Introduction to Soft Matter

Professor Aloke kumar
Department of Mechanical Engineering
Indian Institute of Science, Bengaluru

Lecture 18

Constitutive Equations Cont.

Welcome back everybody to another lecture on introduction to sock matter. And last time, we have started with discussing what are known as constitutive equations. There are very important class of equations, but they differ from the conservation equations. And I had said before that the constitutive equations are approximate and they can often be just phenomenological models for understanding reality. And what better way to see an approximate relationship than the example of the perfect or the inviscid fluid or the perfect fluid that we were discussing it.

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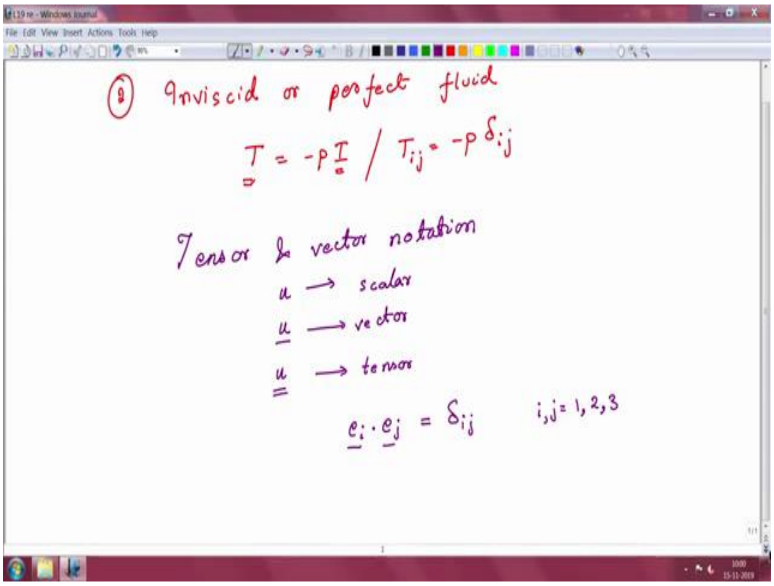

So, here we have also started introducing the tensor and the vector notation. So, I would like to clarify that the notations that we will use here what we will do is since we have started using tensors, we are going to let us say you have a variable u, if we do not, if we just write the variable u then this will imply that it is a scalar quantity. So, for example, pressure here that is we did not without any bar or without any other accompanying symbols. But if you put one small bar underneath, then what, it will imply a vector for us. And the same variable if we put the double bar, then this will indicate a tensor.

Now, just to clarify scalars are actually also considered zeroth order tensor. And a vector is also considered a first order tensor and a tensor is obviously the generalized form of all this.

But in our work, it is important to distinguish that these different quantities between these different quantities. So, we are going to adopt this notation for the rest of this course. And the one of the important quantities we saw was the chronicle delta. So, what is, we will define this also.

Now, chronicle delta is defined as e i. as a dot product of 2 vectors. So, I am going to put 1 bar underneath each of these and these are unit vectors in ith and the jth direction. Now the ith and jth direction independent of each other and hence, so this product is also called delta ij and these are all for Cartesian coordinates. So, we will take i comma j as varying from 1 taking on the values 1, 2 or 3, we are talking about 3 dimensional space.

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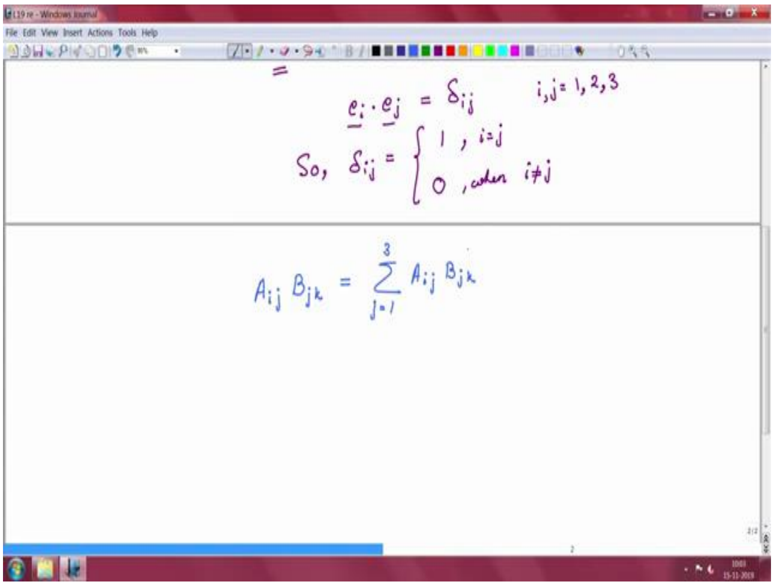

So, this quantity, so this quantity can take on different values and if i and j are the same, if they time stand for the same axis, then what will happen is this is a dot product of the same 2 unit vectors in the same direction. So, it takes on a value of 1 when i is equal to j and when i and j are 2 different vectors or vectors in 2 different directions, then they are also independent of each other. So, the dot product is always 0 and i is not equal to j.

So, we are getting started with rotation. And another important point to be noted with this, what we will be doing is we will be adopting what is known as notation where a repeated index basically implies submission. So, for example, so we have written here Del ij, i and j are 2 different indices. But, we can also write let us say there arises a situation where A is a tensor and when written in competent form let us say it is ij and then it is being multiplied with another tensor, let us say B and this now has 2 components which are indicated as j and

k. Now, you can notice the j is the repeated index here. So, this implies that this entire quantity has to be summed over all the possible values of j.

So, this is basically implying that this quantity is this shorthand expression is the same as this summation. And now, since we have already said j goes from 1, 2, 3, we can write here, j goes from 1, 2, 3. We will keep our attention focus on the 3 dimensional space, so this is good enough for us. So, that brings us to let us, so we discussed the topic of the inviscid or the perfect fluid and you can probably see the chain of thought we are adopting here. What we are trying to do is we introduce the idea of constitutive relationships and we are discussing different forms of constitutive relationships.

But you probably know already where we are going to go, we this is a course on soft matters. So, we are going to look at different constitutive relationships that can hold for viscoelastic materials. And, we are to build up to that, we are trying to look at very-very simple constitutive equations and try to understand those in the process. So, by the end of this class, I hopefully, you will be able to get to a point where we will discuss some of those important constitute relationships that you will require in your study of viscoelastic materials.

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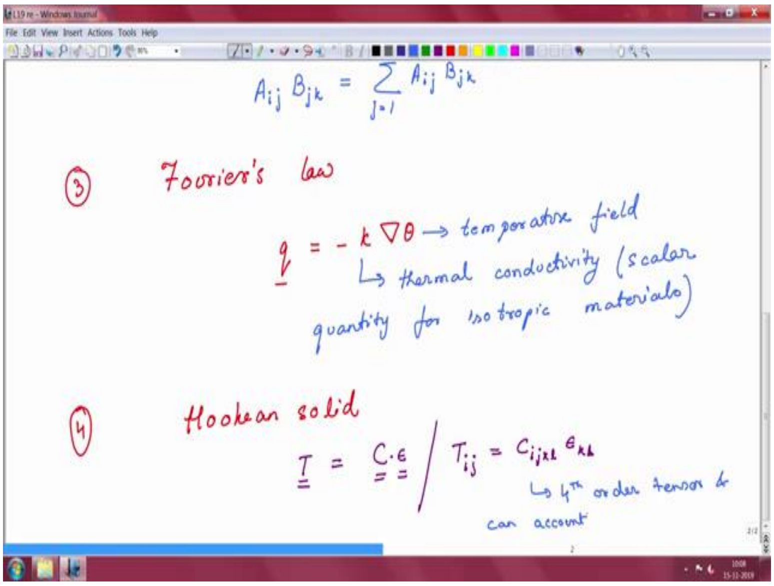

So, the next constitutive equation will be Fourier's law. Now, if you have taken a heat transfer course, this is something that you probably would have seen quite a bit by this time and Fourier's law, if you are already familiar with it, you probably know that it relates what is known as the heat flux with the temperature gradient. So, there is this heat flux is equal to minus k times the gradient of to the temperature field. So, this theta here, this denotes the temperature field and k here denotes thermal conductivity.

I did not write a bar underneath it and the reason for that is it is a scalar quantity for isotropic materials, but if the material is not isotropic, then this k can also become a tensor itself. So, we just because we are just trying to discuss more one and more complicated equations, we do not want to really get into, our objective here is not to understand Fourier's law, so we will leave it at this. The next one is what is known as the famous Hookean solid. Now, the concept of Hookean solid obviously comes from the name Hookean solid is because of being associated with Robert Hooke who wrote a very famous original paper, I believe the year was 1678 and this idea has been developed quite a bit after that.

So, after Hooke's original papers, this idea was developed by many, many, many scientists and around 1800s it was Cauchy who gave the form of the proper stress tensor to be associated with this. So, it was Cauchy, who gave, I believe the equation was given around 1820s. And he identified a stressed answer, which we will call T. And this is true for infinitesimal elasticity. And this stress tensor can be written as the product of 2 other tensors where, I will write down what this is, but before I do that, I just write it in component form.

So, this, now I am writing in the competent form, I do not have to put that. So, this is ij, so this is a stressed instead it is a second order tensor. Whereas C here is actually a fourth order tensor, so C goes from i, j, k, l and the strain tensor epsilon only takes on its again a second order tensor. So, here I will put k and l. So, this is actually C is a fourth order tensor and can account, so we can account, can account for an isotropy and we will just clarify what epsilon here is.

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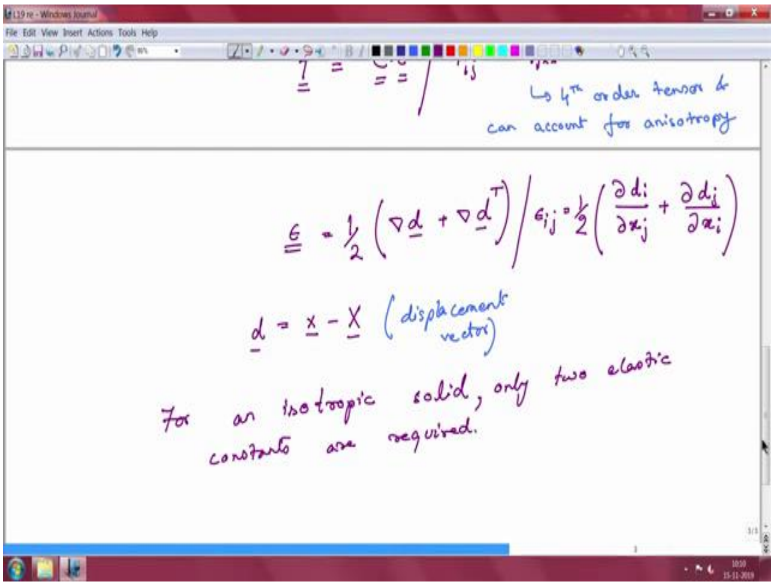

So, epsilon here is the strain tensor. So, that is given by and we will write this in the competent form 2. But I will just clarify what d is, d is the displacement vector. So, I am writing it down here, this is a this. It is displacement vector. And these are the small x and capital X. Capital X denotes as location in the reference configuration x refers to the current configuration and this epsilon obviously here it can also be written in the competent form as we have been doing before and that we will say if it is ij then say it is given by Del of d and this will be the other component here and we add the transpose part. So, this just the indices are just flipped now, so this will now become x I. So, this is the general form but for an isotropic material, so for an isotropic solid, the elastic constants can be reduced to only 2. So, for an isotropic solid only 2 elastic constants are required.

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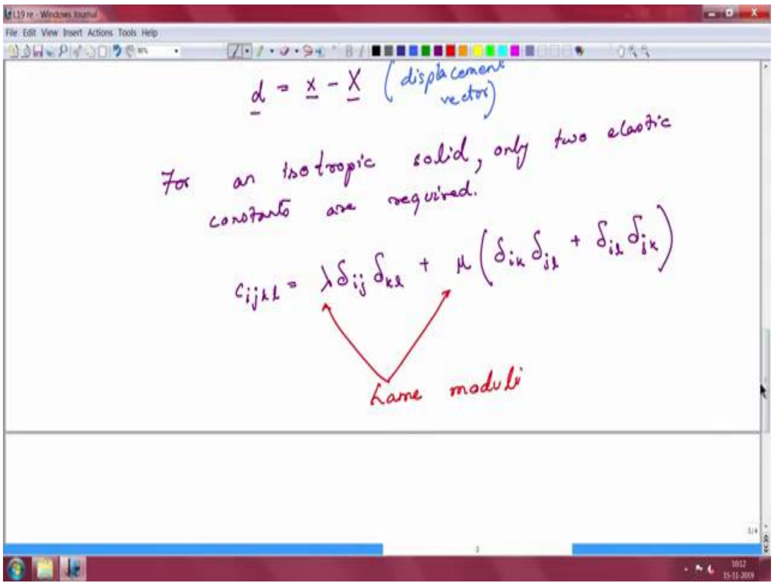

And now and then what happens is that the form of C becomes rather straightforward. And if I have to write it in competent form, I will write it as okay this is sorry I missed one in between c ijkl. Now, this becomes equal to lambda times Del ij and there is another chronicle delta, but this time that it is the other 2 indices. This is k and l will give, so we are using lambda let say mu and this is Del ik. And here this is going to be now the other 2 indices jl and again chronicle delta now you have il. So, if once you have il, what you have left with as j and k, so these where these 2 are also called Lame moduli.

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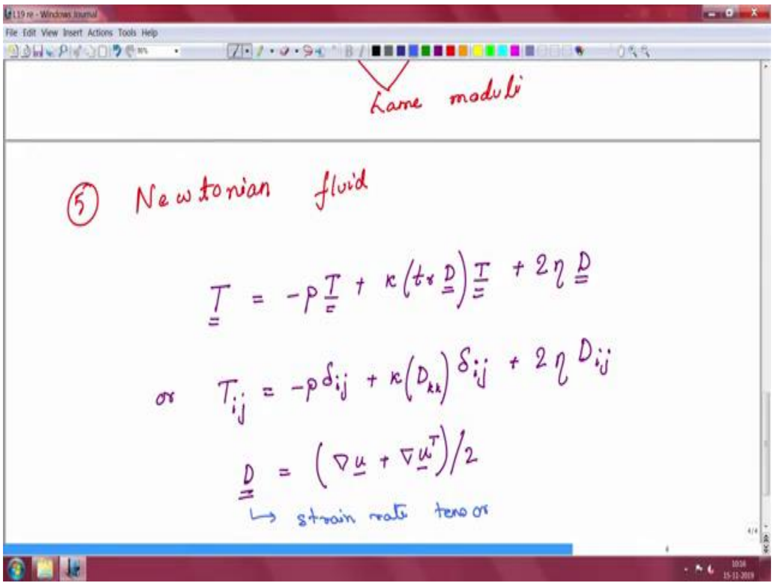

So, this has now brought us to this one was number four. So, from Hookean solid, I may like you can probably guess where we are going to go next right. We do want to discuss the case of the Newtonian fluid. This will be five case of Newtonian fluid. Now, Newtonian fluid is, this is a term that you are probably very-very well familiar with. And the reason Newton's name is associated with it is because very close to Hooke’s original work, I do not remember the exact year right now. But Newton proposed the concept of something called lack of slipperiness which we understand as viscosity and from there on, obviously, the ideas were developed later on.

And we had now we send a Stokes who contributed heavily to the development of the equation, which you today know as the Navier Stokes equation. So, let us discuss the Newtonian fluid and it was Stokes who gave the form for the stress tensor for this particular, for the case of the Newtonian fluid, he was one of the first to give the proper form and he gave the form this goes to Stokes. So, the stress tensor here is now something called p multiplied by the identity tensor and this p is pressure basically, then you have some kappa which is a material property multiplied by the trace of D.

So, this although D is a tensor the trace of D as a number multiplied by the identity tensor plus 2 theta. So, if I have to write it in competent form, we have to write it as or, now by this time you probably are very familiar with it. So, we have minus p, then you have the identity tensor, the identity tensor, it does not have to be written as I comma ij because we already know that this is given by the chronicle delta. So, I will just put delta ij and then you have this kappa times this trace of D. So, the trace of D means the sum of the diagonal elements. So, we know that to create a sum, all you need to do is have a repeated index.

So, what I can do is, I can just write it as D and then I can choose an arbitrary index because this is just going to be summed over. So, I can write it maybe even as ii in this particular case. And you have again the identity matrix. So, you have delta ij, so since I going to write that as delta ij, maybe I will choose k here otherwise no problem. And then you have 2 theta times Del ij. So, we have to now specify what D ij is, so this is a what is called as a strain rate tensor, which is given by the gradient of the velocity plus the gradient the transpose of that divided by 2. So, this D is also called this is the strain rate tensor.

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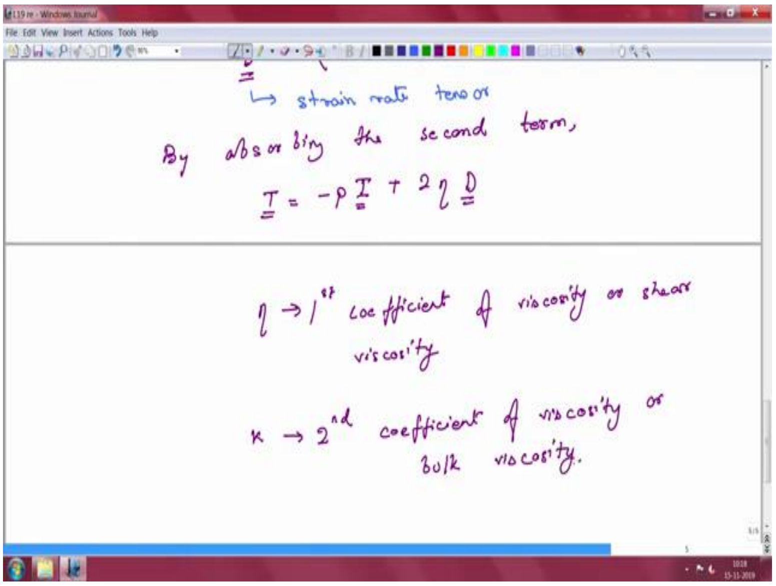

Now, Stokes assume that this kappa is minus two thirds of eta and so that the pure volumetric change does not affect the stress. Furthermore, this particular term the second term here that we see, this kappa, D, k, Del ij term can be absorbed with a pressure term and that leads to more simplified form which is now which can be written as, so I will just say simply by absorbing the second term, we can write it as minus p plus 2 eta. And, so I chose a, by the way, I probably want to clarify here in the Lame moduli I actually ended up using mu as a Lame moduli. So, that is why I introduced another symbol for viscosity. I believe somewhere in the beginning I had used mu as viscosity also.

But now that we have introduced eta, we just keep on using eta for viscosity. Now this eta is also called the first coefficient of viscosity or shear viscosity and this kappa is also called the second coefficient of viscosity or bulk viscosity. So, you can probably see the chain of thoughts is now going to fructify. And we are going to be able to, it is going to be a fruit and we can now introduce some of the more intricate ideas that we want to discuss as part of this course. So, the very fact that we have introduced a Newtonian fluid should beg the question, what is a non-Newtonian fluid? So, we will just quickly write down that.

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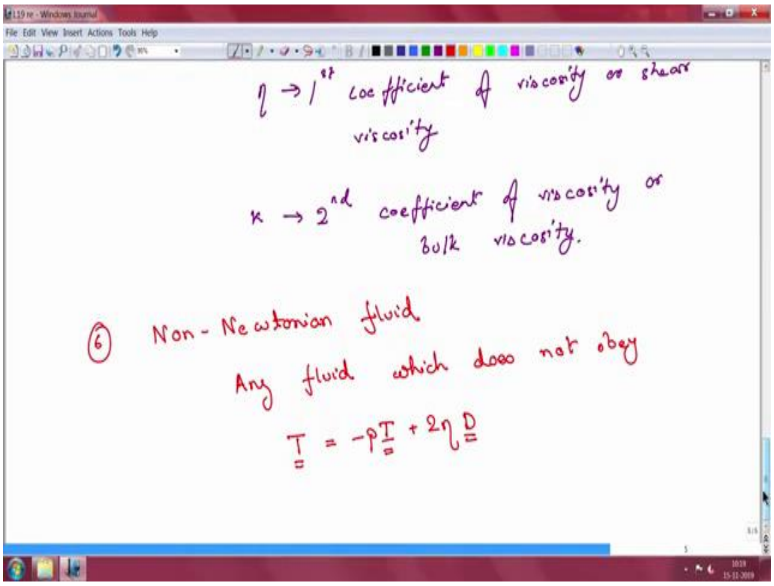

This was just a second, so, this was 5, so, I will say 6 here, non, so a non-Newtonian fluid is basically any fluid which does not obey the equation we just wrote. So which does not obey, when we say it is just does not obey that means the stress tensor does not obey, I think that is easy to understand. So, if it does not obey this particular relationship, we will call it as a non-Newtonian fluid.

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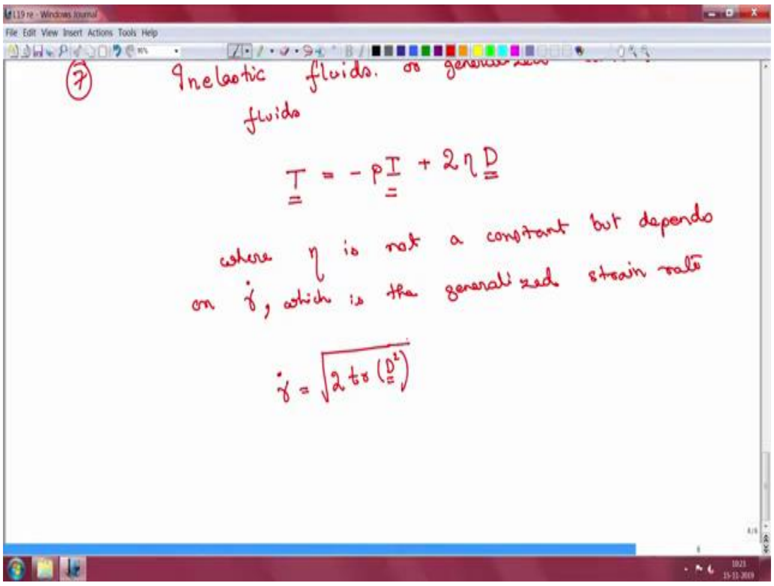

Now when you are discussing for example Navier Stokes types of fluid, we know that the viscosity is a term, but that viscosity does not actually depend on the strain rate tensor. It is independent, viscosity is a material property, it can depend on different conditions for example, temperature, but it does not depend on the strain rate tensor itself, which means that if you keep other things constant or other things unchanged, then no matter how fast the flow is, if just loosely saying that, no matter how fast the fluid is, how fast the fluid is flowing, eta is not going to vary with respect to that. So, here we can now bring in a slightly more generalized idea, which is also called inelastic fluids, inelastic fluids.

Now, when the phenomena dictated by viscosity only then it makes sense to model the viscosity the function accordingly. But in a more generic sense this viscosity term need not be a constant. So, inelastic fluids or generalized Newtonian fluids are fluids for which this extra stress tensor. So, if we forget for example, this part right now, then you have this particular part, the extra stress tensor that is proportional to the strain rate tensor. But now, that there is constant of proportionality eta is allowed to depend on the strain rate itself. So, maybe I can right now, for to write down this equation form I can just say that, this is now such that I can write down the total stress tensor as being equal to pressure term plus 2.

So in a more general, so this eta now is allowed to depend on the strain rate tensor. But, what is this gamma dot that we have written? So here, by the way, this means that eta is just a function of gamma dot. It is not being multiplied by that or maybe let me just eliminate chances of confusion where eta is not a constant. But depends on gamma dot, which is also called, which is the generalized strain rate is something that is derived from the strain rate tensor and it is actually defined as root over of 2 into the trace of D square. So, gamma dot becomes because stress is just a number, even though D is a tensor your gamma dot is just a scalar quantity here.

So, these inelastic fluids they have neither memory nor elasticity. So, we have achieved one complication over Newtonian over the very-very simple Newtonian fluids. And that is we have said that this viscosity term can now be a function of the strain rate tensor but still the entire behavior of the fluid is governed by a viscosity term. The fluid does not have an elasticity or it does not have the ability to store energy in a sense, all the energy that you put in will start to get dissipated immediately.

So, it is important to understand that is why we call it the inelastic fluid, which means elasticity is absent. So, in a sense, you can see how we are building it up, we introduced the idea of Newtonian fluid and then we said, there is something called the inelastic fluid or the generalized Newtonian fluid and from here on, we can build 2 more complicated ideas. So, one of the easiest ways to understand this is through a diagram.

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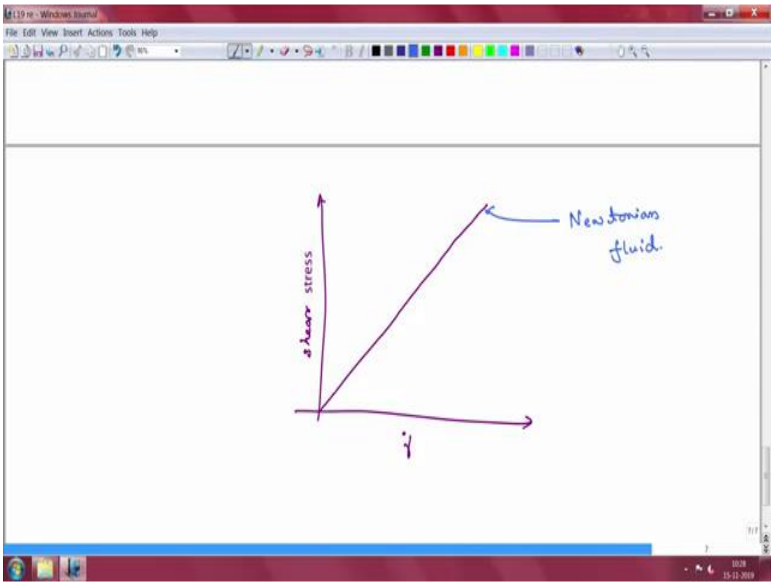

So, what we are going to try and do now is, we were going to create a plot and on this axis, I have gamma dot, which we discuss as the strain rate, the generalized strain rate. And here we are going to plot shear. Now for a Newtonian fluid, what do you think could be the relationship between the two? See, in the Newtonian fluid, the viscosity term is constant. It does not depend on gamma dot. So, the relationship between the two is actually the shear rate is simply proportional to gamma dot and the proportionality constant is truly a constant for all values.

So, this relationship is given by line. So, when you plot this and this represents the behavior of a Newtonian, so this is where we are going to leave it off for this today's class. And for the next in the next class, we are also going to look at the same diagram what we are (try) going to try and do is plot some of the behaviors of some of the other fluids for example, what we will call the shear thickening and the shear thinning fluids, which are examples of what we just saw, generalized Newtonian fluid or inelastic fluids. So, we are going to stop here for today's class and then we will carry on from where we left off. Thank you.