system, perspective that we started with basis vectors in an arbitrary curvilinear coordinate in three dimensions.

So, let us see. So, remember where we had this expression where we had grad of, let us say, this could be represented as a sum over i equals 1 to 3 e i at by the scale factor hi, partial phi, partial Ui. So, let us use this important identity that we developed and using this let us try to find out what that if phi is the ith curvilinear coordinate, let us say, remember this ith curvilinear coordinate? Like, we had u1, u2 u3 in three space and we were thinking about u1 equals to c1, the surface in that three-dimensional space.

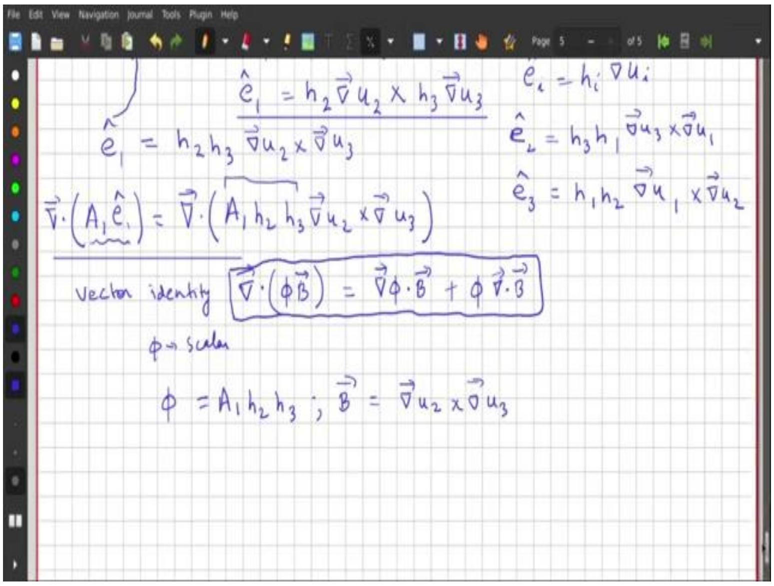
So, let us say if phi equals ui, what will this equation become? So, this equation will become grad phi, instead of phi, we have ui and this will become ei at ihi, where if we substitute phi equal to ui, the sum will only survive for the component where like, phi is ui, so the other the cross terms will not be there because like the derivative of let us say u1 with respect to u2 will be 0 because u1 and u2 are orthogonal, and they are independent coordinates.

So, grad of ui will become ei hat by hi, where this hi is a scale factor. So, the magnitude of grad of ui is the magnitude of ei hat by hi. So, then, this becomes, since this is a unit vectors, so this becomes 1 over hi. So, this is a very important expression that we got and you see this relates somehow the contravariant components with the covariant components, because if you see the right hand side e serve i, this represents the contravariant components but the grad ui, this is the normal component, so this represents the covariant basis.

So, let us keep in mind this very, this identity will become very handy as we develop further ideas. So, let us now what do we have? Let us say we want to write the divergence operator. So, let us say we are in divergence of A1 e1 vector. So, if you see, this we can write using, so let us say we use this middle expression.

If we use this middle expression, we can write e1 as e2 cross product e3 and then using this expression, which we just now derived, where we had ei hat is hi grad ui, this we can rewrite as h2 grad u2 cross h3 grad u3. So, this becomes, so this is e1 cap. So, therefore the e1 vector becomes h2 h3, so these are the scale factors in h2 and h3 multiplied by grad of u2 cross product grad of u3. So, that is what e1 is.

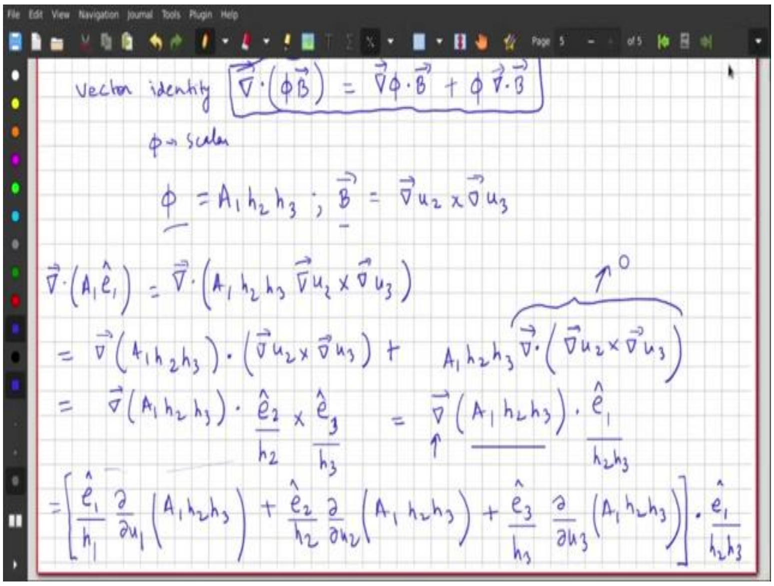
(Refer Slide Time: 5:29)



And let us use this e1 and let us substitute it here to expand the divergence. So, this will become the divergence of A1 e1 cap, this is equal to a divergence of e1, h2 h3, grad u2 cross grad u3. And remember this identity is very, this we have, similar identities we can write for the other direction. So, we can write, let us say e2, e2 is h3, h1, grad u3 cross grad u1, just using the cyclic nature of 123 and similarly, for e3, we can write e3 equals h1, h2, grad u1 cross grad u2. So, let us see now. So, let us remember vector identity which we know from before, vector identity, which is basically the equivalent of kind of the product rule of calculus.

So, let us say we have the divergence of a scalar function let us say, y times another vector let us B. So, phi is a scalar and B is a vector. So, this becomes, remember this is like the product rule, so it is like the grad phi dot B plus phi divergence of B. So, if we use this identity for A1 e1 where we recognize that what is phi and what is B, so, if you see phi, like so, if we compare let us say phi B, we identify that phi, if we represent phi by this term, which is A1 h2 h3 and B vector by a cross product of grad u2 cross grad u3.

(Refer Slide Time: 8:16)



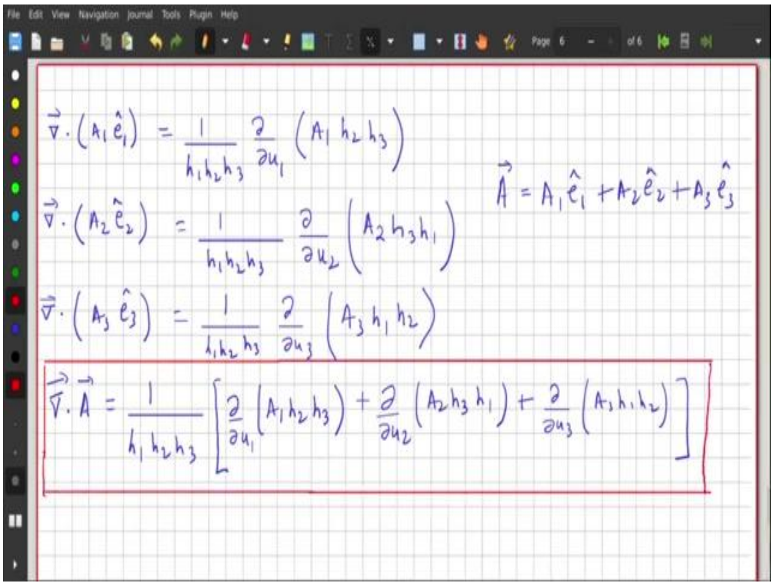
So, recognizing this is phi and B and using this vector identity we can rewrite the divergence of A1 e1, so, that is the divergence of A1 h2 h3 grad u2 u2 cross grad u3, that is what we had before and now using the rule of this vector and identity, recognizing appropriately what is phi and what is B, we can expand this as the grad of A1 h2 h3 dot product with grad u2 cross grad u3 plus A1 h2 h3 divergence of vectors is grad u2 cross grad u3.

And now realizing what this can be, so, this is like the divergence of curl or something you can think about, this is like. It is not strictly like that, but you can think about in this direction and we can show that this will be equal to 0. So, then this expression becomes grad of u1 h2 h3 dot and grad of u2 we can using this identity. This identity we can write as e2 by h2 that can be written like e2 by h2, using the contravariant components instead of the covariant. This will be written as e3 by h3.

So, finally, this could be written as a grad of A1 h2 h3 and e2 cross e3, that is e1 divided by h2h3. So, just like this, we can write the other components basically. But, before writing that, let us expand this thing even further. So, remember the grad operator, the grad operator is defined using this, this del operator acting. So, this is the grad acting on a scalar functions, this is the sum over i equals 1 to 3 ei by hi.

So, using that we can expand this basically, we can expand that. And that becomes e1 by h1 partial u1, e1, h2 h3 and we have a sum, so three components e2 by h2 partial u2 A1 h2 h3 and then plus e3 over h3 partial u3 A1 h2 h3 and this entire thing dot product with e1 h2 h3. So, if you see we are just dotting it with e1, that is the only component which will survive. So, this is the thing that will survive, others will become 0.

(Refer Slide Time: 12:34)



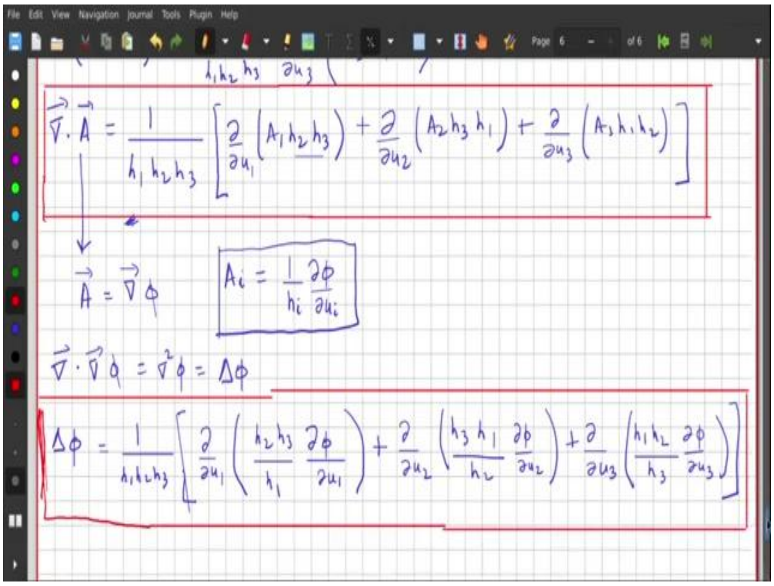
So, the orthogonality of e vectors- e1 e2 e3 conditions. Then this will become divergence, let us say divergence of A1 e1 cap, this is equal to you see, h2 h3 and h1 is down. So, it will become 1 over h1 h2 h3 and then this will become partial respect to u1 and this will become A1 h2 h3 as a result. Similar expressions we can have for let us say the other components, let us say A2 e2, I am just using the cyclic nature of things, we can write it as h1 h2 h3 and this will be partial partial u2 and this will become A2 and you see the cycle 123.

So, this will have the cycle 231 h3 h1. And similarly, we can write divergence of A3 e3 which is 1 over h1 h2 h3 and then this will become partial u3 and this will become A3 e3 and instead of, so that will become h1 h2. So, that is the divergence of A1 e1 cap A2 e2 cap and A3 e3 cap and now identifying like we can write the vector A in the contravariant basis like A1 e1 cap plus A2 e2 cap plus A3 e3 cap, we can take all the all the things together to find out the divergence of the vector field in an arbitrary curvilinear coordinates with scale factor h1 h2 h3 as this.

So, this is h1 h2 h3 and then we will have as a sum basically, so as you can see, it become partial u1 A1 h2 h3 plus partial u2 A2 h3 h1 plus partial u3 A3 h1 h2. So, this is the general expression for the divergence of a vector written in an arbitrary curvilinear coordinate system where we realize h1 h2 h3 as the appropriate scale factors in the 123 direction respectively. So, this is the expression that we were targeting to have.

This is a very, very general expression as you will see and this expression we will see, use it for various coordinate systems to get the expressions for the divergence operator in the corresponding coordinate system.

(Refer Slide Time: 15:56)



So, now, if in this expression, let us say if we substitute instead of A, if we substitute grad of any scalar, that means, what we have in my mind is that EI is basically remember from partial phi partial Ui, the ith component of the… this is what we are talking about, grad of phi. So, that is the components.

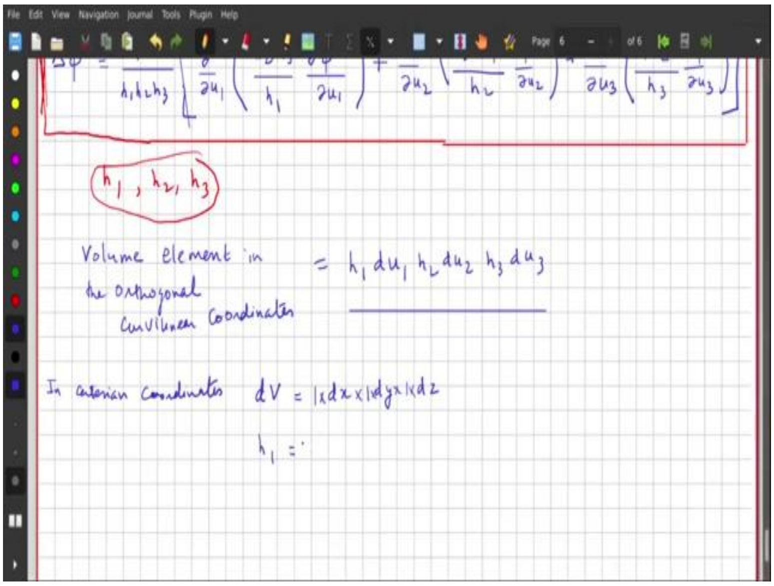
So, if we recognize this, that Ai is this, this is the gradient of phi. So, this is the thing that we are thinking about, this expression basically, if you see here, instead of f, we have A 1 over hi. So, that is the idea that we are thinking. So, if we substitute A is equal to grad phi, you see, the left hand side becomes the divergence of the gradient of phi, so this becomes the Laplacian operator that is square phi or that can be written as this is a Laplacian of phi, so we can have a general expression for the Laplacian of any scalar field.

So, the general expression for the Laplacian of a scalar field is given by, just becomes 1 over h1 h2 h3, it naturally follows from this expression which we had and this will become partial u1 and instead of A1, we will have 1 over h1 partial phi partial u1 basically. So, we will have h2 h3 which are these two things, and then we have one h1 from here, and then we will have partial phi partial u1. And like that for all the others, we just substitute as Ai s appropriately here.

So, this will become partial with respect to u2, and this is h3 h1 by h2 partial phi partial u2 and plus partial with respect to u3. And this is h1 h2 by h3 partial phi partial u3. So, that is what we have as a general expression for the Laplacian. So, we have written the general expression for the divergence and we have written the general expression for the Laplacian of a scalar field.

So, the divergence has been a vector and the Laplacian of a scalar field in arbitrary curvilinear coordinate. So, these two expressions are very powerful. And as you see, the only thing that we require to get to a specific form is knowing what are the scale factors.

(Refer Slide Time: 19:14)



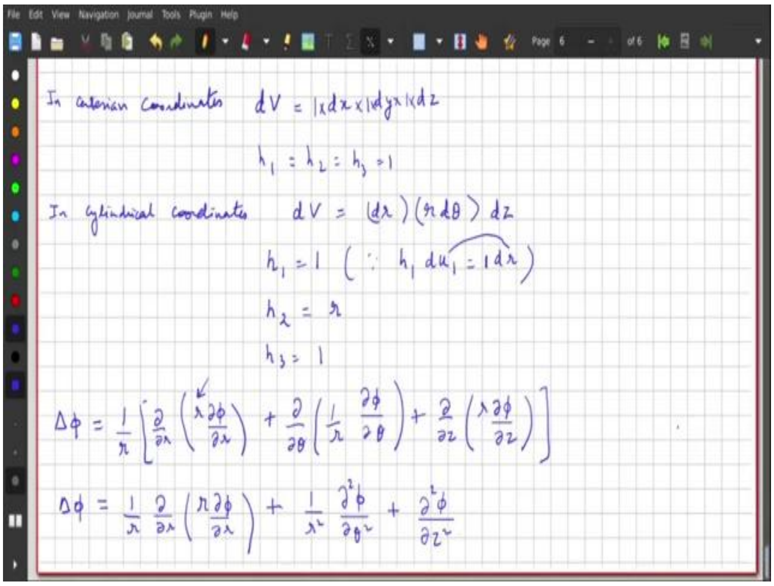
So, if we know what are h1, h2 and h3, we can get the specific expression for the individual coordinate system. That is the power of this, this method and the generalized that we are trying to build. So, now we have these two expressions and using these we can write the divergence of a vector field and the Laplacian of a vector field in arbitrary coordinate systems.

So, we can write the divergence and the Laplacian in a particular coordinate system, , this specific coordinate system and that is what we will be doing now in seeing the specific cases. So, before that, let us write what is the volume element in the orthogonal curvilinear coordinates that we used? And that is h1 du1 h2 du2 h3 du3. That is the volume element. So, if we compare this with the volume element in each coordinate system, which we had before, we will be able to figure out what h1, h2 and h3 are in general.

And another thing to, why these two expressions are so powerful? It looks very long, but these are very, very simple, there is a pattern to it. If you see like h1, h2, h3 and then partial with respect to u1, A1 and then 123, 231, 312, the cyclic nature is there. So, that makes it very easy to like keep this thing in mind. So, let us say for example, in Cartesian coordinates, we realized what was the volume elements, the volume element dV was dx dy dz. And if you recall this, we can rewrite like this, where we realize the individual h1 h2 h3, the scale factors all are one.

So, h1, h2, h3 all are one and just if you substitute h1, h2 h3 all equal to 1, you will see divergence of the vector as well as the Laplacian, you will get the familiar fields, the familiar Cartesian expressions.

(Refer Slide Time: 21:58)

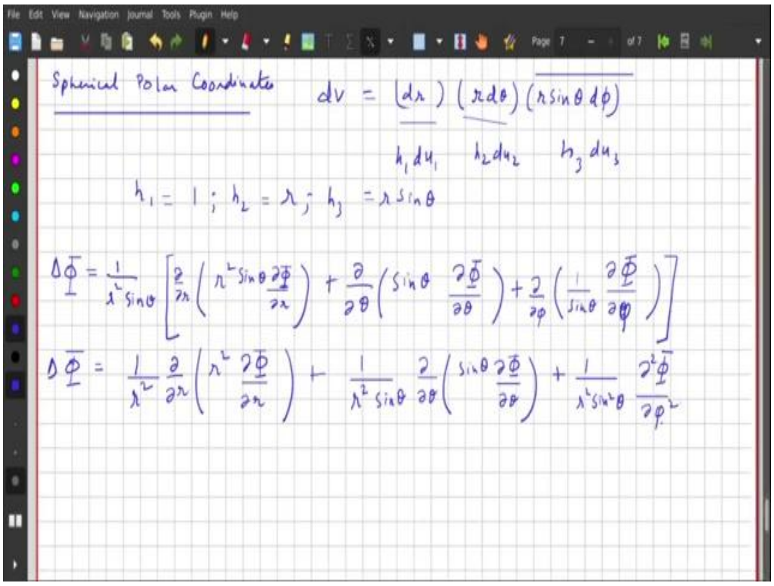


So, let us do another case where let us say in cylindrical coordinates, what is the volume element? Let us write the volume elements. So, dV is in cylindrical coordinate is dr into r dr or no, if you remember this is d r into rd theta. This was dr into rd theta that was in the azimuthal direction and times dZ. And if we compare this now with this expression, we will see that h1 is 1 since what we are doing is h1, we are realizing h1 du1 is basically dr where we recognize e1 is R and the factor that that sits in front of dr is 1.

So, h1 is 1. Similarly, we will get h1 is 1, h2 is r and h3 is 1. So, if we use the values of h1 h2 h3 in these two expressions, we will get the appropriate expression for the divergence as well as the Laplacian. Let me write the Laplacian operator in cylindrical coordinates. So, that will become 1 over r, because remember you have 1 over h1 h2 h3 and you see h1 and h3 is 1 and h2 is r. So, the product is r and this becomes del del r of r del phi del r. You see, because that is h2 h3 by h1 and this h2 is r, this r comes from that h2 plus del del theta 1 over r del phi del theta plus del del Z r del phi del Z.

And if we know, just open the brackets we will have Laplacian of the scalar field, phi in cylindrical coordinates is given by 1 over r del del r r partial phi partial r plus 1 over r square into phi del theta 2 plus delta phi delta del Z square, so that is the Laplacian in cylindrical coordinate systems. Similarly, we can write it for spherical polar coordinates. So, let us do that.

(Refer Slide Time: 25:10)



So, let us write it for spherical polar coordinates. Let us identify what was the volume element. So, if you recall the volume element dV was dr into r d theta into r sine theta d phi, now this was the azimuthal, this was the zenith term and this was the radial term. And if with this we recognize with h1 du1 h2 du2 h3 du3, we realize that h1 is 1, h2 is r and h3 is r sine theta. And if you use this h1 h2 h3 you can show that this will be, just substitute h1 h2 h3 in this equation and out pops the Laplacian of a scalar field in spherical polar coordinates.

Because of this will become 1 over r square sine theta del del r of r square sine theta del phi del r plus del del theta because you are using also phi for a coordinate variable, so let us use capital phi as a function to avoid any (())(27:00) confusion. So, that we will have del del theta of sine theta phi del theta plus del del theta 1 over sine theta del phi del theta.

This grad of phi is 1 over r square del del r of r square del phi del r plus 1 over r square sine theta del del theta of sine theta by theta plus 1 over r square sine square theta and del 2 phi by del phi 2. That is laplacian in spherical polar coordinates. We saw how to get each of the Laplacian operators in cylindrical as well as spherical polar using the general expression that we derived and same thing can be done for the divergence.

So, that is what we wanted to show, a general way of writing these two expressions are the things that we wanted to understand and then we wanted to show how general these two expressions are for the divergence as well as the Laplacian of a vector and a scalar respectively in arbitrary curvilinear coordinate systems. So, that is for this segment and we will see you in the next segment with some other materials. So, thank you.