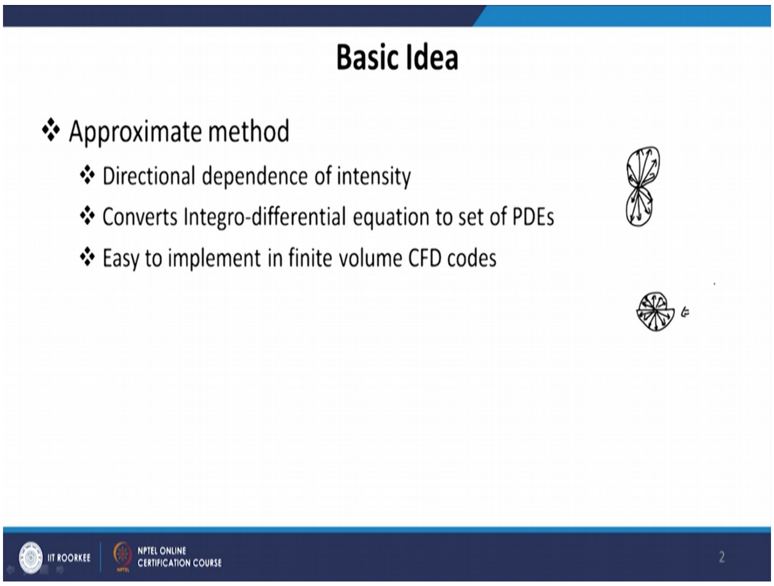
Radiative Heat Transfer
Prof. Ankit Bansal
Department of Mechanical and Industrial Engineering
Indian Institute of Technology - Roorkee

Module - 5
Lecture - 22
The Method of Spherical Harmonics (P N Approximation) - I

Hello friends, we are discussing approximate methods for the solution of radiative transfer equation. In the same category, today we will discuss the method of spherical harmonics, that is P N approximation method. As discussed last time, the approximations in the solution of radiative transfer equation may be 3-fold. In the first category, we simplify the geometry of the problem. So, we tried to solve radiative transfer equation for 1-dimensional medium.

Either the plane parallel slab or the cylindrical medium. In the second category, we approximate the properties of the absorbing medium. That is, we assume that the kappa or the absorption coefficient is very small optically thin case or it is large optically thick case. Today, we will discuss the method of spherical harmonics. And this method is basically the third, second, third type of method, where we basically divide the intensity or approximate the intensity in directions.

(Refer Slide Time: 01:44)



So, let us suppose, at any point in the medium we have actual intensity distribution which may not be symmetric. So, the intensity can be divided into 2 parts: the 1 component moving in the upward direction and the second component moving in the downward direction. And it may not be symmetric or isotropic at all. We saw for the case of 2 approximate methods in the last class.

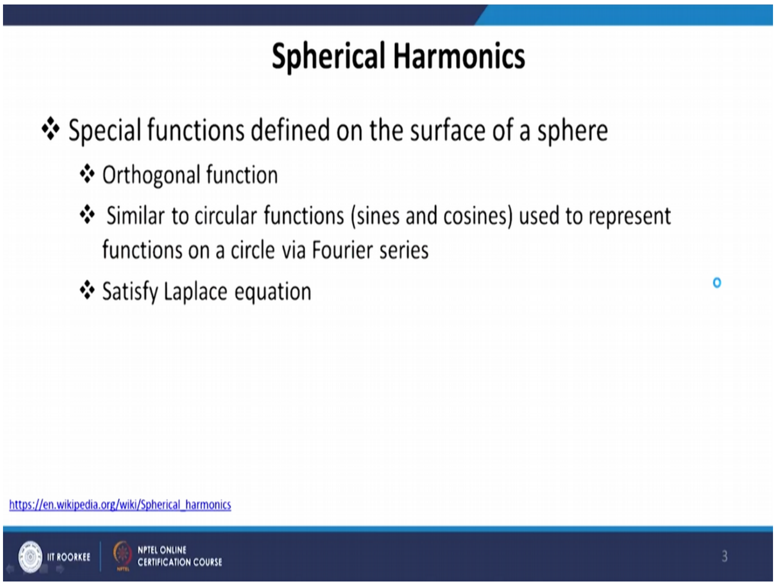
That is the Schuster-Schwarzschild approximation two-flux method and Milne-Eddington method that is a differential approximation method. The intensity distribution was assumed to be isotropic with different magnitude in the upward direction and downward direction. So, we assume that the intensity is isotropic. But the magnitude of intensity in the upward direction is different and the magnitude of intensity in the downward direction is different.

So, this was assumed for the 2 method that we discussed in the last class. Now, the basic idea of the method of spherical harmonics or P N approximation method is similar. We have to define the directional dependence of intensity. The advantage is similar to what we get in the two-flux method or differential approximation method. That is our integro differential equation is converted into a set of differential equations.

And the P N method is very popular. It is implemented in many commercial codes like Fluent and Star CCM. The reason is that, we know how to solve this differential equations. And it can be easily implemented in CFD codes. So, we start with the idea of the spherical harmonics. So, most of us are comfortable and familiar with Fourier series. So, in Fourier series, we have circular functions like sines and cosines.

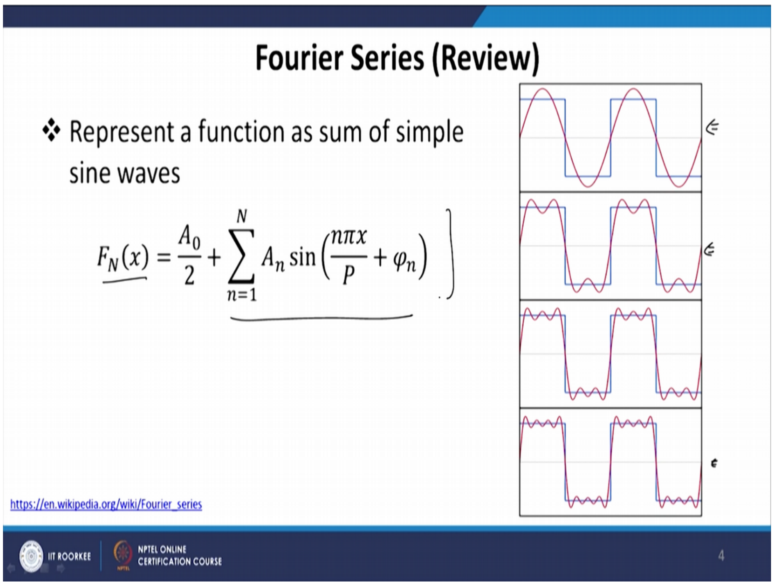
And with the help of the circular functions, we use the circular functions in a form of series to represent any function. So, Fourier series is defined in 2-dimension. Circular functions like science and cosines. And as a series of this sines and cosines functions, we can write any function fx. Okay. Now in 3-dimension, on the surface of a sphere which basically we want to represent our intensity, we use similar representation. And this representation is called the spherical harmonics.

(Refer Slide Time: 04:39)



So, spherical harmonic series basically is used to represent intensity in the 3-dimensional space on the surface of a sphere. And this, as we will see, satisfies the Laplace equation in spherical coordinates.

(Refer Slide Time: 04:56)

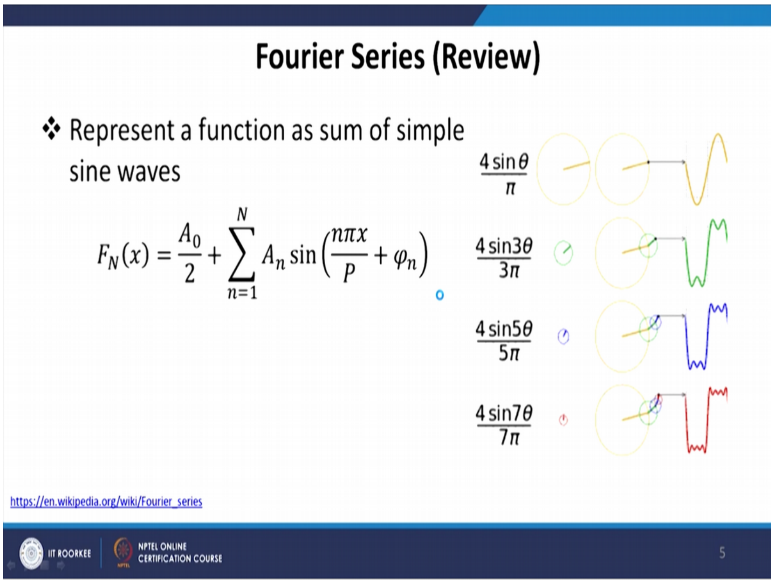


So, before I go into the details of the method of spherical harmonics; if we can understand clearly how the Fourier series is used to represent a function, it would be easy for us to understand the method of spherical harmonics. So, on this slide you see a sines and cosines series representing a square wave. So, we have a square wave. And if we use only first term of the series; so, we have an infinite series or it the series may have n terms.

So, the function is periodic. That is, the square wave is a periodic function. And this periodic function we want to represent using a series of sines and cosines. So, if the first term is written, we see that we get a sine wave and the sine wave is not able to fit the square wave very accurately. If we to take 2 terms, we get a better agreement. And this wave agrees or represents the square wave much better than the single term of the series.

We keep on increasing the terms in the series and our result gets better and better in agreement with the square wave. So, this is how basically we can represent a function F N x, a periodic function F N x, where N, subscript N represents that there are total N terms that have been take in the series. And this function which represents the square wave in this example is used to, is represented by this series. And this series is called Fourier series. To give you a clear idea why this is called circular function;

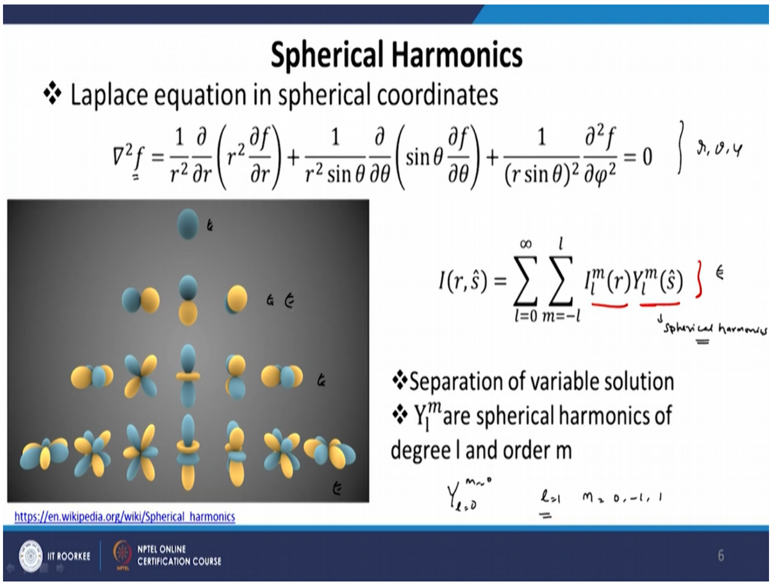
(Refer Slide Time: 06:39)



(Video Starts: 06:39) What you see on this slide is basically the variation of the y magnitude of the wave, as the pointer moves on the circle. Now, the pointer speed along the circle is changing. So, here we have the rotation at sin theta upon pi, here we have sin theta and so on. And based on the speed of the rotation, the wave is better able to represent the square wave. And we see that, if we can combine 1 or more, 2 or more terms of these circular functions in a series, then the function can be represented very accurately.

So, this is in 2-dimensional space. And we are representing **(Video Ends: 07:29)** a function using circular function. Now, intensity we understand is a 3-dimensional quantity. That means we, the intensity varies over the entire solid angle 4 pi and it varies on the surface of a sphere. That means, we represent it using spherical coordinates. So, we can, if we can represent this intensity in the form of a series just like a Fourier series, we probably can convert our integro differential equation into some form of differential equation.

(Refer Slide Time: 08:03)



So, this is basically the idea. Now, first of all we will understand what is spherical harmonics. And then, we will go for how the spherical harmonics can be used to solve the radiative transfer equation. So, in spherical coordinates, the Laplacian of a scalar, del square f, where f is now a scalar function; So, intensity is the, not a scalar function, it is a vector function. But, let us understand first for the scalar function.

So, Laplace of a scalar function, del squares f in spherical coordinates depends on its coordinates r, theta and phi. So, it can be written as, 1 by r square del by del r, r square del f by del r + 1 upon r square sin theta del by del theta of sin theta del f del theta + last term 1 by r sin theta square del square f by del phi square. And this will be = 0. So, in spherical coordinates, this is the Laplace equation.

And the solution of this Laplace equation will basically contain or comprise the spherical harmonics. So, using the method of separation of variables, we can solve this equation in spherical coordinates. We will have a term that will be purely dependent on r by the method of separation of variable. And 1 term which will be purely dependent on direction. Now, direction contains both theta and phi.

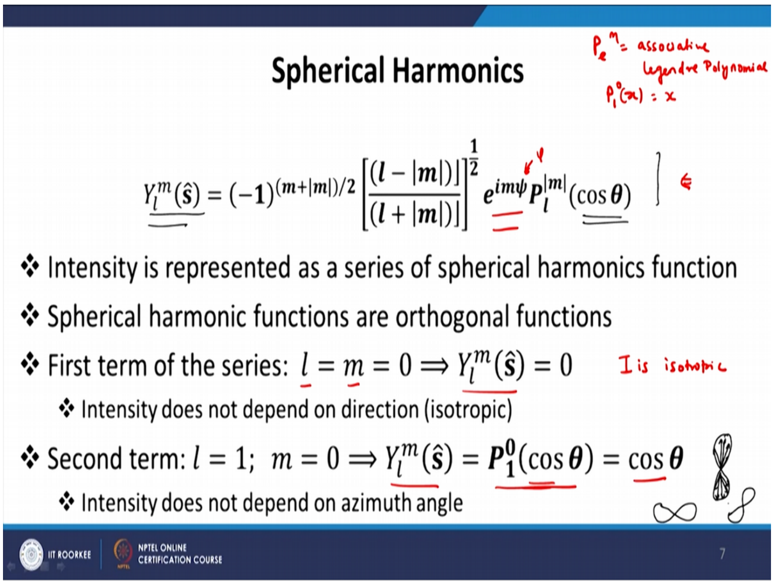
So, we have what we called method of separation of variable solution for this type of problem. Okay. Now, the second term that depends on direction is called the spherical harmonics. So, this term is basically the spherical harmonics. And how they look like? If you plot this function, the spherical harmonic function, it has a familiar shape that you must have studied in twelfth class, which has shapes like s, p, d, f orbital shapes in chemistry.

So, the first term of this series of Y l m. That is, l is = 0 and m is = 0. The first term of this series has symmetric shape. That means, it does not depend on theta or phi, polar and azimuthal angle. And the shape is spherical. And in chemistry it is called s orbital. Similarly, the second term l is = 1 has 3 values. The values of m will be 0, – 1 and 1, for l is = 1. So, for each value of l, there will be 12 + 1 values of m. And the shape is like p orbital shape.

So, there are 3. And the shape is of type p orbital in chemistry. If you go for higher order terms, l is = 2, the shapes are of d orbital and there will be 5 terms. And similarly, there will be 7 terms if you go for l is = 3, that is the f orbital. So, l is = 0 is s; l is = 1 is p; l is = 2 is d; and l is = 3 is f. So, what does it represent? If we take only 1 term in the series of spherical harmonics, that is only 1 term in this equation is taken, then, we are assuming that the intensity is going to be isotropic.

That means, intensity does not depend on theta; does not depend on phi. It is independent of polar angle and azimuth angle. If we take 2 terms, first and second term, then for both the terms the azimuthal angle does not appear. So, the second term, the intensity variation that looks like p orbital is a function of theta only. It does not depend on azimuthal angle phi. So, idea here is that, if we can represent intensity as a series of these type of functions, then we should be able to capture the effect of intensity variation much more accurately than what we have already done in the case of two-flux approximation or the differential approximation.

(Refer Slide Time: 12:20)



So, the solution of Y l m, the term Y l m looks like this expression. And this can be easily derived by applying the method of separation of variables. It contains the value of l and m, where l is the degree of the spherical harmonics function and m is the order of this spherical harmonics function. So, l and m represents the degree and order of the spherical harmonics function.

In s, in chemistry, they have their own meaning as angular momentum and magnetic moment. But here, l is basically defines the degree of the spherical harmonics function and m represents the order of the spherical harmonics function. It has dependence on polar angle cos theta and it has dependence on the azimuthal angle psi. Here, just do not get confused with the notation. Basically, phi and psi are basically the same here.

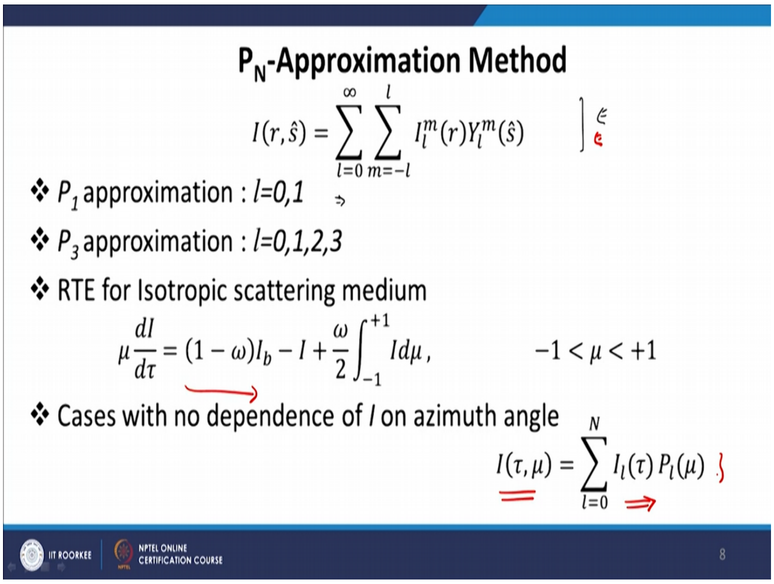
Sometimes, we use phi in the spherical harmonics function. So, spherical harmonics functions are used to represent the intensity, just like Fourier series. If we just retain the first term, then l is = 0, m is = 0 and we get Y l m, the angular dependence of intensity is 0. That means, intensity I is isotropic for the first term. And if we take the first term, then l is = 1 and m is = 0. From this equation, just substitute l is = 1 and m is = 0 and you will get Y l m; that is, angular dependence of intensity as P 1 0 cos theta or simply cos theta.

Now, what is P here? P is basically the Legendre polynomial. Okay. So, P 0 1; and cos theta is the argument. Okay. So, P l m is basically associative Legendre polynomial. So, this is the associative Legendre polynomial and it has a; again P l m is the degree and order of this polynomial. And the first value for l is = 1 and m is = 0 is simply; and x is the argument. Okay. So, this is the function, Legendre polynomial is a function in variable x.

And the P 1 0 value is basically given by simply x. So, P l m root cause, the argument cos theta is simply = cos theta. So, the second term depends on cos theta. And this variation, if we plot the variation of this term where cis theta, the shape will look like that of a p orbital, that is like this. And what it basically says is that the intensity variation is following a distribution of cos theta. And it does not depend on azimuthal angle.

And we see that there is very little intensity in the grazing angles. The maximum intensity is in the normal direction. And the intensity decreases. So, the intensity is maximum at theta is = 0. And for theta is = 90 degree, the intensity is 0. So, this is the 1 term. Okay. But there will be similar terms in y and z coordinates also. So, we will have 3 such terms. And all these terms are then added together in the Fourier series.

(Refer Slide Time: 16:16)



So, the P N approximation method, where N means how many terms in the Fourier series, actually are retained. So, we have written the intensity. That is, the intensity variation at any point in the space, will be function of radius r and direction s, which where direction depends on theta and the azimuthal angle psi. And which can be written in terms of a series or spherical harmonics.

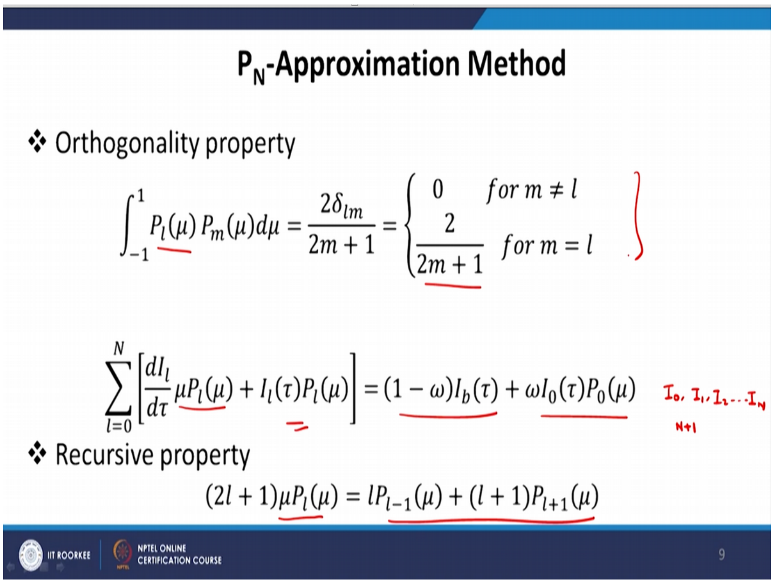
Now, if we retain the terms for l is = 0 and 1, the method is basically called the P 1 approximation method. And if we retain terms for l is = 0, 1, 2 and 3; that is, up to order 3, the method is called P 3 approximation method. So, depending on what is the largest value of l retained in the series, the method is called by the largest value used. So, if 1 is the largest value of l used, it is called P 1 approximation.

If 3 is the largest value used, it is called P 3 approximation. Now, let us substitute this intensity which is approximated as a series of spherical harmonics into the radiative transfer equation for isotropically scattering medium. We know the governing equation is given by this equation. So, this, in this equation, we will substitute for the expression of intensity in the form of series. That is, we will put this expression.

Here we will make 1 simplification. If we are using P 1 method, we have already observed that the intensity has no dependence on azimuthal angle. And if we take a special case of 1-dimensional medium, then also the intensity will not be a function of azimuthal angle. Okay. So, the dependence on m will basically drop down. And we can simply write down intensity which will be a function of now tau and mu only.

That is, polar angle and space coordinate tau. And we can write down in a simplified form the same series of spherical harmonics, as I l tau P l tau, where P l is basically the associated Legendre polynomial. Now, these polynomials we observe have a special property.

(Refer Slide Time: 18:52)



This observes the polynomial Legendre polynomial, they have the property of orthogonality. That means, for different values of l; that is, if we take 1 value P l mu and other value p m mu, l and m not same, then their integration P l times P m over the argument d mu will simply be = 0.

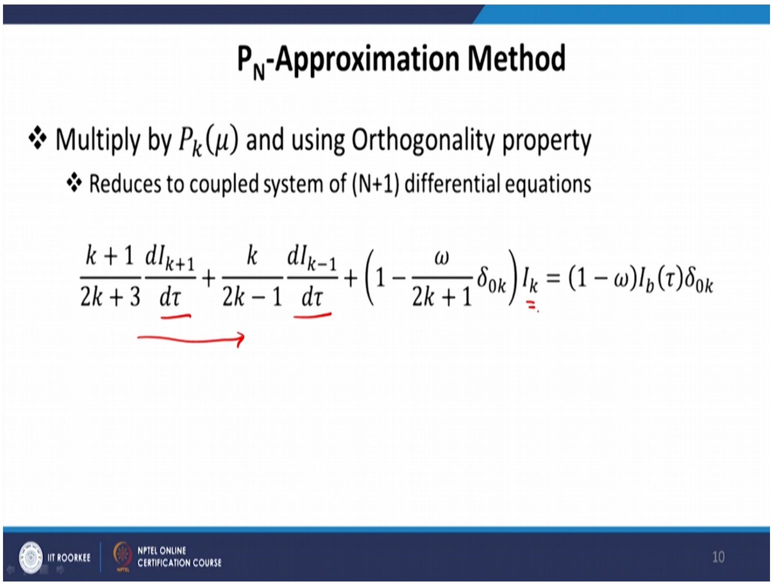
So, this is the orthogonality property of these special functions. And if the value of l and m are same, then we get the value of the integral as 2 over 2m + 1. So, we substitute the expression of intensity in the radiative transfer equation. And what we get is this equation. So, d; summation l is = 0 to N, d I l by d tau mu times P l mu + I l tau P l mu. And right-hand side, 1 – omega I b tau + omega I nought P nought mu.

Now, this is under the assumption that we have isotropically scattering medium. And intensity is function of tau and mu. So, we have reduced the integro differential equation into a simple differential equation. Now, we have N unknowns, N + 1 unknowns in this expression starting from P 0, P 1, P 2 and P n. So, unknowns here; not P 0 rather I 0. So, unknowns are I 0, I 1, I 2 and so on up to I N. So, there are N + 1 unknowns in this expression.

Now, we have to find out N + 1 equations to solve this, for this N + 1 unknowns. So, what we will do is, we will make use of the orthogonality property of the function. Now, before we apply the orthogonality property of the function, we have to get rid of this term, mu times P l mu. Because, if we do not get rid of this term, we cannot apply the orthogonality property. So, we will use the recursive property of the Legendre polynomial, associated Legendre polynomial.

That is, 2l + 1 mu times P l mu is l times P l – 1 mu + l + 1 P l + 1 mu. Now, orthogonality property can be applied on the terms on the right-hand side, but cannot be applied on the left-hand, term on the left-hand side. So, we will replace the term in the RTE with this, using the recursive property.

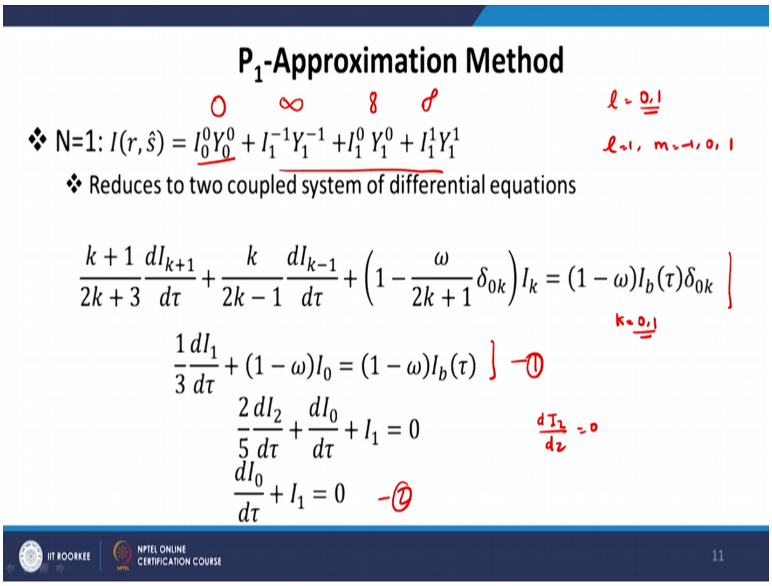
(Refer Slide Time: 21:33)



And then, we apply the orthogonality property. We multiply the entire radiative transfer equation by P k mu and integrate over all the solid angles. So, some of the terms will cancel out as per this relation, for terms where we have l not = k, the term will be 0. And some terms will be retained. The final equation for radiative transfer equation for P N method will be given by this equation.

So, we have 2 derivatives in this term. d I k + 1 d tau and d I k – 1 d tau. And then, we have I k also. So, this is the governing equation for P N method. And this is a differential equation. So, we have removed the integral from this equation and we have got a simplified differential equation. Of course, the number of unknowns have increased, but we can still solve it for any problem relatively easily.

(Refer Slide Time: 22:42)

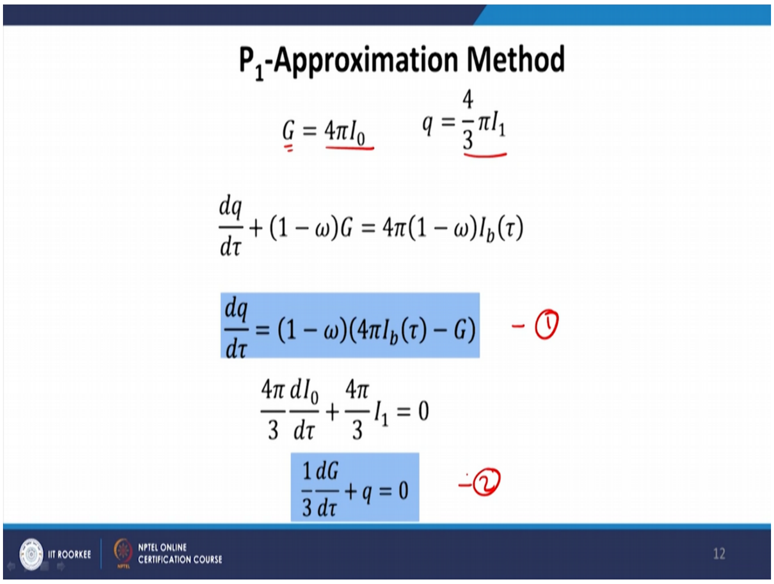


Now, we will take a special case. That is, P 1 approximation method which is mostly used in radiative heat transfer community. So, most of the commercial softwares like Ansys and Star CCM, they come up with this P 1 method. In P 1 method, the values of l retained will be 0 and 1. So, 2 terms will be retained. The first term I 0 Y 0. And the second term will have 3 components. For l is = 1, m is = – 1, 0 and 1. So, there will be 4 terms.

The first term will represented by intensity which is isotropic. The second term will have intensity like this. And similarly, in other direction. So, there will be 3 terms of 3 different intensity variation. Now, we applied the method, the P N solution. And for this is special case, we substitute k is = 0. So, in this case, we will have k is = 0 and 1. We will get 2 equations. The first equation, we substitute k is = 0.

So, we get one by third d I 1 by d tau + 1 – omega I 0 is = 1 – omega I b tau. So, this is the first equation. And then, the second equation we get 2 by 5 d I 2 by d tau + d I 0 d tau + I 1 is = 0. Now, since we have retained only 2 terms, I 0 and I 1, d I 2 by d tau is = 0. So, we get; so, this is equation 1 and this is the equation 2. d I 0 by d tau + I 1 is = 0. So, this is the 2 equations for the P 1 method. We also can easily establish that;

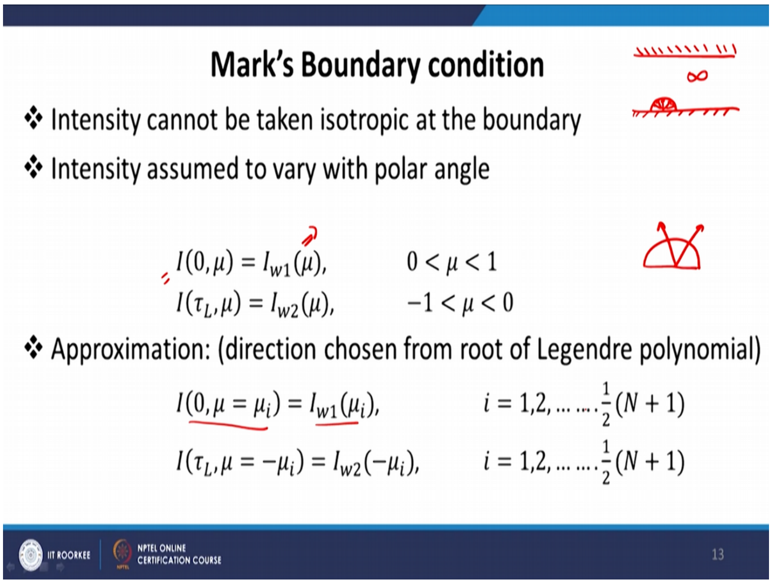
(Refer Slide Time: 24:42)



Incident radiation G is related to the first term I 0 4 pi I 0. And the heat flux is related to the second term 4 by 3 pi I 1. So, this equation, the first equation, we can replace as dq by d tau + 1 – omega G is = 4 pi 1 – omega I b tau. Or taking the G term on the right-hand side, we get dq by d tau is = 1 – omega 4 pi I b – G. So, this is the first equation. And we have found that for two-flux approximation also, we get, we got the similar equation.

The second equation, we multiply by 4 by pi 3. And then, we do that. We get this second equation as 1 by 3 dG by d tau + q is = 0. See, these are the 2 equations for the P 1 method. So, we get 2 equations, 2 differential equations for the P1 method. Now, we have derived this for 1-dimensional case. But the method is not restricted to 1-dimensional case. It can be extended easily to higher dimensions. The only thing is, we have to replace the derivatives appropriately.

(Refer Slide Time: 26:00)



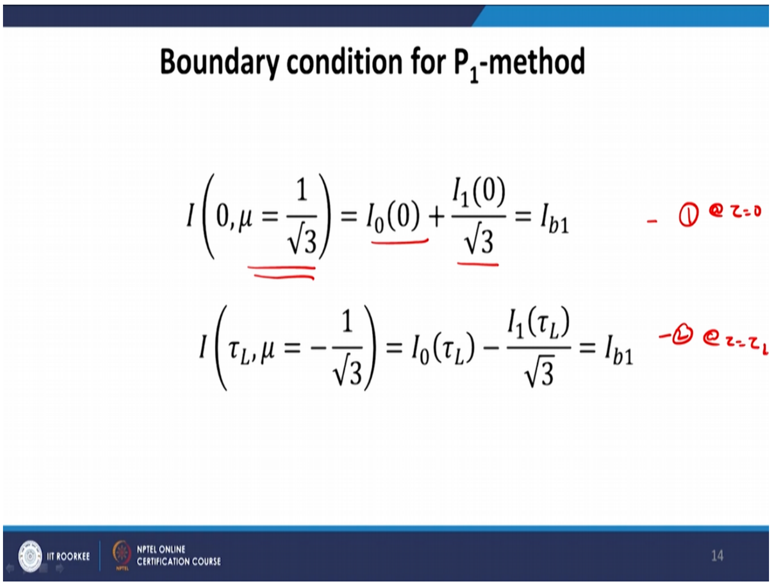
The boundary conditions for the P 1 method can be written as intensity at the bottom wall for this 1-dimensional medium. So, we have 1-dimensional medium. For this, we want to write down the boundary condition for the P 1 method. Okay. So, please remember in two-flux approximation method, we assumed that the intensity is isotropic at the wall. So, this was the assumption we made in the two-flux approximation method as well as differential approximation method.

But here, we cannot take intensity to be isotropic. Because we have written the intensity inside the medium with spherical harmonics. So, inside the medium, the intensity is varying as per the spherical harmonics. So, at the wall also, we should not take it as a isotropic. So, rather the intensity is defined as a function of mu. Okay. So, intensity at the bottom wall depends on mu and intensity at the top wall depends on mu. Okay.

Now, what kind of dependence will be on the mu, that is the polar angle? We will see. So, there are 2 ways of doing it; how the intensity varies with polar angle at the wall. The first is given by Mark. Mark's boundary conditions says that it is not possible to represent wall intensity in all the directions. So, what Mark proposed is that, along some discrete directions, let us say mu I; we have represented the intensity at the wall correctly.

So, in Mark's boundary condition, the intensity is represented by discrete values in discrete directions correctly. So, I 0 at a discrete direction mu is = mu I. The intensity is I w1 mu I. So, that means, if you have a actual intensity; only in discrete direction we have applied the intensity, not in the entire solid angle 2 pi. Now for n is = 1, we will have only 1 direction. For n is = 3, that is P 3 approximation, we will have 2 directions. Okay.

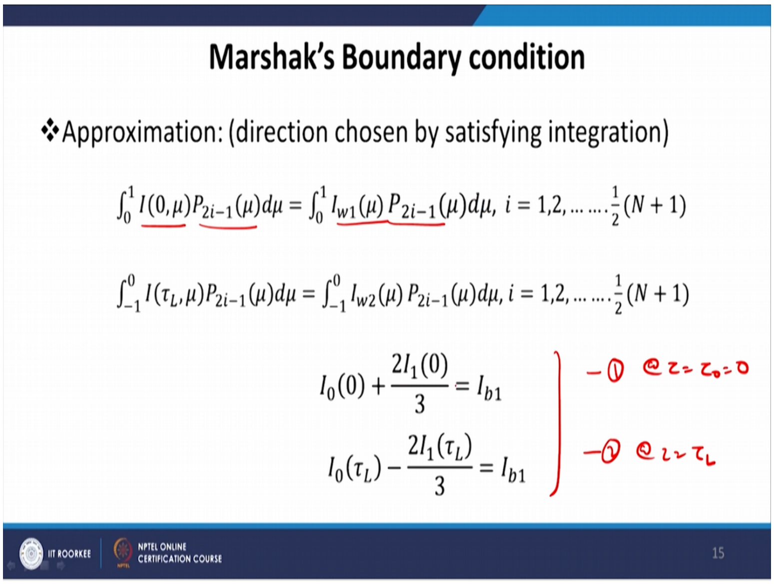
(Refer Slide Time: 28:12)



Now in Mark, these directions are basically defined by Legendre polynomial. And the value of mu comes out to be 1 by root 3. So, the boundary conditions now becomes I 0 0. That is, the bottom wall + I 1 0 by root 3 is = I b 1. So, this is the first boundary condition at tau is = 0. And the second boundary condition is in the direction mu is = – 1 by root 3. And I nought tau L – 1 tau L by root 3 is = I b 1.

This is the second boundary condition at tau is = tau L. So, boundaries are still the problem. We have tried to accurately represent the intensity using spherical harmonics inside the medium, but at the wall we have to satisfy certain boundary conditions. And these boundary conditions are approximate. And in this case, only 1 direction, in only 1 direction we have represented the intensity.

(Refer Slide Time: 29:13)



The other concept in boundary condition is given by Marshak. Now, what Marshak argued that, instead of representing boundary intensity in a discrete direction, as was given by Mark, we should satisfy the wall intensity at the boundary in an integral sense. So, what we, what he did is, basically he took the intensity at the wall multiplied by the Legendre polynomial and then integrated it. And he argued that, this intensity should be satisfied.

So, based on the value of intensity coming from the wall, that is, I w 1. And this is intensity coming from the medium or going into the medium. The 2 should be satisfied at the wall in an integral sense. So, when we do that for the top bottom and top wall, the boundary conditions becomes I 0 0 + 2 I 1 0 by 3 is = I b 1. This is the first boundary condition at tau is = tau 0 is = 0 rather.

And the second boundary condition I 0 tau L – 2 I 1 by 3 is = I b 1. Okay. So, there is slight difference in the boundary conditions. If you look, here we have I 1 0 root 3 I 1 tau L root 3. Here we have 2 I 1 by 3 and 2 – 2 I 1 by 3. So, this is how the boundary conditions between Mark and Marshak has changed. Again, I will highlight the difference between the 2 boundary conditions. One is based on discrete direction.

Mark says that intensity at the wall can be specified or satisfied in only discrete directions. And in P 1 case, it was only single direction. In P 2, P 3 case it was 2 directions. When Marshak said that the intensity at the wall should be satisfied in an integral sense, that means the total integral of the intensity should be satisfied. And based on that, we get this boundary condition. So, we will do some problems in the next lecture. And I will also introduce you the other aspects of this method, this spherical harmonics method. Thank you.