A short lecture series on Contour Integration in the Complex Plane

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Lecture - 22
Inverse Laplace Transform

Good morning, welcome to this lecture on complex variables. So, last class we looked at the L shaped branch cut, that was meant as at a research level not at a classroom level, but still it is very rarely discussed or presented, so I have given hints as to how an L shaped cut can come. Now, today I discuss the last topic in this series which is Inverse Laplace Transform.

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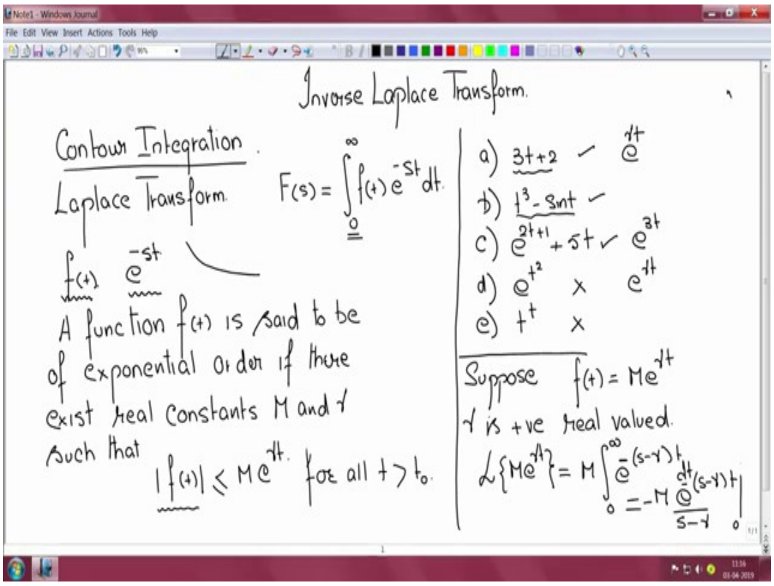

Typically, Laplace transforms are used in engineering in solving ODEs and PDEs and the inverse Laplace transform is used from a set of tables. So, we rarely actually compute the inverse Laplace transform, but it turns out that inverse Laplace transform requires contour integration.

So, I will show you how to get an inverse Laplace transform using contour integration. So, just to begin Laplace transforms, the forward Laplace transform is typically given by capital F, s is the Laplace symbol variable, integral 0 to infinity f of t which is the time signal, e to the power of minus s t d t ok. Our systems are causal, they are excited at time t equal to 0 and so this limit is 0 and it goes off to infinity, ok.

Now, in a simplistic sense the Laplace transform will exist if the function f of t does not grow faster than e to the power minus s t can bring it down ok, e to the power minus s t is a decaying type function. So, if f of t does not grow faster, so much faster that e to the power minus s t cannot pull it down, such functions have Laplace transform, and in engineering it is not a very big issue to find such such responses, ok. All our response is finally decay to 0.

Now, some formal presentation; a function, a function f of t is said to be of exponential exponential order if there exist real constants M and gamma such that f of t amplitude is bounded in this manner; M e to the power gamma t ok, e to the power gamma t is an increasing function, rapidly increasing function. So, f of t amplitude is bounded in this fashion, ok.

For all, so we will write for all t greater than some t 0. So, in this sense if you look at the following functions a), 3 t plus 2 and lets say b) is t cube minus sin t; c) e to the power 2 t plus 1 plus 5 t, then d) e to the power t square and e) t to the power t. If you look at these functions these three are of exponential order. It is very easy to find an e to the power t or some gamma t which bounds this function and this function, and of course, this function is easily bounded by 3 t, but these two are not of exponential order, they rise much faster than any e to the power gamma t.

Suppose, suppose, f of t is indeed equal to M e to the power gamma t, where gamma is positive real valued, ok, then the Laplace transform of M e to the power gamma t is equal to M integral 0 to infinity, e to the power minus s minus gamma t dt which is equal to, minus M e to the power minus s minus gamma t by s minus gamma over the limits 0 to infinity.

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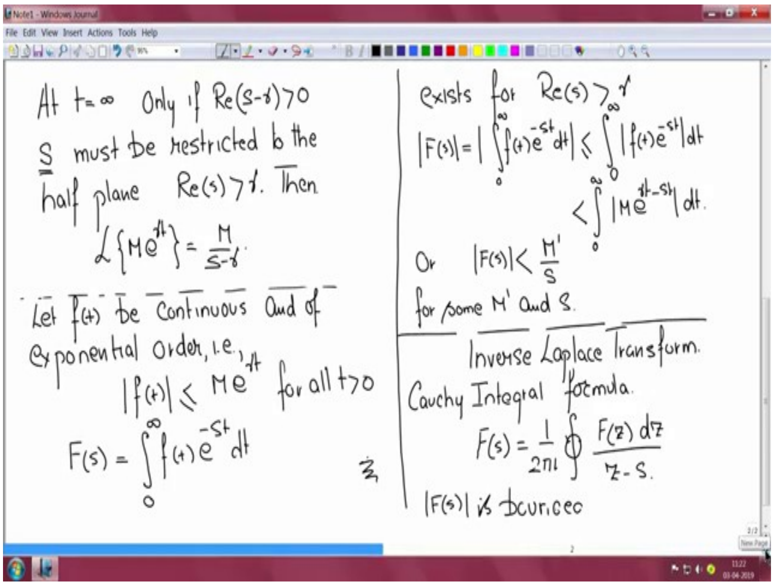

Now, at t equal to infinity, the expression will be bounded; the above expression is bounded only if the real part of s minus gamma is positive, ok, which means s the Laplace variable is must be restricted restricted to the half plane, real part of s greater than gamma. Then Laplace transform of M e to the power of gamma t is equal to M by s minus gamma, ok.

Further let f of t be continuous and of exponential order, that is magnitude of f of t is less than or equal to M e to the power gamma t, for some M and some gamma, let us say for all t greater than 0. Then the Laplace transform f of s is equal to, integral 0 to infinity f of t, e to the power of minus s t dt exists for real part of s greater than gamma.

And we see magnitude of F of s is equal to the magnitude of the integral 0 to infinity, f of t e to the power of minus s t dt, it is less than equal to the integral 0 to infinity, the magnitude of f of t e to the power of minus s t dt, which is less than strictly integral 0 to infinity M e to the power gamma t minus st dt. Or magnitude of f of s is definitely less than m dash by s, for some M dash and some s.

Now the inverse Laplace transform, inverse Laplace transform. We use the Cauchy integral formula to write it, Cauchy integral formula, we write it as F of s is equal to 1 over twice pi i, integral a counterclockwise contour F of z dz, by z minus s, ok.

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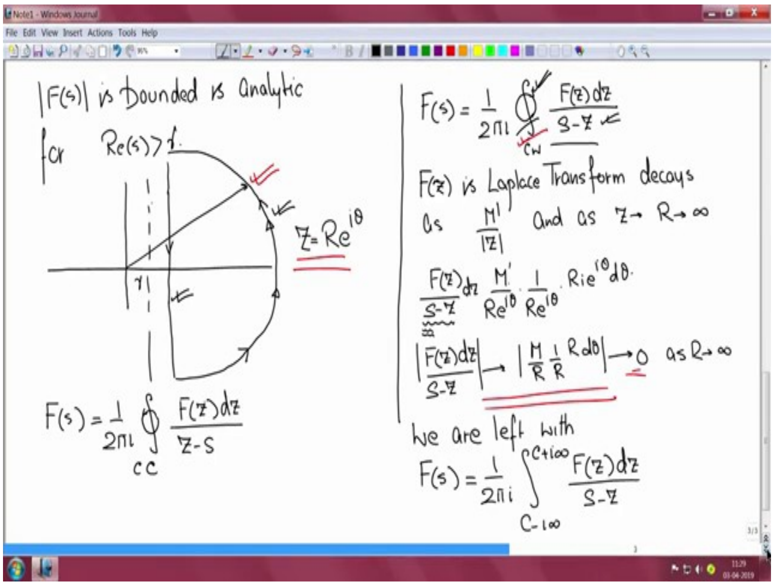

Let us remember that F of s is bounded, is bounded, is bounded; let me change the page, F of s is bounded and is analytic for real values of s greater than gamma. Now let us use, so we are going to look at that integral and let us use the following contour. The contour is this, we have the complex plane and we have gamma over here, here is my gamma. So, I take a contour counterclockwise, so I come down from infinity above to infinity below, I go around this infinite semicircle, ok.

So, let me remind you that what we wrote was F of s is equal to 1 over twice pi i integral counterclockwise F of z dz, by z minus s, ok. Now we switch it to clockwise and change the sign. So, F of s is equal to 1 over twice pi i, we switch it to clockwise F of z dz over s minus z. So, you brought in two negatives, we changed the direction and we change the sign on here. Now, F of z is the Laplace transform and decays as bounded by M dash over magnitude of z as before and z tends to infinity over a radius R the integrand decays.

So, integrand here is F of z, by s minus z, ok. F of z is bounded by M dash over R e to the power of i theta. Then s minus z, s is less important as z tends to infinity. So, I have R e to the power of i theta and then dz is R i e to the power of i theta d theta. So, we are essentially looking at the integrand over this semicircle. So, z is parametrized at R e to the power i theta. So, now, the magnitude of this F of z neither should be here, F of z dz by s minus z behaves as M over R, 1 by R and R d theta which tends to 0 as R tends to infinity.

So, on this contour, this portion of the integral will not survive, on this contour here, on this contour here, this portion of the integral will not survive; that is what we have shown. On this semi circular contour, z is equal to R e to the power i theta, so this integral integrand goes to 0. So, now we are left with the straight line portion so, which is what F of s is equal to 1 over twice pi i. We are to the right of gamma over here, this is a straight line so, we will write that as some C minus i infinity to C plus i infinity F of z dz over s minus z. I will write this again here.

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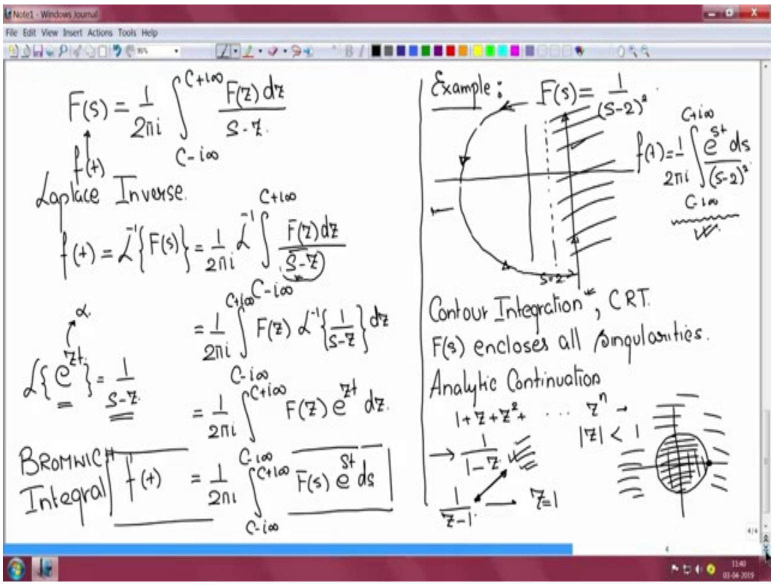

So, what we have is F of s is equal to 1 over twice pi i ,integral the straight line portion C minus i infinity to C plus i infinity, F of z dz by s minus z. Now, let us take the Laplace inverse of this, because it has arisen from time function, ok. So, now the f of t which is Laplace inverse of F of s is equal to 1 over twice pi i, the Laplace inverse C minus i infinity to C plus i infinity, F of z d z by s minus z, ok.

Now, Laplace will inverse will act only on s and not on z. So, what we have here, is 1 over twice pi I, integral C minus i infinity to C plus i infinity, F of z is unaffected, then the Laplace inverse goes and operates on 1 over s minus z.

We can compute the basic Laplace transform of this function e to the power z t and the Laplace transform of this is 1 over s minus z, ok. Normally we are used to seeing alpha over here if you remember your Laplace transforms. So, now the Laplace inverse of this is e to the power z t. So, we get 1 over twice pi I, integral C minus i infinity to C plus i infinity, F of z e to the power z t dz or using another dummy variable, 1 over twice pi i the same integral C minus i infinity to C plus i infinity, F of s e to the power s t d s.

So, that is how the Laplace inverse is computed actually over a straight line segment where F of s is analytic. The straight line segment going from minus i infinity to plus i infinity and this is called the Bromwich integral. Now, let us see; now let us consider take an example, the example is this suppose I have, so lets take an example, the example is this; F of s is equal to 1 over s minus 2 whole squared.

Those of you who remember your Laplace tables may immediately know what f of t is. So, now, here I have a pole at s equal to 2. So, now we said that F of s is analytic in the region as s is greater than some gamma, so you can see that s equal to 2 will be our gamma here, this is s equal to 2 and so now, we have a straight line segment that goes from this way upwards. So, now we have to compute this integral; f of t is equal to 1 over twice pi i ,integral C minus i infinity to C plus i infinity, which is this line, e to the power s t d s, over s minus 2 whole squared, ok.

Now, using contour integration; we have to use contour integration that is the whole thesis of this series of lectures. We are going to use contour integration and again as before, this segment which we want we do not touch, so we will use principles of contour integration and if possible Cauchy residue theorem and compute this integral without touching it, ok. So, now what we do, we close this, we close this in the left half plane of the; we close it in the left half complex plane so, we now have a closed contour.

So, that now the function F of s encloses all its singularities. Now one important point here which I will briefly mention is that in the beginning we said that F of s exists and is analytic to the right of a certain region. So, we should ideally sit only in that region and not extend it to the left, but here is an idea called analytic continuation which is used. I will explain it using a simple example. Suppose we have this geometric series 1 plus z plus z square and so forth and goes off to infinity; the sum we know is 1 over 1 minus z provided that magnitude of z is less than 1.

That means what in the complex plane, you have a unit circle, you have a unit circle, and so you have to be inside the unit circle, so that you have convergence, ok; however, separate from this idea where we sum this geometric series, there exists a function 1 over z minus 1 or 1 minus z which is singular only at z equal to 1 and is well behaved, everywhere else, there exists a function like that, ok.

Now, the idea of analytic continuation says that this 1 over 1 minus z and this 1 over 1 minus z are identical. They are identical, they are not two separate functions, so that is the idea of analytic continuation. So, now this function which is supposed to converge in inside the z equal to 1 radius extends and is singular only at z equal to 1, ok. So, that is the idea of analytic continuation. Now we are going to do this contour integral and as before make sure that the function or the integrand goes to 0 on this arc, we will make sure of that and use the residue theorem. Right now time is running out, so we will continue in the next class.

Thank you.