Optical Methods for Solid and Fluid Mechanics
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Lecture – 02
Vector Spaces and Cartesian Tensors

Welcome back everybody so, we are going to do one more lecture on Optical Methods for Solid and Fluid Mechanics. Now, the last class was just an introductory class, as we had already discussed but I wanted to give you a feeling of what this course is going to essentially cover. So, before I start on the second module I just wanted to do a brief recap of the first module itself.

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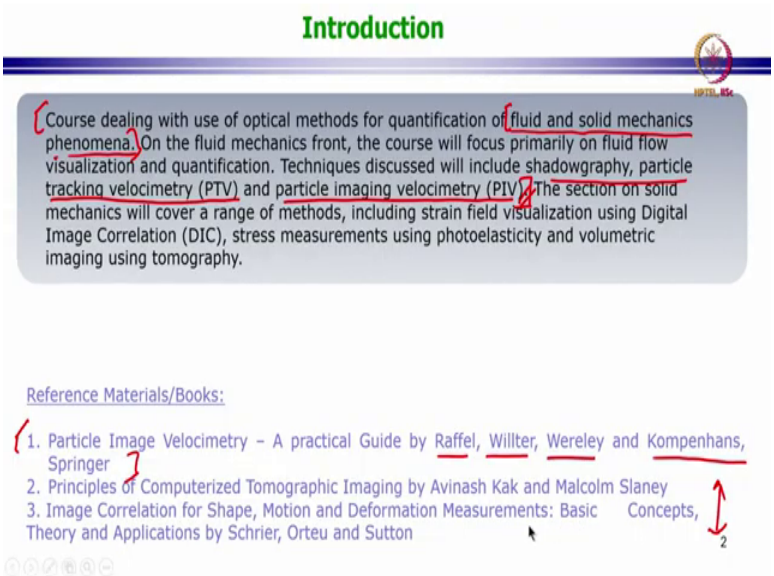

So, we looked at the overall course profile what we are going to do. We looked at the recommended texts for this particular material. And I already told you that I will be using multiple textbooks here and as I go along I will try to give you references for those texts as well.

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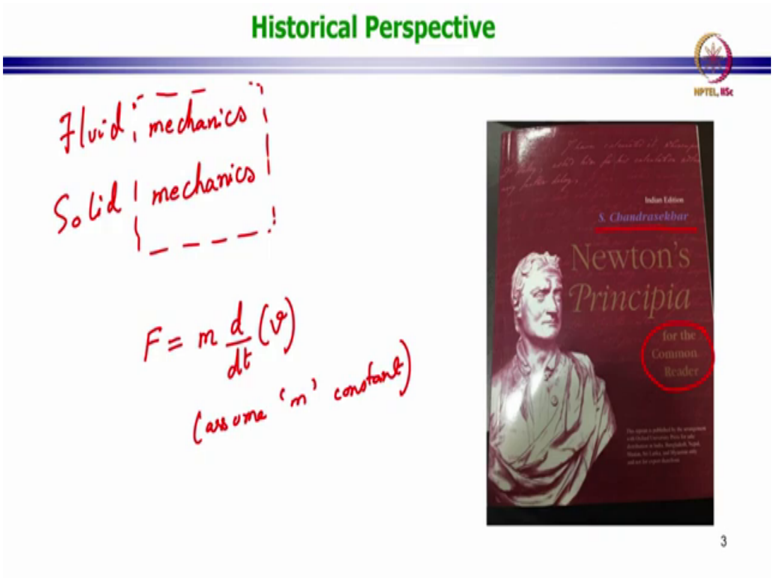

Then, we took a little bit of the historical perspective you from obviously starting from mechanics itself, the birth of mechanics with the three laws and we started reading up on the first law itself and. Here we had Subramanian Chandrasekhar, who very nicely discusses the issues that were are involved in interpretation of the Newton's Laws. So, the idea of the body had come up where what was meant in the original formulation was a rigid body or a point mass.

And then we also discuss that here this is where we are going to have a departure from the rigid body idea. We are going to specifically look at deforming bodies which is all the real bodies do deform.

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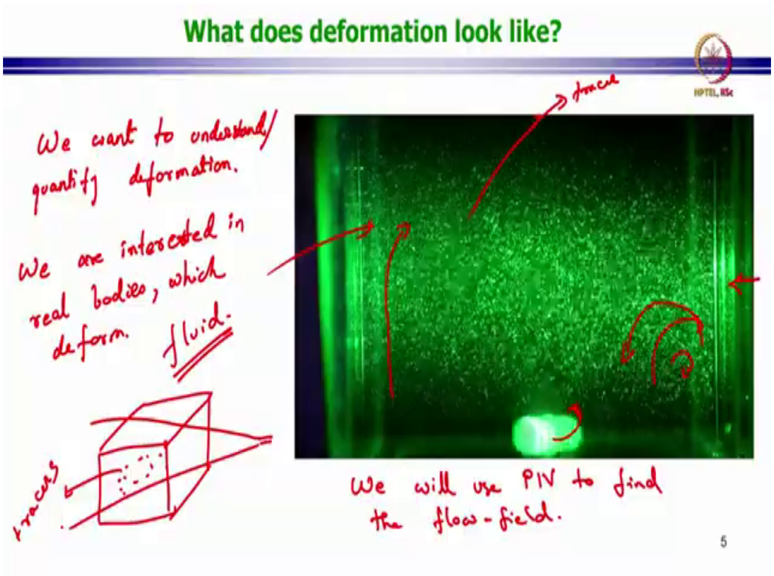

So, with that I showed you some examples. **(Video Starts: 01:35)** This was a video on fluid flow that you saw and I we are going to discuss this obviously this kind of a technique in more detail in the classes. **(Video Ends: 01:44)** **(Video Starts: 01:45)** Then I showed you another video using shadowgraphy, where you had this deformation of a meniscus. So, these are two optical methods, obviously they line up with the intent of the course by itself. **(Video Ends: 01:58)**

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And then we also discussed this photoelastic measurement that I showed you this picture, where you have see these fringes in this plastic. On this beam structure that we see here.

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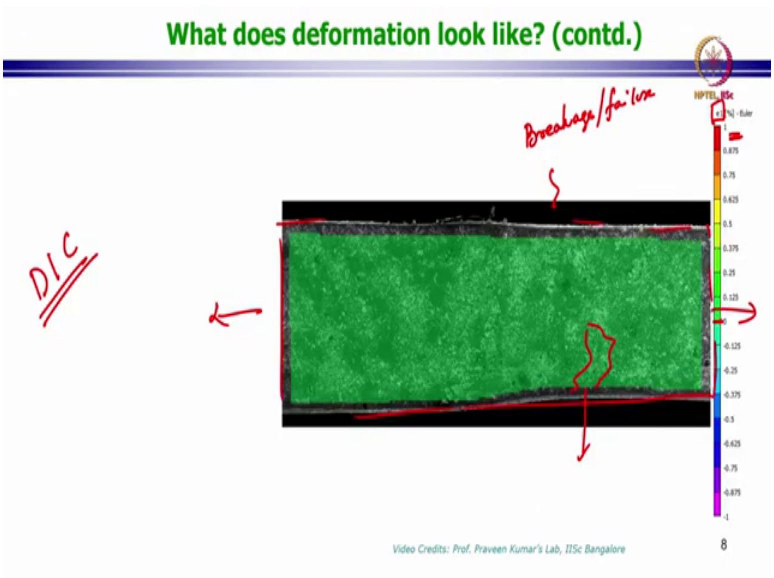

So, and then we saw a digital image correlation **(Video Starts: 02:22)** analysis of the strain. So, we understood some idea. We got some idea of the deformation aspects and I told you that I will be covering more of mostly of the deformations in the fluid mechanics realm, whereas my colleague Kaushik will be covering deformations in the the solid mechanical element. **(Video Ends: 02:39)**

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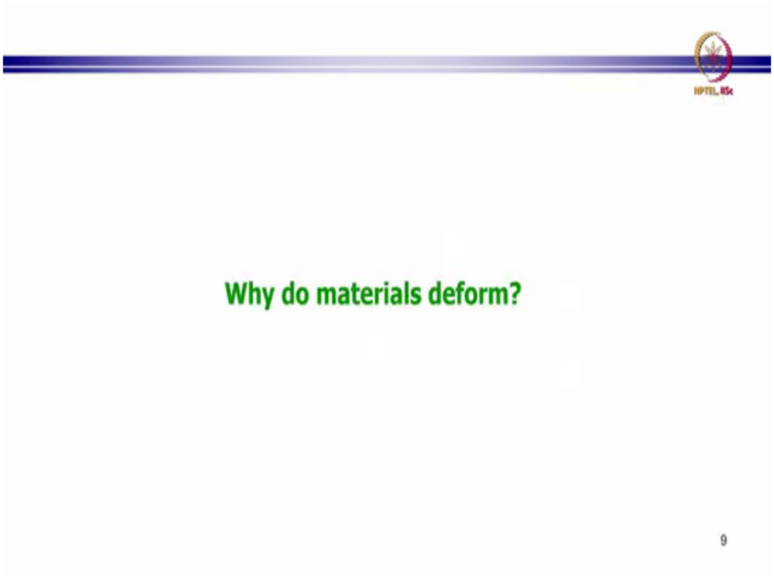

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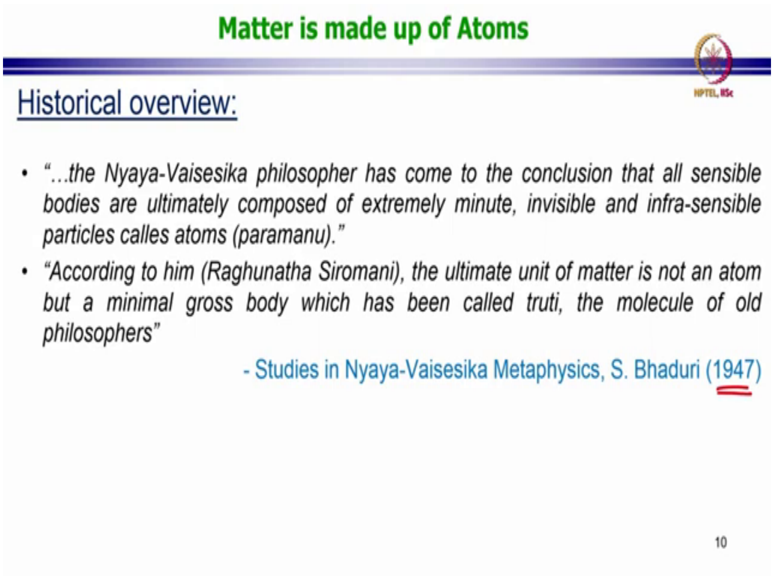

So, we also discussed a little bit about why deformations occur and we talked about the fact that it is because of the constitution of matter itself which is made out of atoms and molecules which has a very, very old history, specifically from India, where we had the Nyaya Vaisesika school which thought of the molecular structure of matter long before the advent of modern methods. And here we have Rutherford and his discovery of with his experimental setup.

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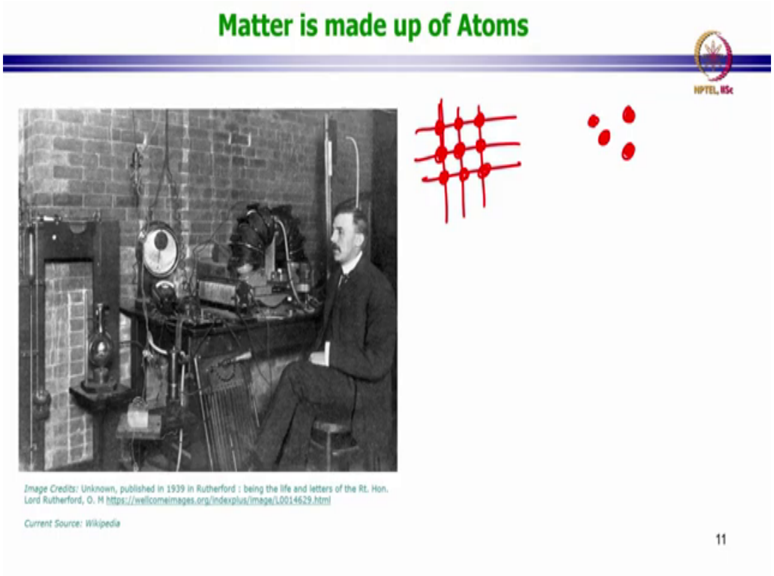

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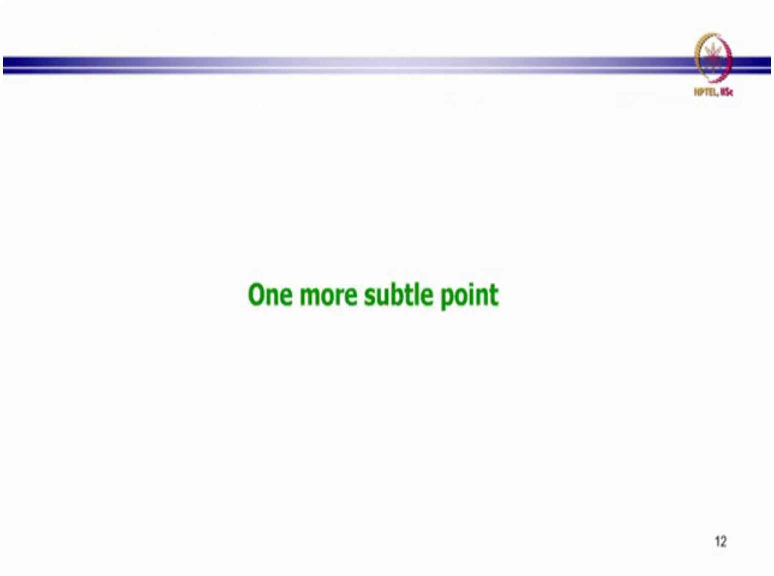

Then we had the understanding of today's understanding of how meta atoms are positioned in either fluids or solids while we are not going to look at the molecular nature of matter. I thought it was prudent to discuss this issue just in the introduction but our techniques will not really involve so much of understanding or seeing the molecular nature by itself.

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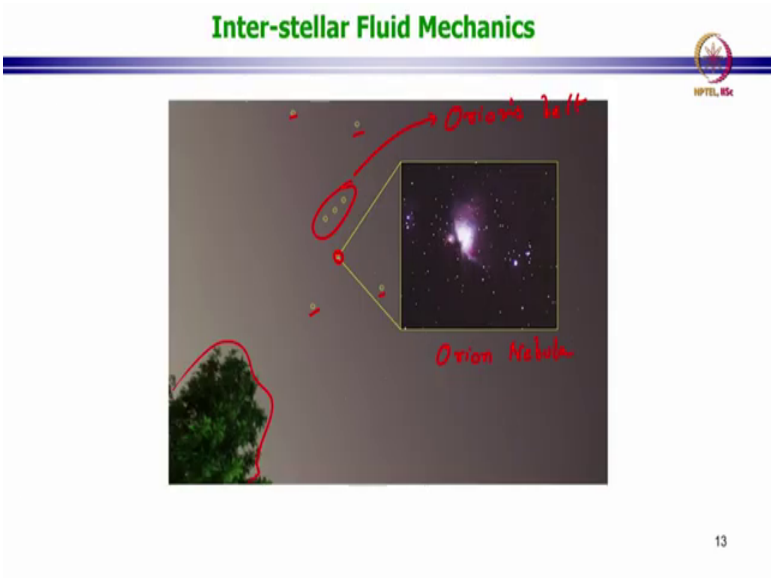

We had one more subtle point that I think beneath here. He wanted me to emphasize upon this point one more time because I think it is a very, very important point and we showed you this image which is. This is a photo photograph taken by myself from by the way Bangalore's Skies. You have this beautiful nebula that you see in this yellow box.

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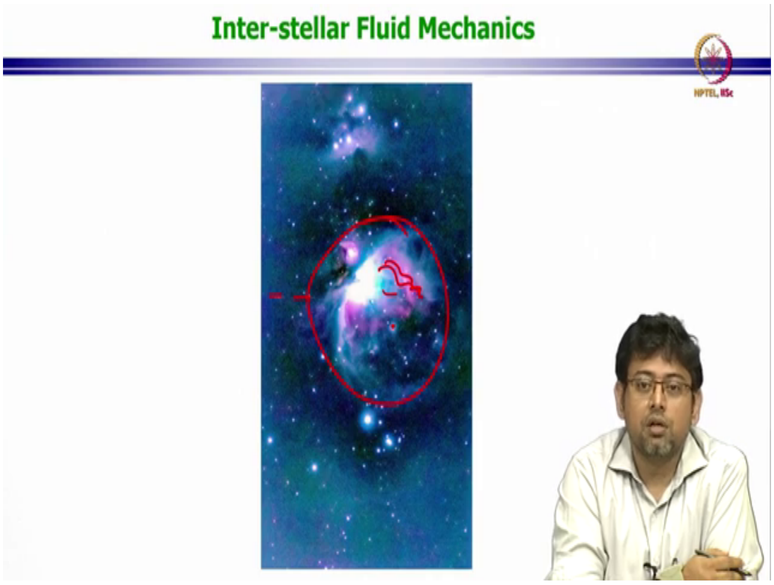

And this is a slightly bigger image of the same. Now, I told you that these are gases that are flowing so that all these colours red, blue, green, etcetera that you see all these different things here, these areas. These are hot gases that are flowing out. These are obviously the stars. And in fact, very highly magnified images from hubble also have shown us that you have bow shock formations in these structures in these areas.

And you have basically gases which are flowing out at many at hundreds and kilometres per second. And but then I had taken the previous image a year ago. And this year this image was taken like a couple of weeks ago from Bangalore. And yet the two images are exactly the same and you do not see any of the flows. So, the question is why not? And the reason of that is a resolution of the system.

I mean what we are looking at this entire area, this spans several light years, probably hundreds of light years. And in that huge times length scale the flow of gases is not making a big difference to the overall appearance, So, it will look exactly sort of the same next year and the year after and here after. So, it is the idea of resolution is an important one. And this course is going to be talking much mostly focusing once we get started on the main course material.

We will see that this is basically about experiments. So, when you set up the experiment, you have to be very careful as to what you want to observe and whether your experimental technique can observe that and it has the spatial and temporal resolution for that. And for the temporal resolution example, we showed the pitched up experiment where you have these drops falling every 10 years.

So, if you are a mythical being which blinks every 10 years, for which every 10 years is maybe the blink of an eye. That being would just see drops of tar fall, every blink of an eye. So, it is very important to set up the experiment the way and ask yourself what you want to observe. So, it is very important in that sense. So, again, this is something that you have to decide you have to have some physical insight into the problem.

And the physics of the underlying physics of the system is very important to keep in mind and it is difficult to give a generic answer to the entire thing. So, you must do this. You must have this understanding before you get started. So that was the key point we hit upon last time. So, what we are going to do now is we are going to move on and we are going to move to the today's module. Sorry this is the old module again.

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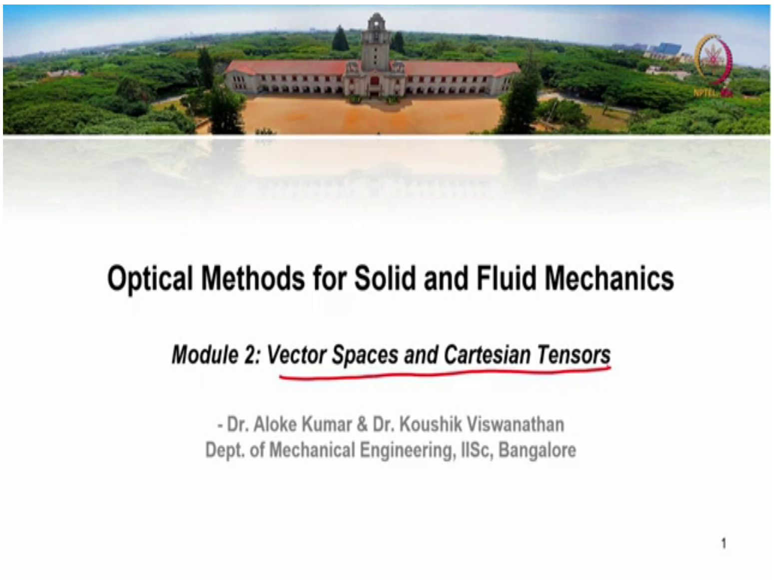

And in this module, we are going to cover some mathematical preliminaries that are required for handling this particular course. Now, in this particular module, we are going to be discussing vector spaces and Cartesian tensors.

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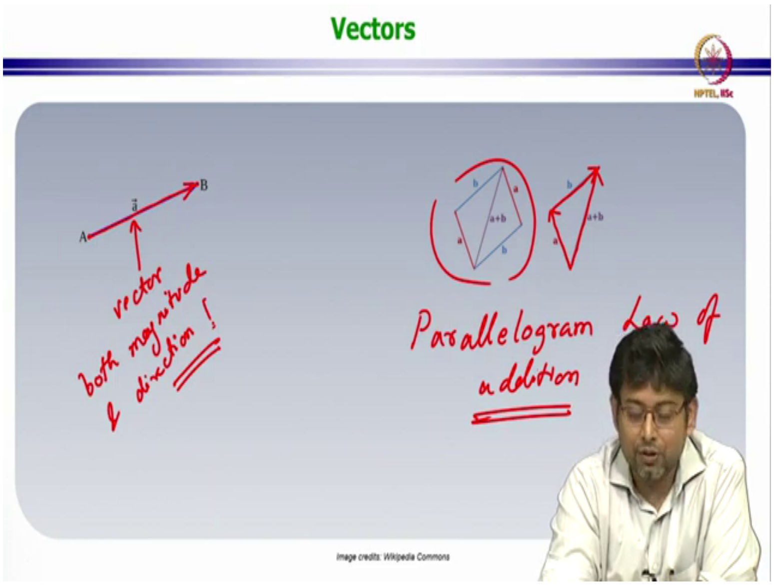

Now vector spaces, so that was the word vector is something I hope almost everybody here is familiar with whoever is taking this course. As a when we are young, we are taught that a vector is an arrow. So, it is something looks like this you have, let us say a point a and you have another point b and you want to draw a vector joining them. So, you draw this arrow and you say that this is now a vector and this has both magnitude and direction.

So, this is the idea of vectors that we learn when we are simple introductory idea of vectors when we learn as children perhaps. Now, this is we are also taught how to deal with vectors?

For example, how to add them? How to subtract them and I hope you also have gone through what is famously known as the parallelogram law of addition. So, here what we have is, for example, I will use this diagram to my right you have a vector a and you have another vector b.

And the sum of the two vectors is illustrated here. And you can through this triangle and you can even do this through the parallelogram. Now, these are things that you are already familiar with and what I want to essentially say that right now is this is the before we get to this kind of an idea of vectors as an arrow and something with the magnitude and direction. We have to understand vectors at a slightly more baser level.

And hence it is there is a need to make this idea a bit more abstract. By the way these images are taken, I have taken down from wikipedia. So that allows me wikipedia comments copyright to use this here with attribution. Now, the reason I did that I could have drawn this here by myself but I wanted to use this is because it shows you that this is how people do get introduced to vectors. This is what a lot of people do across the globe.

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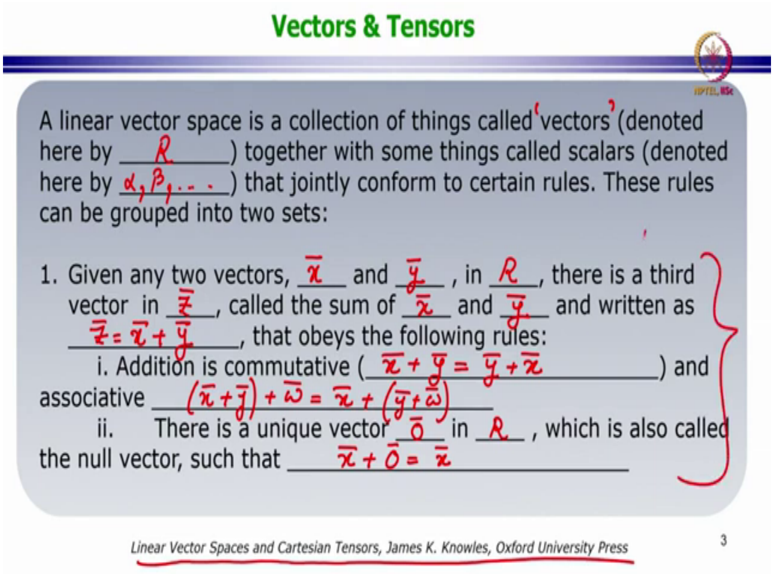

But now what we want to do is move into slightly more abstract idea of vectors. So, what we will do is? Rather than decide what a vector is. We are going to describe what a vector space is? Or what a collection of certain type of vectors is? And we are going to set certain rules for that. And as we set up the rules, what we are going to do is? We are going to say that anything that satisfies these rules is going to constitute a vector.

So, here what I have is a definition of a linear vector space and I have taken this definition from this book which is a rather excellent text right here. Now, this is a very, very small and thin book. If you want to take a look at, it goes into the details of linear vector spaces and Cartesian tensors. I am just going to pick out some of the essentials from there. So, if you have to define and I have taken this definition again from the book we say that a linear vector space is a collection of things called vectors.

So, here we are not defining what a vector says. We are just saying let us assume there is something called a vector then how do you define a linear vector space? So, let us say that there is something called vectors then we define the linear vector space. And the linear vector space is denoted by, let us say the letter R together with things called scalars denoted by let us say the Greek alphabet.

So, let us say alpha beta etcetera. These denote scalar quantities so, these are just well, for all practical purposes. These are just real numbers, they can be complex numbers as well but we will not encounter them in solid mechanics and fluid mechanics so much. So, we can just go ahead with the simple understanding that they are just real numbers. So now, this thing called vectors and these scalars they jointly confirm to certain rules.

And these rules are going to be grouped into two different bunches. The first bunch concerns vector addition. The second one will concern vector multiplication. So, let us see now we have given any two vectors. Let us say a vector x is there. Now, so that we are clear that x is a vector. What I will do is? I will put a bar over it and I will try to be consistent all throughout these lecture nodes.

So that I put a bar over x or any other thing when it is a vector to denote it as a vector. So, let us say there are any two vectors x and y in my vector space are, there is a third vector, let us say we will call it z, called the sum of the two vectors which is obviously x and y here and written as z = x + y that obeys the following rules. So, we are trying to define something called the sum. And here we are going to force it to obey certain rules.

This thing called the sum and the first thing is that the addition is commutative. So, x the how I add this? So, I if I do x + y that is the same as if I do a y + x and this addition is also associative. So, if you are doing, let us say x + y + there is another vector w. Then this is the same as x you first add y and w and then add to x. So, these are, they are commutative and the associative rule further.

We also specify which is there is a unique vector which will call 0 with a bar over it in R which is also called the null vector. And it is a null vector so, such that if you do a sum of any vector with this null vector, it always results in the same vector x. So, these are the rules for addition. So, the rules for addition are done these are.

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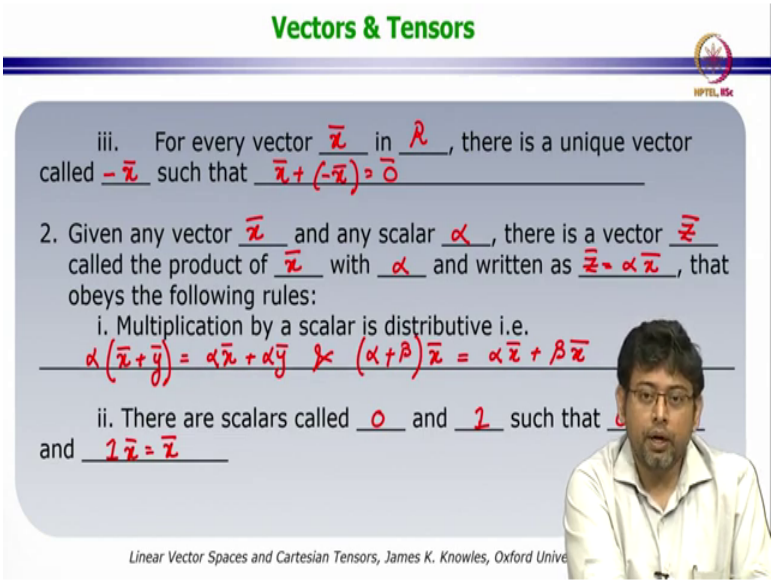

And now oh wait there is one more rule sorry, yes so, we will also define for every vector x in R. There is a unique vector – x such that x + this – x is equal to null vector. So, the null vector there is a way to get the null vector and for any given vector there is a another vector called – x. So that when we add the two you end up getting the null vector. So, with this the rules of addition are done.

Now, we are going to do rules of multiplication, so, let us say given any vector x and any scalar alpha. Now, there is a vector z that is called the product of this x and alpha so, first defining the product of a vector with another scalar. Basically and this is written, as my z is equal to alpha times x bar. That obeys the following rules so, multiplication now is going to be distributive.

So, what I am going to do is? If I multiply, I want to multiply alpha with the sum of two vectors, let us say, x and y. So, I want to achieve this then it basically means I can do alpha x bar + alpha y bar and add that two together and I get the same thing. And one more which is if there are two scalars. Let us say alpha and beta and I want to multiply it with the vector x bar then this is the same as alpha x bar + beta x bar.

So that is one rule now, couple of more things which are will define two important scalars for ourselves. So, there are scalars called 0 and 1 such that when I add 0, when I multiply 0 into x bar I get my null vector. And when I multiply 1 with my vector, I get the same vector back. So, this way I am able to end up getting the rules for my vector space, so, this sets up the rules that are required for defining something called vector space.

Now, what are the things that can become vectors will come to that in a while. I what I want to do is? I want to finish off some definitions. Then I will start writing down different examples for you. Now, when we learned about vectors as an arrow, we also learned about something called as a DOT product. Now, in reality, not all vectors need to have dot products and it may not always be possible to define something like that.

But when you have vectors which do have a sense of magnitude which means in more mathematical terms, we can say that there is a metric which can be associated with the vector. And there is a geometric sense to the entire vector space which basically means I can associate something known as an angle between two vectors. Then I can define something called as a DOT product.

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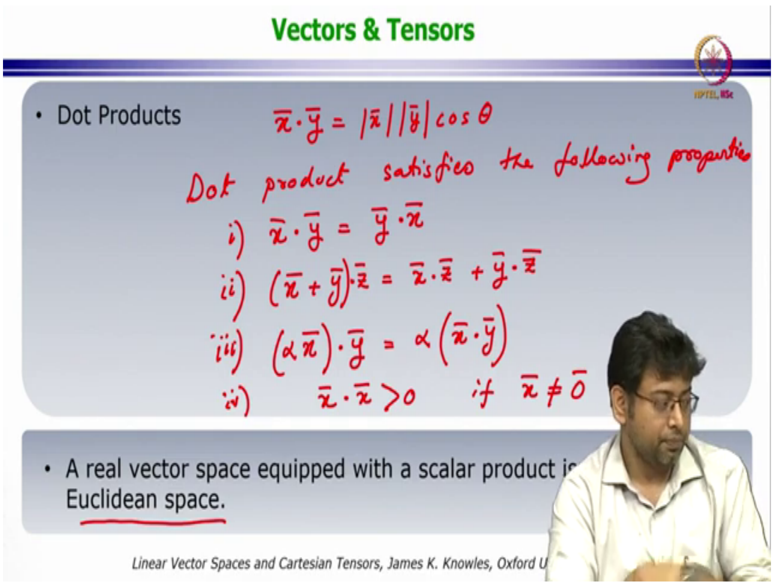

So, dot product is sort of the same as what we have done earlier, as in some other courses. I am sure you are familiar with it. So, we would say there is a vector if you do dot product or two vectors. We evaluate it like this which is x dot, product of x and y of vectors is the magnitude of the vector x multiplied by the magnitude of vector y into cos theta where theta is an angle.

So, this already shows you that we have a metric associated with the vector which is an idea of a magnitude and a sense of angle that is associated. So, now dot product you can quickly check that dot products satisfies some very simple rules, the following properties. So, what are they? One is the order of addition, multiplication does not matter. So, if you have x dot y you can also do a y dot x and the two are the same thing.

But here another one is it is distributive. So, if I do x + y multiply dot producted with another vector called z bar. This is the same as me doing a DOT product of x with z and then adding that to the dot product of y and z. And the third is multiplication with a scalar because that we have not specified yet. So, let us say you want to evaluate this where alpha is a scalar quantity x is a vector dotted with another vector you want to evaluate this expression.

This is the same as I can take the alpha out and I can do a DOT product of the two and evaluate this. You can also show very easily that dot product of us of a vector with itself is always greater than 0 if x is not the null vector. Now, I said before not all vector spaces need to have dot products. So, there will be some vector spaces which will have this sense of metric and the sense of geometry.

So, a particular type of vector space which is equipped with a scalar product, so, a real vector space. real here is implying all the scalars are real numbers vector space equipped with the scalar product is called a real Euclidean space. So, this is an important type of vector space and I wanted to clarify not all vector spaces need to have the sense of metric and a sense of geometry.

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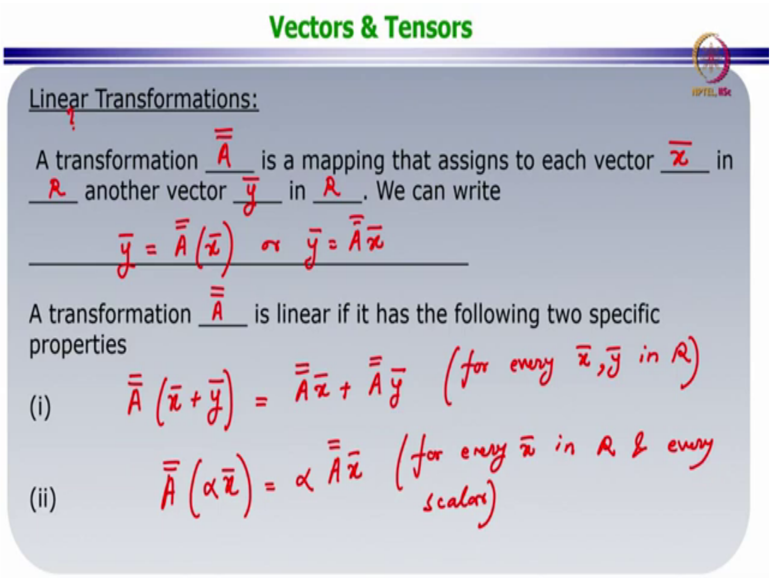

Now, we come to science since we have been discussing the topic of vectors, we also want to understand tenses. Now, in tenses before we start understanding that we want to understand what is a transformation? So, what we are going to do is? We are going to first learn about transformations. So, a transformation, let us say we want to understand linear transformations.

The non-linear transformations are also important and but they are more difficult to understand or they are not always necessary for our case in solid and fluid mechanics, we the most important transformations, are the linear transformations. So, let us say we have a transformation. Let us we will call it A. Now, I have already put a bar over x to denote it was a vector.

Now, A is a transformation and that we are going to operate on vectors. So, I am going to put two bars here to denote that this is something operating on vector. So, it is almost like the transformation is slightly a higher function of some sort where I am using that word a bit lightly. A transformation is a mapping that assigns to each vector x let us say in R another vector y in R again.

So, we can write that this vector y is essentially nothing but my A acting on x or often simply written as y = A x, where it is understood that this writing you have a which is operating on x. Now, there is one more quickly important point that I want to point out is that when it operates on the transformation, when it operates on a vector x which is lies an R. It results in another vector which also lies in R which implies that the linear transformation when it acts on a certain kind of vector should yield a vector of a similar type.

It cannot yield a vector which is very different from itself. So, this is just a transformation and this is not yet linear. So, how do you make it linear a transformation. So, a transformation if it has to be linear it has to be subject to a couple of more properties, specific properties which is let us say you have A acting on x effect the sum of two vectors x and y. Then if the transformation is linear, it can be written as Ax + Ay and this has to be true for every x, y in R.

So, it is not sufficient if this rule applies to only specific vectors, it has to apply to every single vector, vector and another important properties. If we have a scalar multiplication of a vector, so, let us say A is now acting on alpha times of x bar then this is the same as alpha multiplied on Ax bar. So, you apply the transformation A on x bar first and then you multiply it with alpha.

And this has to be again true for every x in R and every scalar. Every scalar alpha if you want to be more specific. Now, this scalars are actually, in more advanced text cells sometimes say that the scalars are sourced from another thing called fields but I am skipping that here which is going to stick to the idea that you sort of understand what scalars are that there is a real number line and these scalars are just coming from that real number line.

So, I will sort of assume that you have an intuitive understanding. If you want, you can look into fields in a little bit more detail, if you want to understand how these scalars are being defined or so strong. So, we have done linear transformations.

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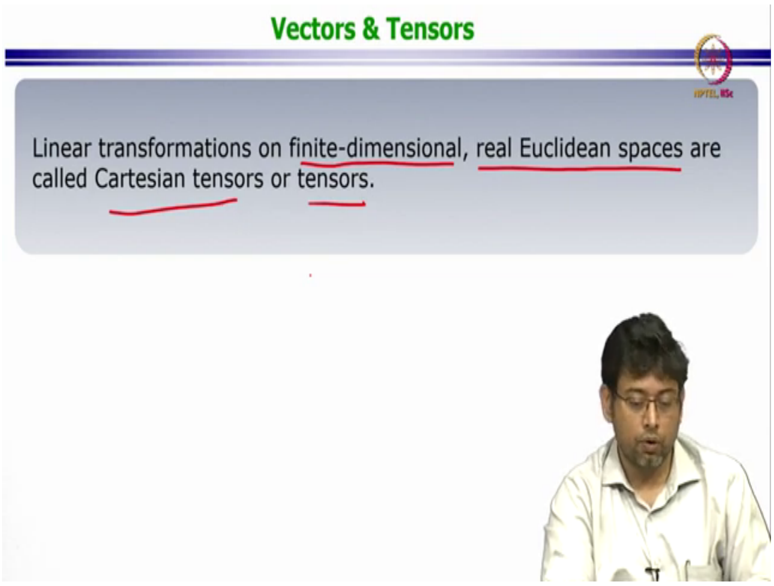

Now, this is a very important definition and we are going to work with this definition. A lot here so, linear transformations on finite dimensional, real Euclidean spaces are called Cartesian, tenses, or just simply tenses**.** So, not all linear transformations are tensors. Linear transformations which are restricted to finite dimensional systems and where the vector spaces are a real Euclidean space which means a DOT product can be defined.

Those tensors are called Cartesian, tensors. So, we are going to see Cartesian tensors quite a bit more as we move ahead. So, now here I am finished with the definitions that I wanted to introduce to you all of you. And what I want to do now is I want to move into some examples, so, I want to show you.

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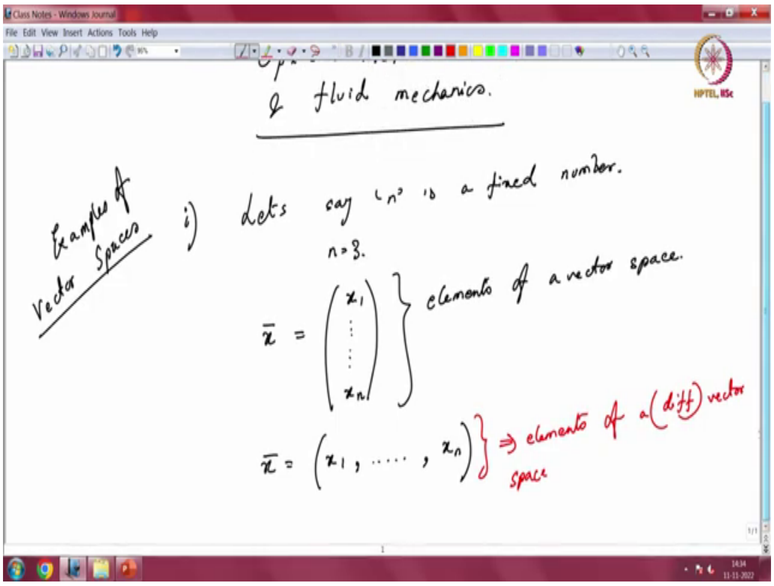

So, we are going to let us just move out and so, optical methods in solid and fluid mechanics is the course we are working on. So, we just went through a whole lot of definitions on vectors and tensors and what we want to do are now examples and illustrate what we just learned. So, let us just do examples of vector spaces. So, number one let us say that n is a fixed number, fixed whole number is the integer.

So, n here is some number that you have already fixed. Let us say n = 3 then, if you define a column vector such that you have these numbers where these are scalars. So, you have x 1 all the way, till x n and whatever n be you change this column. Then this so, it is important that in every vector space this n be fixed. Then, this set of column vectors, all possible set all possible column vectors they are elements of a vector space.

So, this these are elements and every time you change n it results in a new vector space. So, if you go from n = 1 to 2 to 3 to 4 these are all different vector spaces. But when you keep n constant that is one vector space. All possible vectors like that. Similarly, so, this is what we often refer to as a column vector. This is obviously finite dimensional you can see that there is a limit to the number of entries you have.

You can also have a column vector sorry, a row vector where you can arrange everything in a row and vectors like this would also become elements of a different vector space. So, these are. These can also form elements of a different, so, I will just say different vector space. The reason it is important to say different is because when we have our transformations act on these when they act on a column vector.

Let us say the column vector should get transformed into another column vector not into another row vector. It should not go to transform to anything else. Make sure the transformations are proper so that we are always working with the proper vectors.

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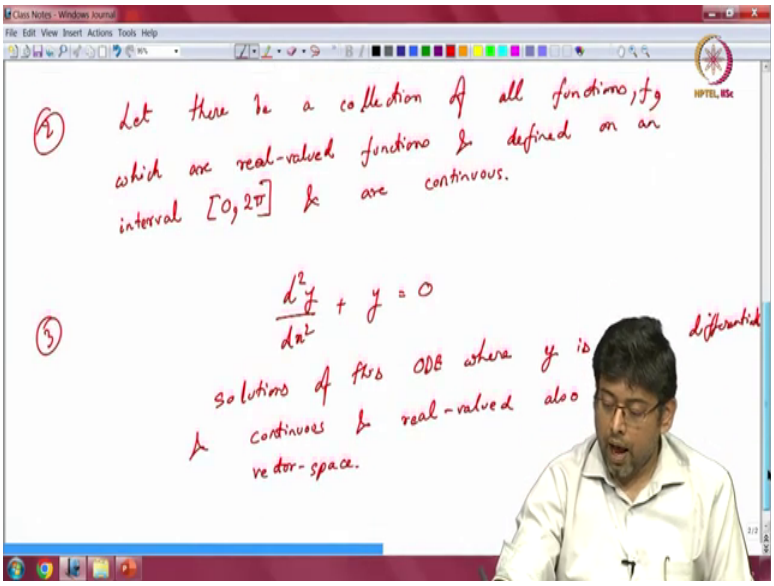

Now so, these are some things that you are already probably familiar with. Let us just see another example. And here what I will definitely try to do is? Make it an example such that I am just going to write it a little bit closer. Example such that it is not going to be it may not look like the arrow diagram that you are sort of accustomed to it. So, let us say let there be a collection of all functions.

Let us say f which are real valued functions and defined on a on an interval. Actually, a close interval 0 to 2 pi and are continuous. So, a collection of functions like this also will constitute a vector space. And this vector space will now be looking very different than the vector space of columns, column vectors or the row vectors that we discussed earlier. You can also define other types of vector spaces.

So, for example, let us say there is this equation let us say, d square y by dx square + y = 0. Now, the solutions of this equation. So, this is a linear of ODE and solutions of this ODE where y is continuous, is actually, in this case double continuous. So, the second derivative exists y is twice differentiable and continuous and real valued also form a vector space.

So, this is we are not going to be using this in our context but it is important to set the tone when we are discussing vector spaces because we want to make sure that we understand vectors and the generality of the idea. So, see the way we went about it, is basically we set up certain rules. And we said anything any vector space where these rules are being followed we will call those things a vectors.

So, you can now go back to the definition and this is what I want you all to do is? Go back to the definitions that we had introduced about linear vector spaces and just make sure that it is correct in this case. So, if you have this solution y of this ordinary differential equation, please make sure that they actually, will satisfy all those criteria that we had set. If they it does not then we will be there, will be a problem there. This will not constitute a real vector space.

So, it is very easy to see in the previous case because say if you have two functions f, f is a real valued continuous function. Then and G is another real valued continuous function. The sum of the two is also a real valued continuous function. And you can define – F, so that it comes to 0 and so, all those things can be done here. So, these are all good examples for us to look at and then we did transformations.

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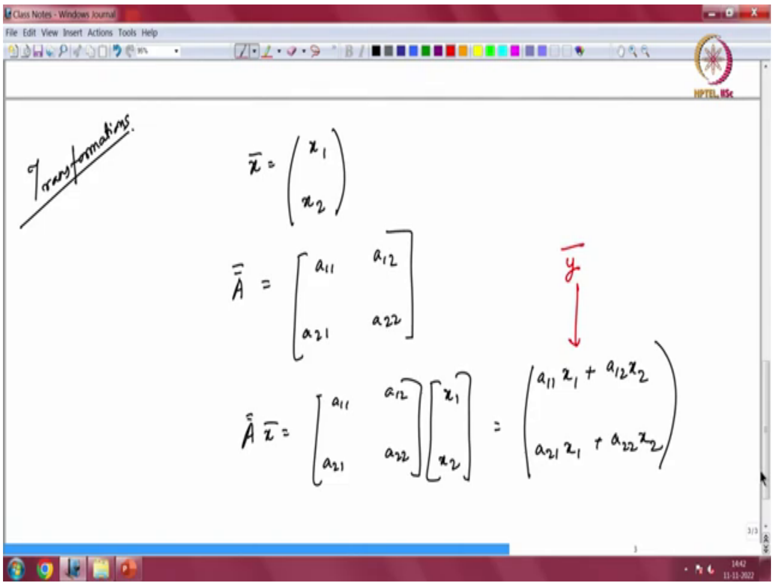

So, we were talking about transformations. So, now in transformations, we can obviously that the type of transformation will sort of depend upon what vector space you are working on? And what do you really want out of it? Now, in our case, what is going to happen is the most common type of transformation we are going to see are going to be matrices. So, let us say you have x as a two dimensional column vector.

So, let us say it constitutes. the two numbers x 1 and x 2. Then a transformation of this type where have this can now become a linear transformation. And you can quickly check that because you can now have this operating on x. So, how to do matrix multiplications and here you can easily see that what you will have is. So, this is basically nothing but your function or your new vector y that we have set.

So, this transformation acts on x and gives you another vector y. So, this type of matrices are going to be very, very important. We are going to encounter them again and again in our work and there is a reason why we are introducing vectors spaces and we are discussing tensors. Because you might want to ask at this point that this is going to be a course on our fluid mechanics and solid mechanics, why are we discussing tensors?

Well, it will turn out that some of the measurement techniques that are going to be there you would want tensors as the output and probably the most common tensor will be your strain rate tensor So, in that case, you want to understand these what tensors are? What they give you? And how they relate to some of the experimental methods? So, we will see, all that today we are going to stop here.

And we will pick up again on tensors in the next class and try to see some real examples of it till now I have just given you some abstract mathematical idea. So, thank you very much and I will stop here today.