Optical Methods for Solid and Fluid Mechanics

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Lecture No # 41
Geometric Interpretations

So we have seen this relation between the 1D for return sum of a projection to a 2D Fourier transform along theta equal to 0 that was fairly simple to establish. So today we will work out the more general case of theta naught = 0.

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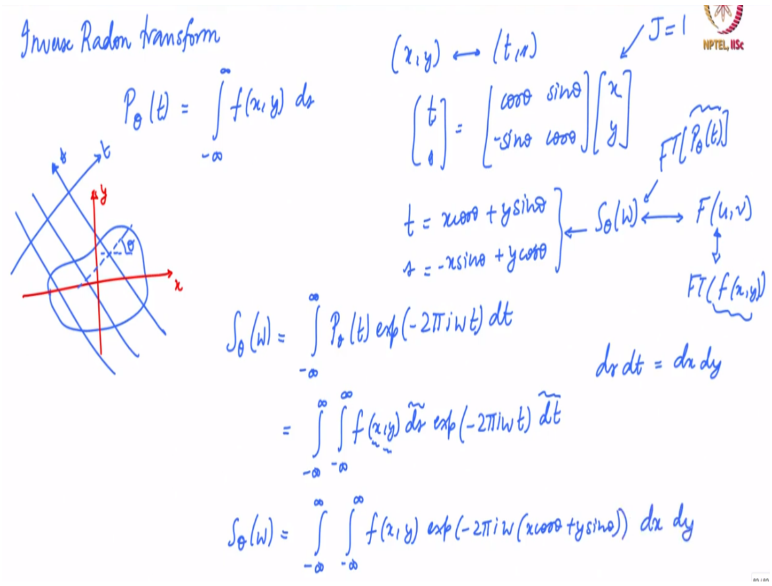

So we are talking about the inverse radon transform and how we can do this. Remember we wrote p theta of t was integral d s going from here to here and, I also mentioned that there are 2 different coordinate systems for clarity I will probably draw them in different colors today. So this is x and this is y and this was t and this was s right so the integral inside the radon transform is integrated over l.

So the radon transform itself is not a function of s it is a function of theta and t and of course the angle that we had over here was, theta right and so and t was in the normal direction right and n hat dot r bar was equal to t that was our equation of the line. And we discussed that you can transform from x, y to t, s using a rotation matrix right. So t s was just this Cos theta sine theta thing times x y.

So t was again x Cos theta + y sine theta we know this because this was our equation of our line of a single, line depending on the distance from the origin in that n bar dot r bar = t whatever. And so s of course is -x sine theta + y Cos theta so if you have not already seen this or if you have not figured it out already based on our discussion in the previous session this is what we will use to relate s theta of t is it of w rather which is the Fourier transform.

So this is Fourier, transform of p theta of t with respect to t with respect to the t variable for a fixed theta right will relate this to F of u v which is the Fourier transform of, f right which is the function we want to determine. So we want to get f we will we have measured p theta so we can take a Fourier transform of p theta and we relate that to the Fourier transform of, f. So we can invert the, Fourier transform of f and then get f right so that is a slightly long-winded approach but it is basically what we will do.

And right now this might seem a little bit mathematical but once I show you what these things look like it will be a little bit clearer and hopefully you will be able to appreciate some of these transformations all right. So let us take a take a look at what, this s theta of w is remember this was integral minus infinity plus infinity p theta of t exponential -2 pi i w t d t this was the 1 D Fourier transform of p theta of t.

And if you write down the expression for p theta of t you will get f of x y going along d s that is what p theta was right times exponential -2 pi i w t d t. So if you look at this is a 2 dimensional integral now, the double integral in t and s right. And you can convert of course x and y have to be written in terms of t and s as well right using this inversion of this guy which is with minus theta but you can convert this entire thing to x and y right.

So remember we said that the Jacobian of this guy was 1 because determinant of the transformation matrix is 1. So d s d t is the same as d x d y so we can, convert this because t and s are being integrated over you can convert them into x and y and that will be integrated over right it does not matter. So now you will just get if you convert it back to x y you will basically get f of x y that will remain unchanged you will get exponential of -2 pi i w times x Cos theta + y sine theta times d x d y that is s theta of w.

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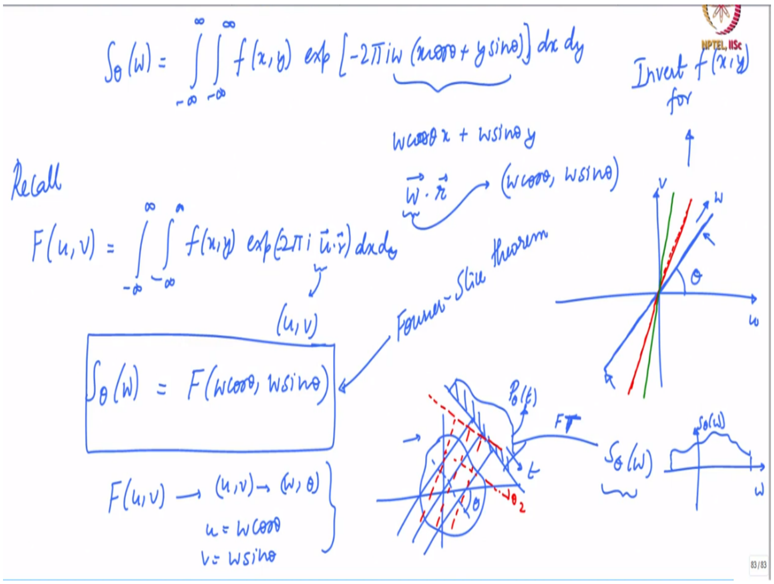

So let, me just rewrite that so s theta of w was a double integral minus infinity plus infinity f of x, y if you go back we had this exponential 2 pi i w times x Cos theta + y sine theta for t times dx times d y. Now if you look at the guy inside the exponential this is the same as w Cos theta x + w sine theta y I have not really done anything I have just reordered the 2 values. So this is, the same as some w bar dot r bar w bar is w Cos theta, w sin theta right.

We are only rewriting what we already have if you go back and recall the 2D to Fourier transform expression this side double integral f of x y exponential of 2 pi i with a minus sign u bar dot r bar d x d y and this u bar now was u, b right. So the actual 2D Fourier transform had u, v corresponding to u bar and, the s theta of w the 1D for it on some of the projection has w Cos theta, w sine theta right.

So this basically means that s theta of w is nothing but f of w Cos theta w sin theta so this is called the Fourier slice theorem. So let us step back a little bit and think about what this means right this is purely algebra to this but up until this point where we need an algebra we have not, really discussed what it geometrically means you have a projection p theta right. So you have a projection p theta of t you take its 1D Fourier transform and you get a value as theta of w this projection remember is taken at an angle.

So you have your object whatever this is your x y direction and then you have a projection taken at some angle and of course the angle is with respect to the, normal but whatever it does not matter right or it can it can go above and come back we will ignore that for now. This is just theta right parameterizing the p and this is your t. So you take this signal whatever you get here the projection signal which is p theta of t for this value of theta you take a Fourier transform of it right which is what we have done here.

And you will get s theta of w if, you do a Fourier transform and from the Fourier slice theorem. It basically tells you that if you go back in the frequency domain of u, v it gives you the value of the 2D Fourier transform along array not only along any array but along the way at angle theta. Because remember this is like f of u, v and if you convert it to polar coordinates u, v 2 w, theta then u = w, Cos theta v = w sin theta.

So this chap s theta of w if you take this function so this will be some value right this is w and this is s theta of w this function is basically giving you the value of the 2D Fourier transform along this ray. So this is your w direction w is like r remember keep this polar coordinate transformation in mind right. So you get one slice of the 2D Fourier transform from a single projection.

Now if you rotate it at another angle you will get another slice let us say you rotate this detector by a little bit more so now it is all like this. So you will get a different p theta of t corresponding to this theta now right corresponding to this normal and this now this angle let us say theta 2 and you will get a corresponding p and you, take over a transform and you get a corresponding slice.

So now you know the value of the 2 D Fourier transform along this line let me rotate a little bit more take another projection you take p theta for that particular theta take off a transform get s theta for that particular theta and you will get another slice in the Fourier space. So in principle if you were to measure the values of the, projections for not 360 degrees but just 180 degrees because you get front and back remember.

If you can measure the value of p theta of t for all of these directions theta going from 0 to pi then you will basically get all you know the information of the 2D Fourier transform along all of these arrays. So now you will have the value of the 2D Fourier transform of the function f of u, v and, you inverse transform that to get f of x y. And that is really all there is to it so whether you have a parallel beam you have a cone beam you have a fan beam you know you have equal space detector equilateral does not matter.

All of that is just a manipulation to bring it to this form there are some geometric maps you can do to bring it to this form and once you come to this form the, Fourier slash theorem tells you how to invert it. So we look at some examples just so that some of this abstract stuff becomes a little bit clearer let us look at a few examples and see how this is actually applied.

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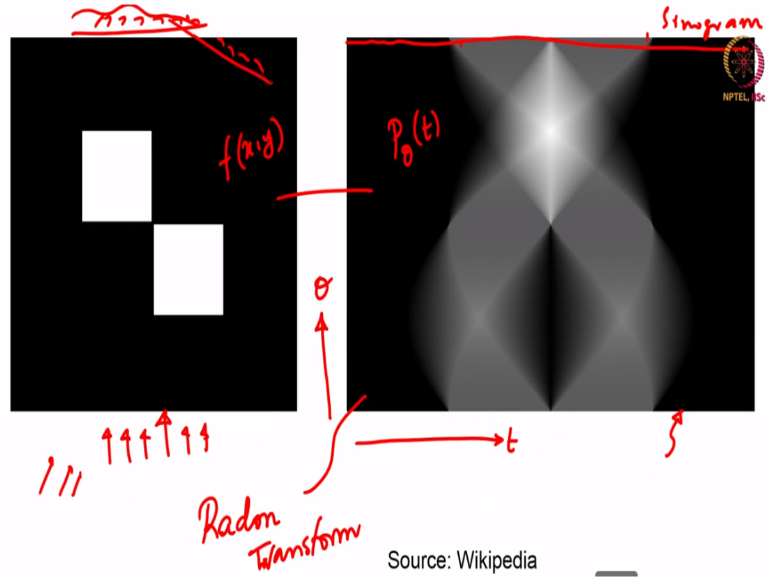

I have a couple of images that are taken you know from Wikipedia you can find them under radon transform. But this is a typical data set that you can generate by yourself. So what we, have here on the left is an actual function so our function is non-zero here inside the white squares and it is 0 everywhere else so this is our cross section. So assume this is highly Cartoonized version of somebody's head and you are doing a cross sectional scan.

So you are passing x-rays like this and you have a detected here and you measure right and you will get intensity, distribution. So if you get you know large absorption you will get a large value something that looks close to white if you get 0 absorption you will get nothing right. You will get a black signal because we have taken minus log whatever if I naught right and so log 1 = 0.

So that automatically removes it is like a dark field image basically so the image you see on, the right is something called a sinogram there is nothing to do with china. It is just that if you take a single spot here you will get a sine wave in this image as you rotate it right but that we will talk about that in a minute so how do you interpret this. So this is your t axis and this is your theta axis so if I start at theta = 0 that means I have my detector aligned, exactly parallel like this and my rays are going exactly like this.

Then I will measure some p theta of t, here depending on how much light is getting absorbed as the stuff goes through right and that will give me the first row of this image. And if you look closely you will see that you have this distinction here this is probably the first cube and this is the second as the first square, and the second square right. And their edges are almost aligned so you see they are almost next to each other.

So then you rotate by a small degree maybe 1 degree or 2 degrees and then you do another scan now you will have to pass light at 1 degree like this. And then you do the same thing measure the intensity and you will get another one the next row on this image. So you stack up the, rows as a function of theta always having t along the horizontal direction this is what considers the sinogram.

And this like I said so this is your f of x y representation and this is your p theta of t representation this also 2D representation both are absolutely equivalent. So if you did not know the left and you only knew the right you can reconstruct using, the Fourier slice theorem together left. And if you only knew the left you can generate the right by just evaluating these line integrals. So that is basically telling you what the rate so the right side image is the radon transforms of the left side image.

And you can you can play with this you can create different types of cross sections on the left image you will get a, corresponding sinogram in the right image.

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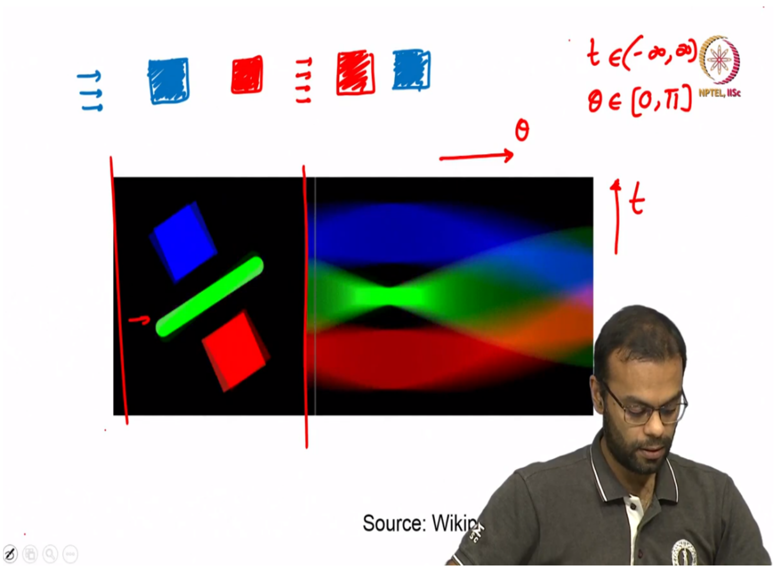

Here is another more you know animated example so this the detector and the you know whatever is not moving but the object itself is rotated equivalently both equivalent and you see how the sinogram is generated right. So this is continuously being animated. So for example if the detector is always keptical indirect is always here and the source is, always here when the source and detector are aligned.

For example you look at the green fellow when the source and the detector are aligned with the green object that means its largest amount of absorption is happening. So you get a very bright image for the green object these three are colored differently just to show you where they are coming from typically these are just intensity, values. So they will only be gray scale images right and the left one will also be gray scale image but this is just color coded to show you which is contributing to with sinogram right.

So the moment you have an alignment with this central one central rod like structure you have a large intensity if you look when it becomes horizontal this intensity will go up and then it goes down again, because the path length is reduced and the absorption is reduced. So this is again typically how a sinogram is generated if you take this axis now is your theta and this axis is your t and it is only being animated to show you how these are generated.

So at each particular instant is one row corresponding to the vertical section here and you can use that and stack them up and if you freeze, this animation fully you will get a full sinogram. And notice again also the information you have for theta = 0 is equivalent to the information you have for theta = pi so you do not need a rotation by 2 pi you only need the rotation by pi. Because for theta = 0 the let us say the red square I am not sure if I can get a blue square but let us try this and the blue square are like this.

So the light coming from here is going to look at both but for theta = pi the blue square is here and the red square is here. Now the light coming here is again going to see the same thing right it does not matter which order they are sealing because it is being integrated over right as the light goes from left to right. So you only need theta equal to you, know from 0 to pi and not from 0 to pi I hope that is clear from this illustration.

So typically the radon transform is t from minus infinity plus infinity and theta from 0 to pi all right.

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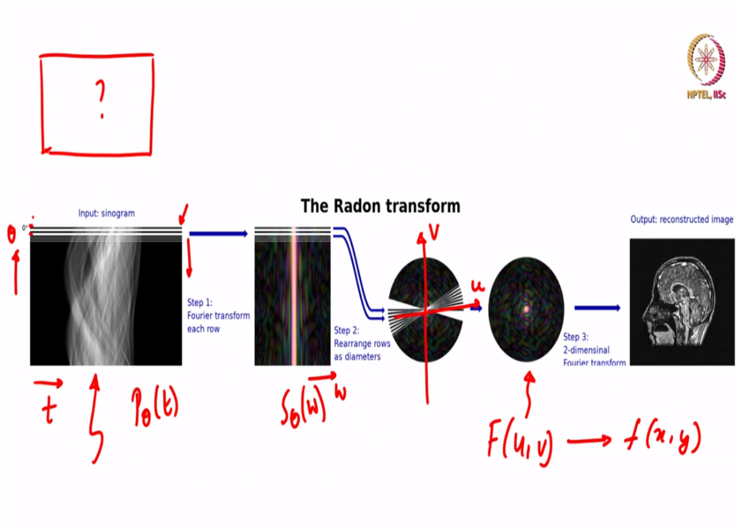

Source: Wikipedia

So now I have here a full summary of the radon transform and so you have initially some cross section which you do not know about so you do not know what this looks like but you only have the sinogram, let us say right for a 2D case. If you do that then the first step is to take a single row in the sinogram and calculates Fourier transform. So you have in this case it should be a column because this is the theta direction.

So you take remember the Fourier transform is in the t direction right strikerism this is t and so you have to take a 1D for a transform and then you take that particular, value of the Fourier transform the s theta. So this is your p theta of t you calculate the Fourier transform you will get an s theta of w so in this case again it is actually the row. So you get theta = 0 you take a rate of some of that you will get one value of s this is the w direction now.

You take this fellow and you stack it along the radial direction in the 2D frequency space so, this now is your u, v and you have this single stack. So this for example this stack gets put here and this tag gets put from here over here and so on depending on the angle. So depending on what the value of the angle is what the y coordinate is or the vertical coordinate is we will say why the vertical coordinate is on the sinogram depending on this value.

You orient this at a suitable angle, in the u, v plane if you do that for the entire sequence of angles from 0 to 180 degrees then you will get all the rays right for various angles 0 to again this also should make it clear that you only need pi. Because for 0 you have this side and this side being radial in both directions and likewise for any other angle you have opposite quadrants the a goes through opposite quadrant.

So if you do, from 0 to pi you will get this full 2D Fourier transform once you have the 2D Fourier transform you invert it take an inverse transform to get f of x y and that will give you your actual image.

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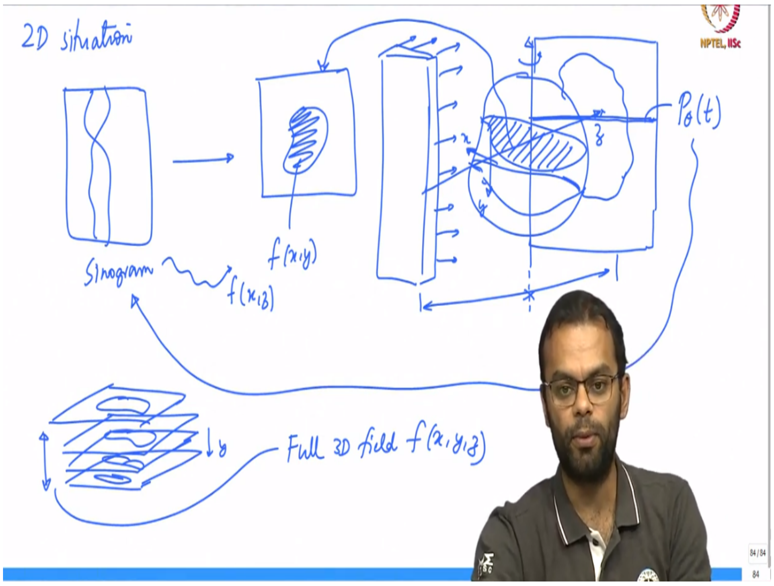

So if you have a simple 2 dimensional situation like this you have a sinogram whatever you can use this to reconstruct the f of x y using this inverse random transform therefore your slice, theorem so that should be fairly clear now. So there are a few different configurations in which you actually get the data we look at that a little bit now and then we will discuss that in more detail subsequently.

But let us say you have a 3 dimensional object this is its cross section the simplest way to actually get data is to get a huge source x-ray source or a line x resource. So, that is generating parallel rays this is again conceptual and interactive is never done this way. But in conceptually this is the easiest way you have a huge source like a huge led source or something and you just take a shot you will get an image some image.

Then you rotate either this about an axis like you do in a CT scan or you rotate the detector and source pair about this axis the object's, central axis and then you get a sequence of images. If this is your let me draw some coordinates here we will call this the x y plane no let us do this let us I think I use z the last time I will change the notation now. So I am going to call this ray direction is one axis and then this is the other axis this is the other axis.

So far I have used x y for the plane earlier, radius z for this for the ray direction let us just stick to that for a second and see where we go with it this will be x and this will be y one of them is x the other is why it does not matter. If you have this coordinate system and you have this 3 dimensional object then how you will generate a sinogram is the following? So for a given value of y according to this coordinate system this is, some given value of y you take the intensity data this will be like your p theta of t.

And then for the same value of y you stack these values of p theta of t for different theta and you will get the sinogram and then you do your usual for your slice business like we discussed and then you will get not f of x y. But in this coordinate system you will get f of x z in this case that is for one, slice. So our actual cross section all that stuff we have been talking about is for a cross section that is this is the same as this and this is in the x z plane.

Now if you have a big source like this you can do the same thing for a different y slice and you can get values over here and get a different y slice over here so you can get different slices your corresponding x z you will calculate. So now you will have a sequence of stacks in this y direction all reconstructed is in the same scheme and you stack them up together and you will get the full 3D field.

That is typically how well it is not typically how it is done but conceptually that is how it is done typically what will happen in a practical situation is? You will not have a big you know tube light type of source, making x-rays because that is almost certainly never happens you will have a point source. And if you have a point source it will not generate parallel rays it will generate a cone beam what is called a cone beam right.

And when you have a cone beam you will start seeing parallax essentially because some beams will go slightly larger distance compared to others. So when you rotate you will not get, parallel information but you will get information depending on the angle with respect to the source position those are things you have to account for we will talk about them at the end there is something called the felt cap algorithm that handles cone beam projections.

But basically the idea is you do some geometry there transform it to a form that looks like this once you bring it to a form, that looks like this you can basically do the usual for your slice stuff and recover the cross section data. So that is basically how you reconstruct from projections from individual projections using the 2D Fourier transform and then invert the 2D Fourier transform to get the actual cross sectional data.

So in the next session I will probably bring some data to show you actual experimental, data and what it looks like and what the corresponding sinograms look like and how you can actually use that for looking at internal structures of objects.