

NOISE CONTROL IN MECHANICAL SYSTEMS

Prof. Sneha Singh

Department of Mechanical and Industrial Engineering

IIT Roorkee

Week:1

Lecture:4

Lecture 04: Harmonic Plane Waves

The slide features a white background with a blue header and footer. At the top left is the IIT Roorkee logo. In the center is the Swayam logo with the text 'FREE ONLINE EDUCATION' and 'swayam'. To the right is the NPTEL logo with the text 'NPTEL ONLINE CERTIFICATION COURSE'. Below these logos, the title 'Noise Control in Mechanical Systems' is written in a large, dark blue font, followed by 'Lecture 4' and 'Harmonic Plane Waves' in a smaller blue font. The presenter's name 'Dr. Sneha Singh' and department 'Mechanical and Industrial Engineering Department' are listed below. At the bottom, there is a wide photograph of the IIT Roorkee main building, a large white structure with a central dome and multiple columns. A small number '1' is visible in the bottom right corner of the slide.

Hello and welcome to the lecture 4 in the series on noise control in mechanical systems. I am Professor Sneha Singh from the Department of Mechanical and Industrial Engineering at IIT Roorkee, and in this lecture 4 we will discuss about harmonic plane waves.

So, let us begin. So, before we begin just a brief recap on what we have done till now in the previous lecture. So, in the previous lecture we derived the linear acoustic wave equation which is a very standard equation that is applicable in almost all sorts of acoustic media and in different types of wave fronts. So, it is independent of the wave front and it is independent of the acoustic media we are dealing with. This equation is applicable in all these situations and hence its derivation was taken care of. and the acoustic wave equation can be given by over here, like this. So, let us in this lecture now we will discuss about

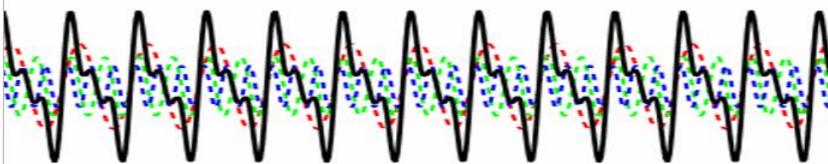
Summary of previous lecture

$$\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = \nabla^2 p$$



Outline

- Harmonic plane waves
- Harmonic plane wave equation



harmonic plane waves and harmonic plane wave equation. So, this is the acoustic wave equation in this format.

Now this is a partial differential equation in the space and time variable. Now we want to find out the solutions to get the equations for acoustic pressure for different types of waves and different types of boundary conditions. So, here the first in order to find out the solutions to these wave equations, we need to have either some boundary conditions or

some definition of the type of wave that is being propagating in the form of sound. So, the very first wave that we will deal with is the harmonic plane waves. So, just to deal with this.

Acoustic wave equation

- Linear acoustic wave equation** for sound propagation in fluid:

$\nabla^2 p - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0$

$c = \sqrt{\frac{B}{\rho_0}}$

← Thermodynamic speed of sound
- Thus, for a homogenous medium the sound wave propagation can be fully defined using the following medium properties:
 - Bulk modulus (B) ✓
 - Mass density (ρ) ✓

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Macroscopic/ bulk acoustic properties of a medium


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$$\nabla^2 p - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0$$

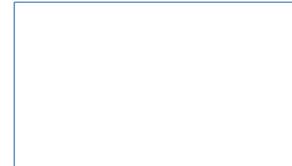
So, in this equation these are the macroscopic bulk acoustic properties of the medium and the thermodynamic speed of sound is given as

$$c = \sqrt{\frac{B}{\rho_0}}$$

So, let us discuss about the first type of waves in the sound waves. And what is the solution for the linear acoustic wave equation, if we use harmonic plane waves as our sort of case study. So, what do you mean by plane wave? Here the name itself says plane. So, it is a wave that has planar wave front. So, what is a wave front? So, wave front is usually defined for any kind of, propagating waves as the surface where the waves have constant phase. okay. So, the surface it is a locus of all the particles which are vibrating in the same phase at any given instant of time and the name suggests that it is a plane waves which means that here the wave front is a plane.

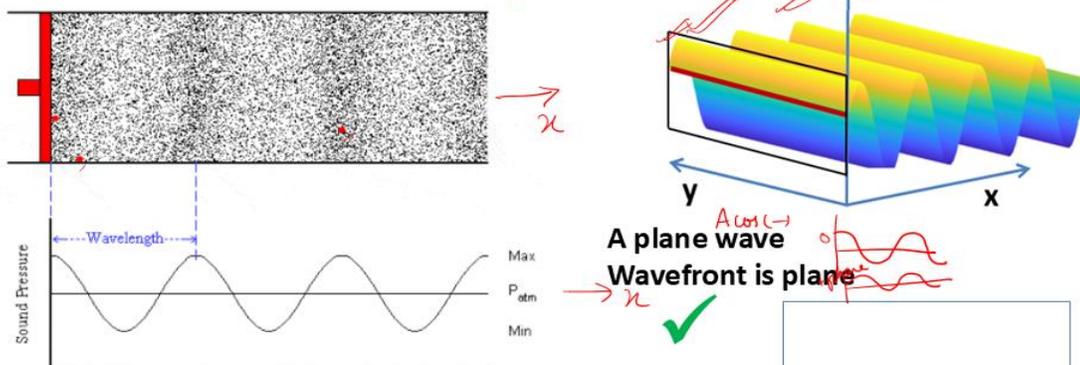
Harmonic plane waves

- **Plane wave:** An acoustic wave in which each acoustic variable (pressure, particle velocity, particle acceleration) has constant amplitude and phase on any plane perpendicular to the direction of wave propagation.
- **Wavefront:** It is the surface where the waves have constant phase.
 - Plane waves have plane wavefront.
- Sound pressure of a plane wave varies in one direction only.



And what plane it is? It is the plane which is perpendicular to the direction of wave propagation. So, to make the definition complete, it is an acoustic wave in which each acoustic variable which means the sound pressure, the particle velocity, the particle acceleration, all of these acoustic variables they have a constant amplitude and they also have a constant phase. So, both they have a constant amplitude and phase on any plane that is perpendicular to the direction of wave propagation. And one important property of this harmonic plane wave is that the sound pressure of this plane wave varies in one direction only and that the sound pressure is independent of the distance of the measurement. So, it

Harmonic plane waves



Sound waves by a reciprocating piston inside a narrow pipe

Source: <https://physics.stackexchange.com/questions/442399/why-should-a-moving-piston-generate-pressure-variation>

has the same amplitude irrespective of the distance it has travelled from the source. So, to make the definition complete and to get a better idea let us have a look at the schematic.

So, this is the harmonic plane waves. One example of that is when we use a reciprocating piston inside a long narrow tube or a long narrow duct or any kind of pipe and we are using this piston which is moving back and forth and creating these pressure waves of compression and rarefaction.

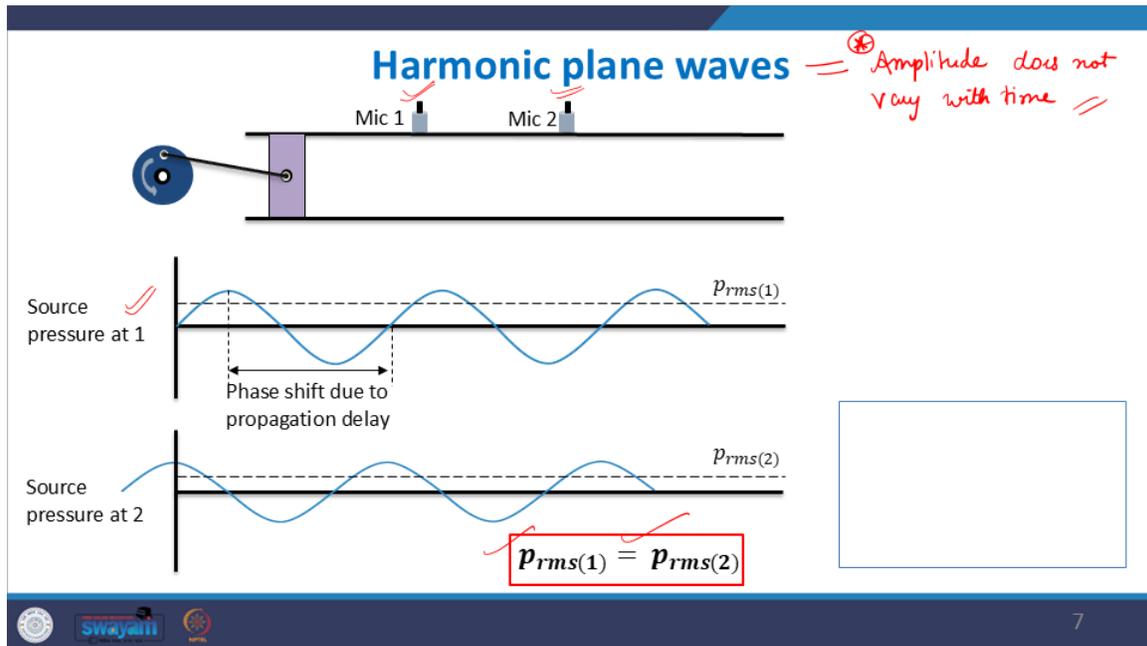
So, what happens in this situation is that it is independent of which point where we are measuring the sound whether it is in the first red dot or the second red dot here or the third red dot you will get a pressure wave of the same amplitude. The amplitude will not vary with the distance. Now, what do you mean by a planar wave front? Let us have a look. Let us say that this direction in which the waves have been created is the x direction. So, in the three-dimensional view, let us have a look at this acoustic wave. Here this is what? We are representing in the wave format the variation in the pressure of the medium, okay So, this is the variation in the pressure of the medium.

So, the definition says that you have to take the plane that is perpendicular to the direction of wave propagation and in that take a snapshot which means that observe the particles at any instant of time. so like at various time snaps or at various time snapshots you want to observe how the particle looks like in that planar wave front in that wave front which is a surface perpendicular to the direction of wave propagation so let's take our snapshot and you would see that at any point in time These particles they are in the same phase. What do you mean by phase? Now many of you who are familiar with wave mechanics and waves in general would know, what do you mean by a phase of a wave. But just to quickly recap this in the course let us say here pressure is having a sinusoidal variation. It could be either a sinusoidal wave or a sine wave or it could be a cosine wave. And the phase simply means at what point we are starting that wave. The solution to this particular equation let us say that the pressure variation we are representing in some form of cos wave or cosine wave. Let us say some $A \cos$ some kind of quantity here it is a cosine wave. Then if the phase is 0 means the wave starts like this. But the wave is continuously propagating in time and we can observe it at any instant of time. We can observe it at time t equals to 0, t equals to t_1 , t_2 at any moment we decide to take a snapshot of the wave. Then it is propagating. So, let us say the wave has propagated a bit further and it has come up like this.

Now, both of these are the same waves, but one is having a phase of zero and one is having some positive phase lag. it has moved further in time. So, over here if you observe in this

snapshot all the particles in this particular plane that we have taken they are all in the same phase of the cosine wave and all of them they have the same amplitude and this is a plane which is the YZ plane So, it is a plane. So, the wave front becomes planar.

So, this becomes a harmonic plane wave, but in the same way if you had any kind of planar surface. So, suppose this is your wave that is propagating like this. and you decide to cut any plane that is perpendicular to the direction of the wave propagation. It is not necessarily it has to be in this particular figure the YZ plane. It could be any plane that is parallel to the YZ plane also and in all these planes if you take a snapshot and observe what the particles are doing at one particular instant of time, you will observe that they are always in the same phase and they always have the same amplitude. This defines as the harmonic plane wave. So if you have understood the definition, so let us see if we can differentiate between a harmonic wave and a non-harmonic plane wave.



Before that, let's say one example I gave you was a reciprocating piston that is generating the waves inside a long tube. Now if we had two microphones to measure the sound pressure at two different locations inside this.

What we will observe is that let us say the pressure at location 1 or the mic 1 is given by this cosine wave. It has a certain RMS or a root mean square pressure and then we measure the pressure at a certain distance away in the microphone 2. We get the same pressure

wave. The only difference is there is a difference in the phase. But the amplitude is same and therefore, the RMS also remains the same.

So, that is what I was trying to illustrate that for a plane harmonic waves the amplitude this is an important property does not vary with time. So, this is a very important property because when we will study about spherical waves, cylindrical waves in our further lectures, we will see that not every kind of wave will have a constant amplitude with space and time. Some waves the amplitude will vary specially with the spatial distance. So, here the amplitude remains the same. Hence, ideally these waves are not getting attenuated with the distance or the space.

Harmonic plane waves

+x direction of wave propagation *-x : direction of wave propagation*

A plane wave in +x direction **A plane wave in +xy direction** **Evanescent wave in +xy direction**

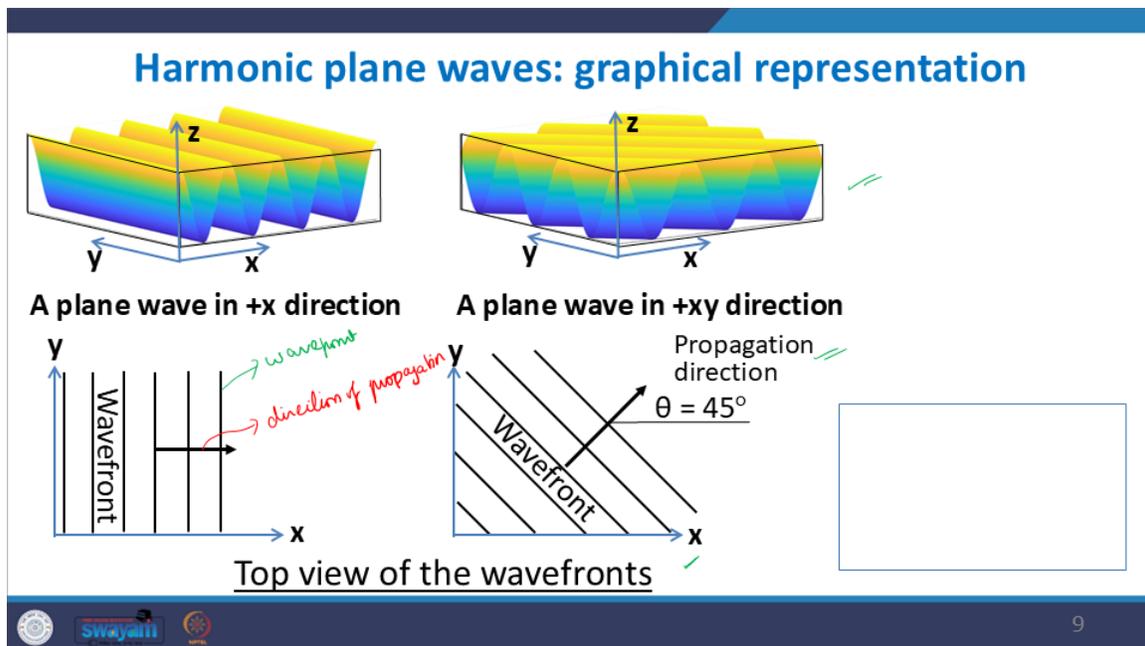
✓ ✓ ✗

- All wavefronts are plane, but is amplitude varies across the wavefront then it is not a harmonic plane wave

So, let us see whether we can differentiate a harmonic plane wave from something which is not harmonic and planar. So, let us start with the first figure as you see here there is a plane that is propagating the direction is x axis this is the same figure that we showed here. So, the same logic if you cut any plane that is perpendicular to x axis which means a plane parallel to the y-z plane at any moment the particles they are always in sync which means they are in the same phase and they have got the same amplitude so this should be a harmonic plane wave but what about this it looks slightly different from the first one. Is this still a harmonic plane wave? Here if you see, first of all we need to identify the surface which is perpendicular to the direction of wave propagation. So, what is the direction of wave propagation here? It is plus xy is the direction of wave propagation. If you cut a plane

that is perpendicular to that which is going to be a plane something like this and so on various parallel planes.

So, in all of this plane what you will observe is that yes the amplitude remains the same, but at the same time if you observe closely the particles they are in phase they are in the same phase. So, this should also be a harmonic plane wave, but what about the third figure here. Here we again have a plane. What is the direction of propagation? It is the negative x which is the direction of wave propagation. And if you cut a plane that is perpendicular to it, the planes which are parallel to y-z plane, at any moment if we take a snapshot what you see is that, over here if you observe in this particular plane. If you observe carefully at any instance the particles still have the same phase. But do they have the same amplitude? The answer is NO. because if you observe the plane in this here, at the very extreme end, closest to the x-axis, they have the smallest amplitude and when it is furthest away from the x-axis or the x-z plane, it is having the maximum amplitude. The amplitude is varying as it is moving away from the x-z plane. So, this should not be a harmonic plane wave because amplitude does not remain constant. and that is true. So, this is actually a evanescent wave which is not in the scope of this lecture series, but it is not a harmonic plane wave. So, let us see some graphical representation of this wave.



Now, in this lecture series for most of the parts I will be representing a harmonic plane wave from the top view how it looks like. It would simply be an arrow with some

perpendicular lines which are cutting the arrow. The way I will represent it is that the wave would be in the form of a top view format where the arrow gives you the direction of the wave propagation and that these lines here so this particular arrow so this is giving you the direction of the wave propagation and the lines perpendicular to it, these they are giving you the wave front. From the top, this is a plane and we are seeing it from the top it looks like a straight line. It is a planar wave front, if it was not planar, it would look like a curve. So, this is how I will be representing. In the same way, suppose using this representation, if you look at this particular figure, again the direction is along the x-y and then we have these wave fronts which are perpendicular to the direction and these wave fronts look like lines. which are perpendicular to the direction of wave propagation. This is the kind of notation or symbol that we will be using throughout this lecture series. Now, because this lecture series is on noise control in mechanical systems.

Plane sound waves in mechanical systems

Examples:

- ✓ Noise inside exhaust pipes
- ✓ Piston noise in ducts
- ✓ Large vibrating machinery surface (sound measured nearby)
- ✓ Large vibrating panels (sound measured nearby)




Source: Google Images


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Everything we put in the context of mechanical systems and machineries. So, why do we want to study harmonic plane waves? Well, the reason is simple, there are two main reasons first of all that harmonic plane waves can be encountered is way in various kind of components in machineries and mechanical systems. The other reason is that, in general there is any kind of real life sound be it harmonic or non harmonic, but any kind of random real life sound it can always be represented as a superposition of harmonic wave fronts of different magnitudes and different directions, but it can be represented it can be simplified

into a superposition of various harmonic plane waves. And hence the study of the simplest form of the wave which is the harmonic plane wave becomes very important. So, what are some of the sources of harmonic plane waves in the mechanical systems. Now, already I have told you that if suppose for example, you are using a reciprocating piston in a long pipe, it creates a planar wave front. Similar kind of system if we explore for example, the noise inside these exhaust pipes, okay. The pipes, these exhaust pipes, they are connected directly to the engine. and the engine is having this reciprocatory motion. It has a piston and it has got its own unique cycle of exhaust and combustion and so on. This exhaust is sort of driving away the noise from the engine which is a harmonic noise. It is sinusoidal in nature and it is through the reciprocating piston action that the exhaust is coming out of this, tail pipe and the tail pipe is long and narrow. So, hence we get planar wave fronts inside the tail pipe.

Now, here I would like to make one differentiation that the noise inside these tail pipes would be a planar wave front, but once it exits outside it becomes spherical in nature. Because, here we have the piston motion creating the waves inside some long narrow tube but once the same sound is coming out it is now the tube has ended and now suddenly it encounters an infinite baffle. Baffle is a term that we will be using here which means that something which is infinite compared to the other components. Here the sound suddenly experiences an infinite baffle and emits or radiates outside into this infinite medium from a narrow tube. It behaves like a point source and its wave front changes from harmonic and it becomes spherical in nature. Similarly, the piston noise, any kind of piston noise inside ducts in the machineries they all are harmonic plane waves. At the same time suppose you had large vibrating machinery surfaces or large vibrating panels and you measure the sound not far away from it, because the further you go the spherical wave front can be observed. You had a large surface and it is sort of pushing away and pulling away these air particles and you are measuring it not very far away from these large surfaces, you will experience harmonic plane waves in these. So, large vibrating panels, roof panels etcetera and the measurement takes place in the order of magnitude of their dimensions, then you can experience harmonic plane waves.

So, already I told you that any kind of real sound be decomposed into a superposition of harmonic plane waves which have varying magnitudes, frequencies and directions. This particular schematic illustrates my point. Let us say we have got a real life sound something like this. It looks like a periodic wave, but does it look like a pure tone or a harmonic plane wave? It does not look like that, but it is what? It is a composition or an addition of these

three harmonic plane waves at the three different frequencies and so on, even very random waves which may be non-periodic in nature, but by the virtue of the Fourier's theorem it can still be decomposed into a summation of harmonic components, each of which can be represented as a simple harmonic plane wave.

Why study harmonic plane waves?

- All real life sounds can be decomposed into a superposition of harmonic plane waves of varying magnitudes, frequencies and directions.

Real life sound ←

Decomposed **harmonic plane waves**

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So, this is our linear acoustic wave equation and now that we know that any kind of real sound be thought of as a combination of various harmonic plane waves, which have

Harmonic plane wave equation

- Linear acoustic wave equation for a plane wave propagating in x direction:

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2}; p = p(x, t)$$

$$p = p_1 + p_2 + p_3 + p_4 \dots$$

↑ ↑ ↑ ↑

↻ generic form of solution for a harmonic plane wave

- A general sound wave can be assumed to be a combination of periodic waves with a harmonic time dependency (Fourier's theorem).
- Thus, it is sufficient to solve for just one frequency component at the time.

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different frequencies magnitudes and directions. and this is by the Fourier's theorem that they can be represented in that format.

So, it is sufficient for us to solve one harmonic solution and then in the same way for all the other waves we can simply add it up. Suppose we had one sound wave let us say:

$$P = p_1 + p_2 + p_3 + p_4 + p_5 + \dots$$

Suppose we could find out what is the solution of a simple harmonic plane wave, what is the solution we are getting with this partial differential equation. Then, these solutions we can feed each of them. We already know the generic form of solution for a harmonic plane wave. Then we can simply substitute and change the different frequencies or the magnitudes or whatever is known to us and we can sum up to get the pressure equation of that complex sound. So, let us find out the solution for the harmonic plane wave.

Harmonic plane waves

- A simple harmonic plane wave travelling in x direction is represented graphically as:
- $p = f(x, t)$ ✓
- Let $t = 0$
- Then, as observed:
- $p(x, t = 0) = A \cos(Bx)$; $A, B = \text{constants}$
- Now, when wave travels by a distance λ , wave pattern repeats itself in space.

Let us start with a 1D solution, a harmonic plane wave that is travelling in the positive x direction. This is the graphical representation of the same which we have already discussed in the previous lectures, what is a wavelength and so on.

Now, this as you can see it is a harmonic plane wave, it is a pure tone component, it has a single frequency, it can be represented as a sinusoidal variation. And here it is a function because the wave is travelling in the positive x direction only. So, it will be a function of x and t. Now, let us try to derive the solution. So, what we will do here to arrive at our

equation is that first we will see what is the nature of the wave front when it is only varying with the space. We take a snapshot at a particular time and see how does the shape look like or what should be the equation like when it is varying with respect to space. First, we take a snapshot at a particular time instant and observe what is the solution when it is only varying with the space. And then in the second case what we will do is we will again observe a single particle. So, in the later half we will fix the spatial distance or x value. So, which means that we fix a particular particle in the medium and we only observe how that particle is moving in the medium and how its wave is varying with respect to time. So, here let us observe it at time $t = 0$. What happens is that we have put $t = 0$. So, it seems like at a particular time instance this is the picture it seems like a cosine wave. So, let us say it has certain amplitude,

$$p(x, t = 0) = A \cos(Bx),$$

it is a cosine wave where it is varying with some Bx . This is a generic form of this wave. Here a and b are constants which we will find out very shortly. Now let us see what is the period of this wave. It is a cosine wave right. So, what is the period of a cos wave? It is 2π . The wave repeats after every 2π intervals.

Harmonic plane waves

$\Rightarrow p(x = 0, t = 0)$
 $= p(x = \lambda, t = 0)$
 $= p(x = 2\lambda, t = 0)$
 $= p(x = 3\lambda, t = 0)$

$\Rightarrow A \cos(0) = A \cos(B\lambda)$

$\pm B\lambda = 2\pi \Rightarrow B = \pm \frac{2\pi}{\lambda}$

$p(x, t = 0) = A \cos\left(\pm \frac{2\pi}{\lambda} x\right) = A \cos(\pm kx)$

Wave number

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So here what we see is that, we have fixed the time as 0, at $x = 0$, whatever the pressure is, if you see what is the period of this wave, if you see that suppose this is the acoustic pressure at time $t = 0$ and $x = 0$, that same thing repeats after a certain interval at this

point. So, the pressure here is the same as the pressure here and in the same way if we extend it is same as the pressure here and so on. So, after every λ we are getting the same pressure. So, basically just like a cosine wave has got a 2π as the time period or the period over which it repeats its pattern let us not call it time period for the timing it is a spatial dimension. This cosine wave repeats after every 2π radian and here what is this 2π , when the wave is completing one full wavelength. So, after every wavelength it is repeating the pattern. Which means that this thing inside here Bx , it is repeating

and because it is a cosine wave it is repeating with respect to space it can be propagating in the positive x direction or it can propagate in the negative x direction. We will take a general solution and add a (\pm) sign to it so B times of λ from here.

$$A \cos(0) = A \cos(B\lambda)$$

$$\pm B\lambda = 2\pi \Rightarrow B = \pm \frac{2\pi}{\lambda}$$

Wave number

- **Wave number or Wave constant (k):** Number of complete wave cycles in radians per unit spatial dimension.

$$k = \frac{2\pi}{\lambda}$$

- This constant is a very important wave parameter.
- SI unit: rad/m

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Now, if we remember when we discussed about an important wave parameter called as the k which is the wave number. What is a wave number? By the definition it becomes a number of radians per unit spatial distance which we found it to be,

$$k = \frac{2\pi}{\lambda}$$

So, this thing here is plus minus k. So, if we substitute this value in this particular equation,

$$p(x, t = 0) = A \cos\left(\pm \frac{2\pi}{\lambda} x\right) = A \cos(\pm kx)$$

we have found that this is what bx is nothing but $(\pm)k$ times of x because b is plus minus k . So, now we have gotten the Solution, if the wave is only varying with respect to space and constant with time, but that is not the case, the wave is varying with time. So, now, we will observe a single particle and see how it moves over time. So, we will follow the same approach. This parameter we have already discussed, it is the number of complete wave radians per unit spatial dimension. The SI unit, so this is important, here k is the wave number. The SI unit is radians per meter. So, here before we go and find out the solution for the time wise variation an important relationship needs to be derived.

Wave number

Relation between wave parameters:
Wavenumber & frequency ?

$k = \frac{2\pi}{\lambda}$ $f = \frac{c}{\lambda}$
 $\omega = 2\pi f = \frac{2\pi c}{\lambda}$

\Rightarrow $\omega = kc$ \Rightarrow $k = \frac{\omega}{c}$

$k = \frac{2\pi}{\lambda} = \frac{\omega}{c}$ ✓


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A relationship between the wave number(k) and the frequency(f).

$$k = \frac{2\pi}{\lambda}$$

$$f = \frac{c}{\lambda}$$

So, the angular frequency (ω),

$$\omega = 2\pi f$$

$$= \frac{2\pi c}{\lambda}$$

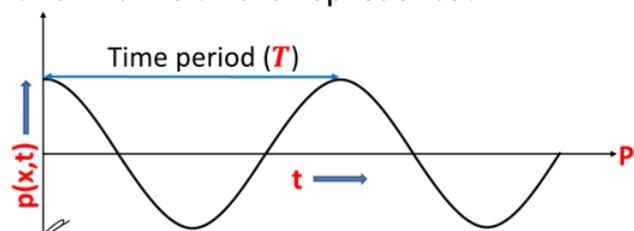
$$\omega = kc$$

$$k = \frac{\omega}{c}$$

These are the two important relationships between them which will be using when we derive the time equation this relationship here which is same as this relationship here so that is an important relationship.

Harmonic plane waves

- A simple harmonic plane wave in time axis is represented graphically as:



- Let $x = 0$ ✓
- Then, as observed: ✓
- $p(x = 0, t) = A \cos(Ct)$; $A, C = \text{constants}$
- Now, when wave travels by time by T , wave pattern repeats itself in time.


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Let us see how it varies with respect to time, we follow the same approach let us observe a particle at $x = 0$ and see how does the waveform look like, even that particle is doing that sinusoidal motion up and down back and forth with the same amplitude which we represent its displacement of the particle so here this is representing the displacement of the particle with respect to the time. The wave front again looks like this. Let us observe what happens. So, at $x = 0$, if we see this, it also looks like a cos wave. let us say some,

$$p(x = 0, t) = A \cos(Ct); A, C = \text{constants}$$

it is varying with respect to time. We will again find what are these constants and we will follow the same approach.

Harmonic plane wave equation

$\Rightarrow p(x = 0, t = 0)$
 $= p(x = 0, t = T)$
 $= p(x = 0, t = 2T)$ & so on
 $\Rightarrow A \cos(0) = A \cos(CT)$
 $CT = 2\pi; C = \frac{2\pi}{T} \Rightarrow C = \omega$

$\Rightarrow p(x, t) = A \cos(\omega t \pm kx)$

Or, $p(x, t) = Ae^{j(\omega t \pm kx)}$

Equation of a harmonic plane wave travelling in x direction ('+' = -X direction, and '-' = +X direction)

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Here, the wave is repeating the pattern after every time period T. So, whatever was the pressure here would be the same pressure here and would be the same pressure after again. So, after every T, it is repeating the pattern.

$$\Rightarrow A \cos(0) = A \cos(CT)$$

and this happens when this becomes equals to 2π because 2π is the period of cos wave.

$$CT = 2\pi; C = \frac{2\pi}{T} \Rightarrow C = \omega$$

and see here like for example, when we are discussing the solution with respect to space we had added a \pm sign that is because we want to incorporate the solution of both a wave travelling in the positive x direction and the wave travelling in the negative x direction. But can we do that with time? We cannot move backwards in time. Time is unidirectional. So, we are not incorporating this \pm thing.

So, this becomes omega. So, now we can sum up both these solutions because these are the solutions with two independent variables. We simply have this solution,

$$p(x, t) = A \cos(\omega t \pm kx)$$

Here plus is when the wave is travelling in the negative x direction and the minus comes when the wave is travelling in the positive x direction.

So, we can represent it as this particular waveform. So, this becomes our harmonic solution to the linear acoustic wave equation. Now, in acoustics many a times what we do is that we have to deal with lot of because we are finding a solution to a differential equation. So, differentiation, integration, multiplication these become some of the very basic operators which we have to do continuously while dealing with wave mechanics. It is a good practice to represent the same solution in an exponential format. The reason being simple that if you represent it in the exponential format, then the differentiation and the integration becomes easy. As an acoustics engineer is that we can solve the various kind of equations using this exponential format, do the differentiations, integrations, divisions, multiplications and the final solution that we get, the real part of it represents the actual wave. That is the approach we follow.

Harmonic plane wave equation

- Let us cross-check if $p(x, t) = A e^{j(\omega t \pm kx)}$ satisfies the differential wave equation: $\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2}$

$LHS = \frac{\partial^2}{\partial x^2} [A e^{j(\omega t \pm kx)}] = \frac{\partial}{\partial x} [\pm A jk e^{j(\omega t \pm kx)}] = \frac{\partial}{\partial x} [jk p] = j^2 k^2 p = -k^2 p$

$RHS = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} p = \frac{1}{c^2} \frac{\partial}{\partial t} [j\omega p] = \frac{1}{c^2} j^2 \omega^2 p = -\frac{\omega^2}{c^2} p$

$k = \frac{\omega}{c} \quad \underline{\underline{RHS = -k^2 p}}$

Let us quickly cross check whether this satisfies the solution that we are now assuming. We have found the solution; we are converting it into the exponential format. Can it satisfy this? Let us have a look.

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2}$$

So, left hand side is what? It is,

$$\begin{aligned} LHS &= \frac{\partial^2 [A e^{j(\omega t \pm kx)}]}{\partial x^2} \\ &= \frac{\partial}{\partial x} [A j k] \end{aligned}$$

So, when you do differentiate it partially with respect to x, what do you get? The sort of constant with respect to x which is this k is coming up. So, what is the answer? The answer is simply, after the first differentiation you will get,

$$\begin{aligned} &= \frac{\partial}{\partial x} [A j k e^{j(\omega t \pm kx)}] \\ p(x, t) &= A e^{j(\omega t \pm kx)} \end{aligned}$$

So, we can also say that,

$$LHS = \frac{\partial}{\partial x} [j k p(x, t)]$$

Again, differentiating with respect to space we get the same way we again have another constant which is j k. So, j k multiplied by j k.

$$\begin{aligned} &= j^2 k^2 p(x, t) \\ LHS &= -k^2 p(x, t) \end{aligned}$$

and j square is minus 1. So, this is our left-hand side. Now, let us solve again for the right-hand side.

$$RHS = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2}$$

Let us have this solution here. So, when you differentiate it with respect to time, this ω is what? This is coming outside. This is what is with time. After first differentiation, what you will get is,

$$RHS = \frac{1}{c^2} \frac{\partial}{\partial t} [j \omega p(x, t)]$$

Again, differentiating it you get this twice,

$$\begin{aligned}
 RHS &= \frac{1}{c^2} j^2 \omega^2 p(x, t) \\
 &= -\frac{\omega^2}{c^2} p(x, t) \\
 &= -k^2 p(x, t) \\
 RHS &= LHS
 \end{aligned}$$

So, yes, this equation satisfies our plane wave solution.

Harmonic plane wave equation

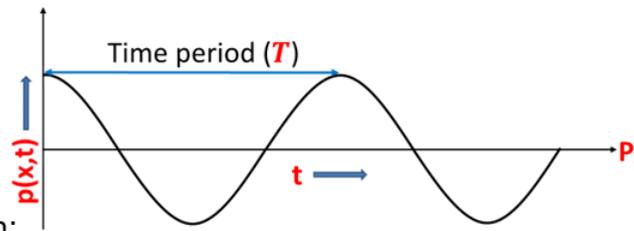
Equation of a harmonic plane wave travelling in x direction:

$p(x, t) = Ae^{j(\omega t \pm kx)}$

Using the Linear Euler's Equation:

$$\rho \frac{\partial \vec{u}}{\partial t} = -\nabla p \quad |s| \ll 1 \Rightarrow \rho_0 \frac{\partial \vec{u}}{\partial t} = -\frac{\partial p}{\partial x}$$

Putting the value of p in the modified Linear Euler's equation:



this is the equation of a harmonic plane wave traveling in the x direction. And now if we use the euler's equation which we had used in the previous lecture so you can refer to my previous lecture to see how this equation came about and this the rate of condensation is very small because the change in the density is not very significant. So, this can be written as simply

$$\rho \frac{\partial \vec{u}}{\partial t} = -\nabla p$$

$$|s| \ll 1 \Rightarrow \rho_0 \frac{\partial \vec{u}}{\partial t} = -\frac{\partial p}{\partial x}$$

we will have simply this nabla simply becomes del by del x for a 1D wave. So, let us see we have this equation and we have this.

Harmonic plane wave equation

$p(x, t) = Ae^{j(\omega t \pm kx)}$ & $\rho_0 \frac{\partial \vec{u}}{\partial t} = -\frac{\partial p}{\partial x}$
<<Solve>>

$\frac{\partial \vec{u}}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} = \mp \frac{1}{\rho_0} jk p$

$\vec{u} = \int_0^t \mp \frac{1}{\rho_0} jk p dt = -\frac{jk}{\rho_0} \int_0^t p dt = \frac{-jk}{j\rho_0 \omega} p$

$u(x, t) = -\frac{k}{\rho_0 \omega} p = -\frac{p}{\rho_0 c}$

$\Rightarrow \frac{k}{\omega} = \frac{1}{c}$


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Now, we want to solve it to get an equation. So the thing is that we have already gotten the equation of the acoustic pressure. Now we want the equation of the particle velocity. Particle velocity also is a sinusoidal variation with respect to space and a sinusoidal variation with respect to time.

let us see how we can get the particle velocity from these two equations.

$$p(x, t) = Ae^{j(\omega t \pm kx)} \quad \text{and} \quad \rho_0 \frac{\partial \vec{u}}{\partial t} = -\frac{\partial p}{\partial x}$$

$$\frac{\partial \vec{u}}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x}$$

What I will say here is that, because there is already a minus sign here. So, let me indicate it in this way which means like this. Whatever the sign was it has gotten inverted.

$$= \pm \frac{1}{\rho_0} jkp$$

$$\vec{u} = \int_0^t \pm \frac{1}{\rho_0} jkp dt$$

So, we have plus multiplied by plus if it was minus it would be minus while differentiating the constant coming outside would be minus. again, we integrate it we get a minus. So, twice we multiply so the sign will remain intact which means that the final solution if it was negative in the beginning it will be negative afterwards as well if this was negative here.

$$= -\frac{jk}{\rho_0} \int_0^t p dt$$

$$= -\frac{jk}{j\rho_0\omega} p$$

$$= -\frac{k}{\rho_0\omega} p$$

$$\vec{u}(x, t) = -\frac{p}{\rho_0 c}$$

That is our solution.

Harmonic plane wave equation

- Equation of acoustic pressure of a harmonic plane wave travelling in x direction:

$$p(x, t) = Ae^{j(\omega t \pm kx)}$$

+ = wave going in -X direction
- = wave going in +X direction

- Equation of acoustic particle velocity of a harmonic plane wave travelling in x direction:

$$v(x, t) = -\frac{Ae^{j(\omega t \pm kx)}}{\rho_0 c} = \pm \frac{p_{\pm}}{\rho_0 c}$$

p_+ = acoustic pressure of forward propagating wave
 p_- = acoustic pressure of backward propagating wave



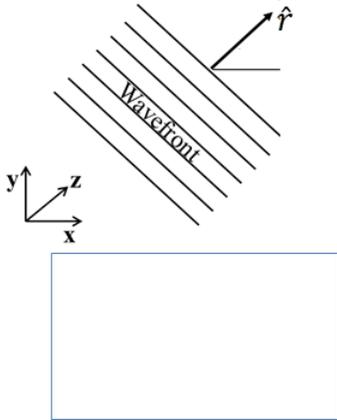
So, in a harmonic plane wave, this is how the velocity and the pressure functions are related which is given by this condition here.

when the acoustic pressure is moving forward and negative when the acoustic pressure is moving backwards okay, this is how the velocity and the pressure are related in a harmonic plane wave Now, suppose we had, this was the solution when the plane wave was moving in the positive x direction or a negative x direction. Suppose instead of moving in just an x direction in the 3D it can be moving in any arbitrary direction which whose position vector we are representing as \hat{r} or \hat{r} cap.

Harmonic plane wave equation

- The harmonic plane wave equation in an arbitrary direction, \hat{r} :

$$p(\hat{r}, t) = Ae^{j(\omega t \pm k\hat{r})}$$
- Spatial direction of wave propagation (\hat{r}) is:
- $\hat{r} = x\hat{x} + y\hat{y} + z\hat{z}$
- Where, $\hat{x}, \hat{y}, \hat{z}$ are unit vectors along X, Y and Z axis respectively



So, this \hat{r} so this \hat{r} is what it is the direction vector of the harmonic plane wave. So, this is a generic harmonic plane wave which we can write as some x component in the direction of the x axis, some y component in the direction of y axis and some z component in the direction of the z axis. So, these become the unit vectors x, y and z respectively in these axis respectively. So, suppose k is the overall wave propagation vector. How do you define the wave propagation vector of a harmonic plane wave? It is a vector. So, a vector will have a magnitude and it will have a direction. So, this is the vector whose magnitude is the same as the wave number k , but whose direction is along the direction of the wave propagation. So, here the wave is propagating along the direction of \hat{r} which means k is in the same direction as \hat{r} and it is in this direction. So, it will also have its own component because it is a vector. It will have a certain component in the x-axis direction, in certain

Propagation vector

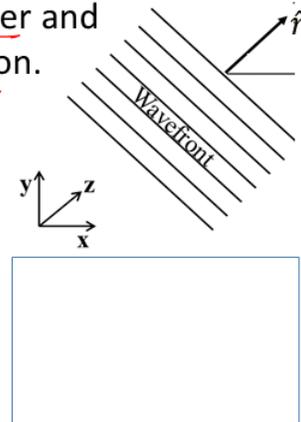
- Let us define the **wave propagation vector (\vec{k})** as the vector with magnitude same as the wave number and direction along the direction of wave propagation.

$$\vec{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z}$$

- Therefore,

$$|\vec{k}| = k = \frac{\omega}{c}$$

$$\left(\frac{\omega}{c}\right)^2 = k_x^2 + k_y^2 + k_z^2$$



component in the y-direction and certain component in the z-axis direction, okay. So, this is a generic representation of this k vector. The magnitude of this k vector is k , which is our wave number ω/c .

And we know that the magnitude for any vector is the square of the magnitude of the vector will be what? It will be the sum of the square of the component in all these three directions.

$$\left(\frac{\omega}{c}\right)^2 = k_x^2 + k_y^2 + k_z^2$$

This is an important relationship to remember for a harmonic plane wave travelling in a generic direction k . So, now we have the k vector the k vector is given by this and we have the r vector which is given by this what is this here in this generic equation this is k vector dot r . So, we are doing a dot product of the k vector and the position vector r . So, we simply do the dot product of this thing. And the dot product of this thing. if you do the dot product their respective x components get multiplied the respective y components get multiplied and so on. So, when you do that ultimately you end up with this solution here. So, this becomes the solution for a general harmonic plane wave. that may be travelling with along a certain direction in the 3D such that the direction of the vector has a k_x component in the x axis, the direction vector has got a k_y component in the y axis and the direction vector has got a k_z component in the z axis. and these equations hold true.

Harmonic plane wave equation

- The harmonic plane wave equation in an arbitrary direction, \hat{r} :

$$p(\hat{r}, t) = Ae^{j(\omega t \pm \vec{k} \cdot \hat{r})} = Ae^{j(\omega t \pm k_x x \pm k_y y \pm k_z z)}$$

$$\hat{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$p(\hat{r}, t) = Ae^{j(\omega t \pm \vec{k} \cdot \hat{r})} = Ae^{j(\omega t \pm k_x x \pm k_y y \pm k_z z)} \quad \hat{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\left(\frac{\omega}{c}\right)^2 = k_x^2 + k_y^2 + k_z^2$$



So, with this we end up our lecture on the harmonic plane wave and we derive this harmonic plane wave solution at any generic direction. So, thank you for listening.

Thank You