Radiative Heat Transfer
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Lecture – 18
Solution of Radiative Transfer Equation

Hello friends, In the last lecture, we discussed the derivation of radiative heat transfer equation. We discussed how the radiation is attenuated as it passes through a participating media. It is subjected to scattering, absorption. It is also subjected to augmentation by emission and In-scattering. The governing equation that we derive had 7 parameters including time.

And if we assume quasi-steady problem and ignore the time coordinate the total number of parameters will be = 6. So the equation is basically an integral differential equation and in itself is extremely difficult to solve. So in this lecture we will see how we can solve this problem. Normally we do lot of simplifications while solving this type of problems. There are 2 approaches.

In first approach we take the exact problem but we derive an approximate solution method and in the second approach we make a simplified approach and try to solve this simplified problem using some exact methods. So there are large number of method that have been developed to solve the radiative transfer equation. We have exact methods for very simplified geometry.

For example, plane parallel slab which is a one dimensional problem and similarly in cylindrical and spherical coordinates. One such problem we solved in the previous lecture to find out the intensity of radiation coming out of an isothermal sphere. In this lecture we will discuss the exact solution in plane parallel medium for problems having higher dimensionality and higher complexity. We have approximate solution method such as spherical harmonics P_1 method.

The discrete ordinate method the finite volume method and the discrete transfer method. So these are some of the major approximate methods that are used to solve radiative transfer equation in higher dimensions or complex geometry. Then based on the energy balance similar to what we discuss in the case of energy transfer between parallel plates. We have some methods which solve radiative transfer equation not exactly the radiative transfer equation.

But rather the radiation problem in participating media we have the zone method, the imaginary plane method, the Monte-Carlo method. These methods solve radiative transfer problem in participating media without actually solving the radiative transfer equations. So these are basically the energy balance method which directly or indirectly rely on the view factor or exchange areas. So we will discuss as we go along these types of methods also.

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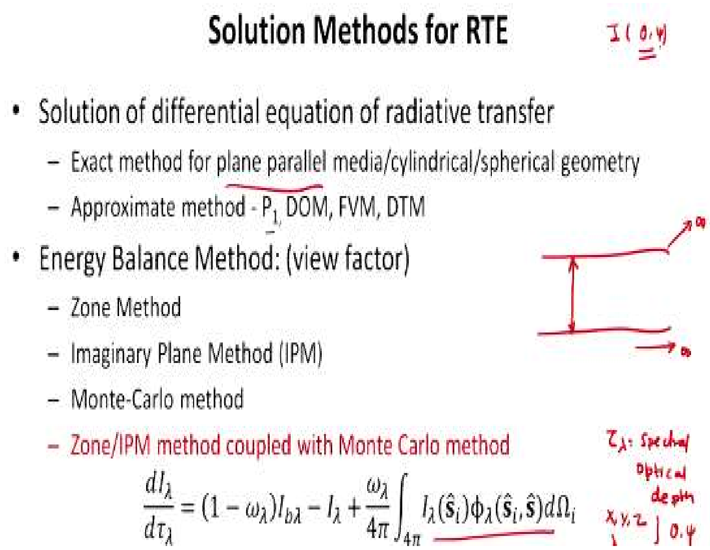

The focus of this presentation this lecture is the solution, the exact method solution in plane parallel media. So plane parallel media is basically a medium which is one dimensional. So we have one dimensional medium two parallel plates. The thickness of the plates in the plane of the board is infinite and the length of the plates or the width of the plates is also infinite. So there is only one finite dimension which is the separation distance between the two plates.

So for this plate for this configuration the solution is relatively simple. We can solve the equation of radiative transfer RTE. So recalling from the previous lecture the RTE is given by $\frac{dI_\lambda}{d\tau_\lambda}$. So this is in non-dimensional optical thickness so the denominator τ_λ is non-dimensional form. $\tau_\lambda$is optical spectral optical thickness. So the equation is $\frac{dI_\lambda}{d\tau_\lambda} = (1-\omega_\lambda)$, ω_λ was single scattering albedo, $I_{b\lambda}$ the spectral black body emissive power $-I_{b\lambda} + \frac{\omega_\lambda}{4\pi}\int_{4\pi} I_\lambda(\hat{\mathbf{s}}_i)\phi_\lambda(\hat{\mathbf{s}}_i,\hat{\mathbf{s}})d\Omega_i$. And then this integration this results from the scattering In-scattering. Soϕ_λ is the scattering phase function and in such form this equation is basically an integro-differential equation. On the left hand side have a differential and the right hand side has an integration. So we call it an integro-differential equation and it is very difficult to solve.

Now we have 6 coordinates or 6 parameters independent parameters here.

The three parameters are the spatial coordinates x, y and z and we see that as an engineer we are used to solving the partial differential equation in 3 dimensional space. So x, y, z is not an issue as far as solution of this equation is concerned. The one other parameter is the wavelength. Now we will solve this problem for a single wavelength and then we will integrate over all the wavelength.

So the calculation of this radiative transfer equation or the solution of this radiative transfer equation is not as such complicated by spectral variable λ. The λ makes the solution expensive that means we have to solve the solution the equation for large number of wavelengths and then find the sum. So the solution is expensive. It takes time to solve the problem for all the wavelength.

But as such for each wavelength the problem is independent and wavelength does not create as much complexity into the solution of the problem as is introduced by the two directional variables θ and ψ. The polar angle and azimuthal angle. So the major complication in the solution of this equation comes from the directional dependence of intensity. This directional dependence of intensity also appears in the integral part of this equation that is the right hand side the last term.

The scattering phase function introduces how the intensity will be redistributed after scattering. So that is basically a directional quantity and similarly this particular government equation is for a given path and we have to integrate this equation to find radiative heat flux and divergence of heat flux over all the solid angles. So the major complication in the solution of this equation comes from or from the dependence on θ and ψ that is the directional dependence of intensity.

So if somehow we can simplify the solution or the dependence of this intensity on theta and phi. If this relationship can be simplified by either assuming that I does not depend on ψ or I does not depend on ψ or I does not depend on any of the parameters, the problem can be simplified significantly. So we will see how this relationship can be simplified for some standard geometry. The problem that we are going to discuss is a plane parallel medium.

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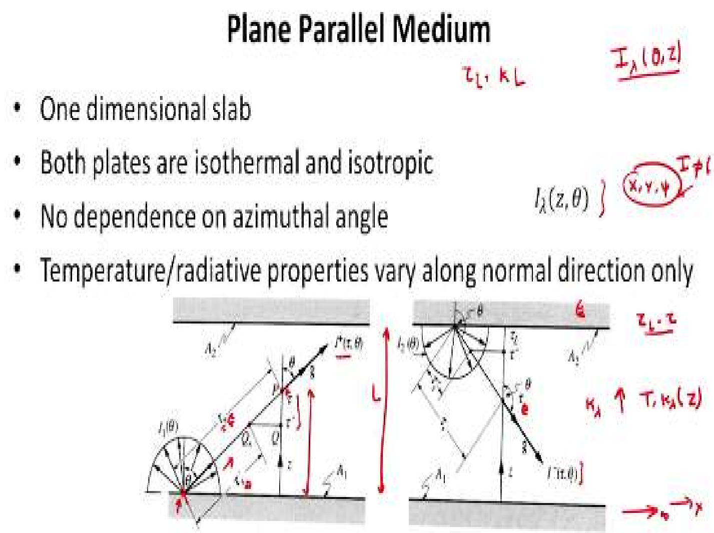

So plane parallel medium as I explained it has infinite depth into the board plane of the board and the width of the plate is also infinite. So this is called one dimensional slab. Now the two plates that bounds this medium are isothermal and isotropic that means anywhere on the plate the properties are same that means emittance of the surface is not a function of the x-coordinate it is isotropic.

So plates are isotropic the emittance also is isotropic it does not depend on the location, it does not depend on the location. It also does not depend on angle, so we have diffuse surface. So when we assume that then we can say in this configuration the intensity will be a function of θ and z only okay. So intensity is a function of z and θ only and it does not depend on any coordinate like x, y and ψ.

So intensity is not a function of these variables okay. So it depends on only two parameters θ and z and with this we can simplify the problem significantly. The space the medium between the two plates with a gas with absorption coefficient κ. So this space is filled with gas with absorption coefficient let us say the κ_λ. The spectral absorption coefficient is κ_λ and it is uniform, the temperature of the medium is also uniform.

But it may vary in the direction in the z-direction. So temperature as well as κ_λ that is absorption coefficient is a function of z only okay. So it is a plane parallel medium there is no dependence on x coordinates. So T and κ_λ the absorption coefficient in temperature are function of z only okay and under this consumption the intensity is going to be a function of z and θ only.

So now understanding the two images that are shown here. At any point P we are basically interested in finding the intensity. So we have to find out the intensity at any point P now point P is suspended in space. It will receive radiation over the entire solid angle 4π. So we divide solid angle 4π into two parts. One radiation coming from below that is from the bottom plate and the gas between point P.

And the bottom surface and second part of the intensity that reaches point P is basically from that medium and surface above point P that is the top plate and the medium between it. So we have divided the problem into two parts. In the first part the intensity originates from the point on the surface one okay towards point P. It undergoes emission so from here to here it undergoes emission from the medium.

It also has contribution from radiation emitted by the surface. So it undergoes all the four processes attenuation by absorption and scattering and augmentation by In-scattering and emission. Now we use the coordinate system. So starting from the point of origin of the ray we use coordinate τ_s and τ to represents the position of any point between the origin and point P.

So any point on this let us say Q has location or optical path length as τ_s' and the optical path length of point P has value of τ_s. So this is τ_s so this τ_s is basically the optical path from the point of origin of the ray to point P that means κs and τ_s' is any point on this ray let us say Qs $\kappa s'$. So that is the path length of point Qs.

And this is along the ray now along the normal direction we have the coordinate system tau and τ'. So τ is basically the optical distance of P from the plate the vertical optical distance from the plate. So this is the distance τ non-dimensional distance, optical distance τ and similarly the coordinates or the distance of point Q will be τ'. And same thing we can have for the intensity coming from upper side.

So we represent the coordinate in a similar fashion the only thing is now the coordinate of this point P will be $\tau_L - \tau$ where τ_L is the total distance of the plate. So this distance let us say total distance is L so τ_L is simply κL okay. So this is basically the notation. One last thing in the notation the intensity that originates from the bottom surface and moves in the upward direction is represented by I^+ it is a function of τ and θ.

And the intensity that comes from the top surface and moves in the downward direction is represented by I^- it is a function of τ and θ.

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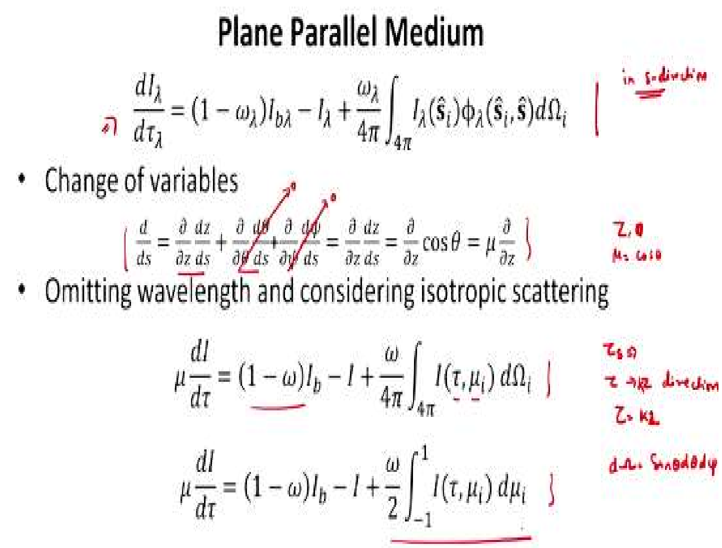

So we start we have to solve for these two intensities I^+ and I^- and then use these two intensity to find out the quantities of our interest that is the radiative heat flux and divergence of heat flux. So these two quantities are of our interest. So we start with this radiative transfer equation in non-dimensional optical coordinate because this is given in optical path along the radiation travel.

We have to now change the variables in our coordinate system that is z and θ okay. So we apply the chain rule so d/ds= partial derivative with respect to z dz/ds then partial derivative with respect to theta $\frac{\partial\theta}{ds}$+ partial derivative with respect to the azimuthal angle ψ, $d\psi/ds$. Now we apply the chain rule. Now there is no dependence on ψ so the last term in this will be=0.

Now $\frac{d\theta}{ds}$, $\frac{\partial\theta}{ds}$ is basically the variation of θ in the s direction and if you look at it along this direction the θ is not going to change so θ is constant. So $\frac{d\theta}{ds}$ is also=0 okay so this quantity is also 0. So we are left with this will become $\frac{\partial}{\partial z}\frac{dz}{ds}$. Now $\frac{dz}{ds}$ is nothing but cos theta. So $\frac{dz}{ds}$ from this you can see this is *dz* and this is *ds* and this is θ. So $\frac{dz}{ds}$ is simply cos theta we will represent $\cos\theta$ by μ just for notation purpose we have introduced a variable μ which is=$\cos\theta$. Now our *d/ds* has become= μ times *d/dz* because other partial derivatives are not there. So we have changed the variables. Now we will simplify our radiative transfer equation. So from here from this equation now we have changed the variable from τ_s okay this was in s direction.

So this radiative transfer equation was in s direction now we have changed the coordinate so this will be in τ or z direction. So we have changed the radiative transfer equation in terms of coordinates where now d tau is simply κz. So τ is nothing but κz okay. So this is the vertical optical path length okay and the right hand side $(1-\omega)I_b - I + \frac{\omega}{4\pi}$.

Now I have also changed the intensity in terms of writing of the path length S_i I have written the intensity which is now a function of z and θ. So z dependence comes in τ and θ dependence comes in μ. I okay and we have to integrate our all the solid angles dΩ_i. One thing you should note is that I have drooped the subscript λ and the reason is just for simplification purpose.

But the equation is basically valid for each wavelength just for convenience we have dropped λ, but it is understood that the equation is valid for each and every wavelength. Now $\mu\frac{dI}{d\lambda} = (1-\omega)I_b - I$ and removing the azimuthal angle or integrating over the azimuthal angle. So $d\Omega$ we know is basically $\sin\theta\, d\theta d\psi$. So we have already integrated our $d\psi$ there is no dependence of intensity on ψ so we can take it out.

And the equation is simplified to the right hand side simplifies to $\frac{\omega}{2}\int_{-1}^{1} I(\tau,\mu_i)d\mu_i$. So we have to integrate our all the θ directions. So this was the simplification of radiative transfer equation.

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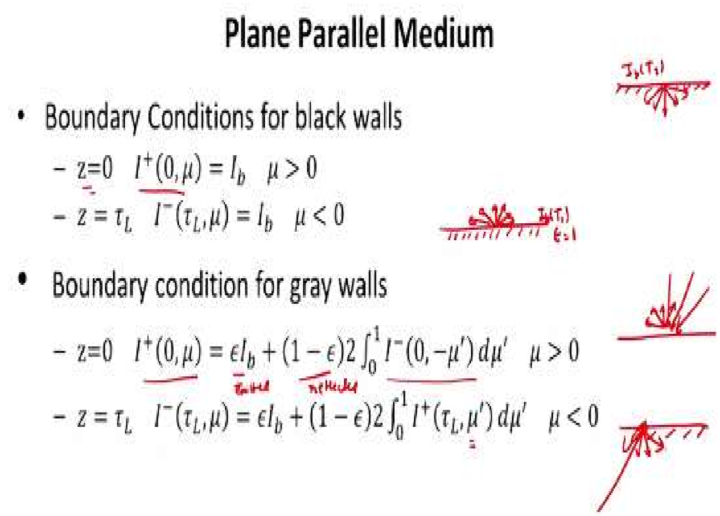

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Before we solve this equation formerly. Let us have a look at the boundary condition at the bottom boundary that is z=0 so this is my bottom boundary. We assume let us say it is black that means ϵ=1 then the intensity is going to be isotropic uniform in all directions. So this is going to be intensity so our I^+ that is intensity in vertical direction that is moving up at τ=0 and $\theta = \mu$ is simply I_b there is no dependence on theta as the wall so intensity is isotropic.

And similarly at the top wall also so this is I_b at local temperature let us call T_1 and this is I_b at local temperature T_2 and this is the plate the top plate and the intensity is going to be isotropic that is uniform in all direction. Now if our plates are not black then the intensity or the boundary condition will not be simply the algebraic equation I^+=I_b or I^-=Ib then it will be an integral equation and we have already discussed this in length heat transfer from surface is governed by Fredholm kind of equation kind of equation if the plates are gray.

So same thing we will have so I^+ now at the surface the bottom surface in any direction mu is going to be a function of μ will be black body emissive power epsilon I_b and the refracted power. So this is the emitted now it has two components emitted part and reflected part and it is in this reflected part we have the integration over all the solid angles. We have taken $-\mu$ why because the reflected radiation basically includes radiation coming from outside and then reflecting back.

So we are taking back so we are taking μ in the negative direction that is downward moving μ downward moving direction for the reflection part and same thing we do on the top surface. On the top surface this is the top surface radiation moves in the upward direction. So we take positive value here I^+ and then it is reflected back okay. So we see that for gray surfaces for a medium bounded by gray surfaces the result is going to be little more complicated than the case where the medium is bounded by black walls.

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Solution of Plane Parallel Medium

$$\mu\frac{dI}{d\tau} = (1-\omega)I_b - I + \frac{\omega}{2}\int_{-1}^{1} I(\tau,\mu_i)\,d\mu_i$$

$$\mu\frac{dI}{d\tau} = -I + S(\tau)$$

isotropic scattering $\Phi(\mu_i,\mu) = 1.0$

$S(\tau,\mu)$

- Integrating Factor approach

Integrating factor = $e^{\tau/\mu}$

$$\frac{d\left(e^{\tau/\mu}I\right)}{d\tau} = \frac{1}{\mu}e^{\tau/\mu}S(\tau)$$

$$\frac{d\left(e^{\tau/\mu}I^{+}(\tau,\mu)\right)}{d\tau} = \frac{1}{\mu}e^{\tau/\mu}S(\tau)$$

I^{+}

I^{-}

Now we will solve so this is our equation. Let us call the first and third term combined as source term to simplify our notations. We just write I $(1-\omega)I_b$ which is augmentation by emission and the last term which is augmentation by scattering. So this two augmentation term we put as $S(\tau)$ the scattering term. Now I_b is not a function of μ and this scattering term is also not a function of outgoing direction.

So we have basically S as a function of τ only and this is only for isotropic scattering okay. So this is valid only for isotropic scattering that is where scattering phase function which is function of μ_i and μ is simply=1 that means there is no dependence on outgoing direction. Otherwise if it is not isotropic scattering then the source function will be a function of τ and μ.

But because the scattering is isotropic the source function is just a function of τ and it does not depend on μ. So the equation in simplified notation is $\mu\frac{dI}{d\tau} = -I + S(\tau)$ where $S(\tau)$ is the source term that includes emission augmentation and scattering augmentation. Now to solve this equation this differential equation we will use the approach by multiplying by the integrating factor.

So the integrating factor we have taken is the integrating factor = $e^{\tau/\mu}$. So with this integrating factor we multiply and we see that the differential equation is transform to the form here $\frac{d(e^{\tau/\mu}I)}{d\tau} = \frac{1}{\mu}e^{\tau/\mu}S(\tau)$. And we can split this intensity as I said in 2 parts I^+ and I^-. I^+ is intensity moving upward and I- is intensity moving downward. So we just are interested in finding the solution of I^+ and I^- separately. So I have I as I^+ as a function of τ and μ.

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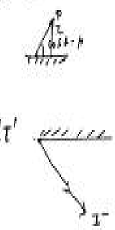
Solution of Plane Parallel Medium

$$e^{\tau/\mu} I^+(\tau,\mu) - e^{0/\mu} I^+(0,\mu) = \frac{1}{\mu}\int_0^{\tau} e^{\tau'/\mu} S(\tau')d\tau'$$

$$I^+(\tau,\mu) = I^+(0,\mu)e^{-\tau/\mu} - \frac{1}{\mu}\int_0^{\tau} e^{-(\tau-\tau')/\mu} S(\tau')d\tau'$$

- Similarly

$$I^-(\tau,\mu) = I^-(\tau_L,\mu)e^{-(\tau_L-\tau)/\mu} + \frac{1}{\mu}\int_{\tau}^{\tau_L} e^{-(\tau-\tau')/\mu} S(\tau')d\tau'$$



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And when we integrate this we get this solution. So this is the solution of the differential equation that we multiply it by the integrating factor. So $e^{\tau/\mu} I^+(\tau,\mu) - e^{0/\mu}$ where τ=0 has been substituted that means at the bottom wall. So τ =0 at bottom wall and $I^+(0,\mu)$ is the boundary conditions. So we are substituting the boundary conditions here= $\frac{1}{\mu}\int_0^{\tau} e^{\tau'/\mu} S(\tau')d\tau'$ So we have to take the integration of source term in all directions.

So $e^{\tau'/\mu} S(\tau')d\tau'$. So all the directions we have to basically include and we have to integrate from 0 to τ. And then if we simplify it we divide by $e^{\tau/\mu}$ both left and right hand side we get $I^+(\tau,\mu) = I^+(0,\mu)$ that is the boundary condition $e^{\tau/\mu} - \frac{1}{\mu}\int_0^{\tau} e^{-(\tau-\tau')/\mu} S(\tau')d\tau'$. So this is the solution of equation of radiative transfer for any wave starting from bottom surface and moving in upward direction at an angle θ where $\cos\theta = \mu$ and this distance is τ.

So at this point P the intensity I^+ moving in the upward direction at an angle θ the distance of this point P from the plate is τ the intensity is given by this result okay. So similarly on point P any radiation travelling from the upward plate the top plate at an angle θ it will be moving in downward direction the intensity $I^-(\tau,\mu)$ will be = $I^-(\tau_L,\mu)$ that is the boundary condition at this point $e^{-(\tau_L-\tau)/\mu} + \frac{1}{\mu}\int_{\tau}^{\tau_L} e^{-(\tau-\tau')/\mu} S(\tau')d\tau'$ and this integrated over the source term.

So this is the equation of intensity in the two direction at point P in the upward direction and in the downward direction. Now we have to find out the expression for radiative heat flux. So once we know how the intensity is distributed finding the expression for heat flux is very simple.

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Solution of Plane Parallel Medium

$q = \int_{4\pi} I \hat{n}.\hat{s}\, d\Omega$

$$q(\tau) = \int_0^{2\pi}\int_0^{\pi} I(\tau,\theta)\cos\theta \sin\theta\, d\theta d\psi = 2\pi \int_{-1}^{+1} I(\tau,\mu)\mu d\mu$$

$$= 2\pi\left\{\int_0^1 I^+(0,\mu)e^{-\tau/\mu}\mu d\mu - \int_0^1 I^-(\tau_L,\mu)e^{-(\tau_L-\tau)/\mu}\mu d\mu\right\}$$

$$+2\pi\left\{\int_0^1\left[\int_0^{\tau} S(\tau')e^{-(\tau-\tau')/\mu}d\tau' - \int_{\tau}^{\tau_L} S(\tau')e^{-(\tau'-\tau)/\mu}d\tau'\right]d\mu\right\}$$

$$\nabla\cdot q = \kappa(4\pi I_b - G)$$

$G = \int_{4\pi} I\, d\Omega$

$$G(\tau)=2\pi \int_{-1}^{+1} I(\tau,\mu)d\mu = 2\pi\left\{\int_0^1 I^+(0,\mu)e^{-\tau/\mu}d\mu + \int_0^1 I^-(\tau_L,\mu)e^{-(\tau_L-\tau)/\mu}d\mu\right\}$$

$$+2\pi\left\{\int_0^1\left[\int_0^{\tau} S(\tau')e^{-(\tau-\tau')/\mu}d\tau' + \int_{\tau}^{\tau_L} S(\tau')e^{-(\tau'-\tau)/\mu}d\tau'\right]\frac{d\mu}{\mu}\right\}$$

So we have to integrate over all the solid angles and multiply by $\cos\theta$. So we remember that q is basically intensity dotted with normal vector and integrated over the entire solid angle 4π. So that is the expression for heat flux so that is what we have done for n.s we have multiplied by $\cos\theta$ and then solid angle $d\Omega$ is $\sin\theta\, d\theta d\psi$. If we write in terms of μ that is $\cos\theta = \mu$ it will become simply $2\pi\int_{-1}^{+1} I(\tau,\mu)\mu d\mu$ okay. Substitute the value now we can split this integral into $I^+$and I^-.

So I^+will be the value from 0 to 1 and I^- value will be -1 to 0 okay. So after substituting the expression for $I^+$and I^- in this equation the final solution for heat flux is basically 2π then integration over the I^+similarly integration over I^- and the two source terms. These are the two source terms one term for each $I^+$and I^-. So the solution for this plane parallel plate configuration has been found analytically in this form.

Although there are still some integration that are appearing here especially the integral in the source term $e^{-(\tau-\tau')}$. So these integrations have to be found before we actually have a solution. Similarly, the incident radiation $G(\tau)$ can be calculated and we know that we have to find out the value of G before we can solve for the divergence of heat flux $\nabla.q$ because the expression for $\nabla.q$ is simply= $\kappa(4\pi I_b - G)$ and the expression of G can be derived in a similar fashion.

We define G as 4π integrated over solid angles simply I times dΩ okay. So this is the expression for the incident radiation that is intensity integrated over all solid angles and we get the similar result except we have μ in the denominator here and instead of – sign we have a + sign here. So the expression of the q and G they are pretty much very much similar to each other. Now So as I said the solution is complete as of now the only unknown now left in this solution are basically the integration that are appearing in these terms.

Especially the integration of this exponential term that we have to find out. So every time we solve this problem it is very difficult to calculate these integrals. So researchers, mathematicians have evaluated these integral. In fact, this integral are very popular in radiative heat transfer we call it exponential integral and they are very popular. They have been pre calculated and tabulated in books.

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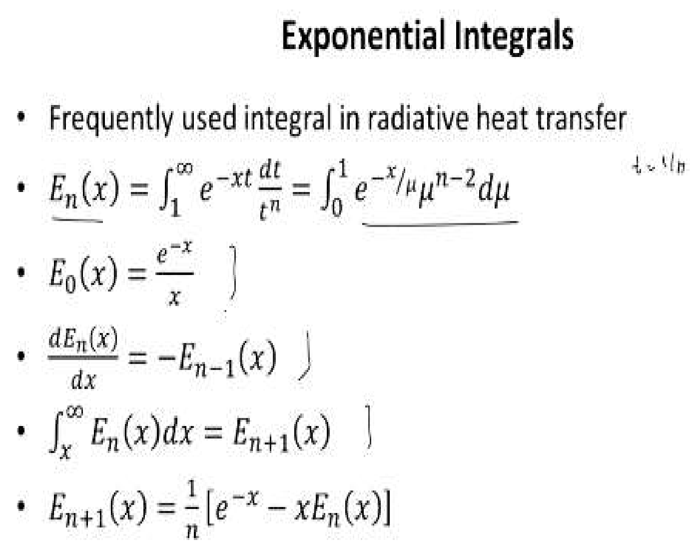

So we define exponential integral $E_n(x)$ as= $\int_1^\infty e^{-xt}\frac{dt}{t^n}$ where t is a dummy variable here in the integration. So this is called exponential integral of order n in variable x. So $E_n(x)$ is exponential integral of order n in variable x. We can use another dummy variable μ. So by just substituting t as=$1/\mu$ we get this definition of exponential integral which is very similar to what we have found in our results. So we see that 0 to 1 exponential term.

And we can relate this exponential integral to the solution of radiative transfer equation. Now the advantage of tabulating this integration is that we can very easily solve the problems analytically. Exponential integral is very powerful it has lot of properties that can be exploited. For example, the derivative property derivative of this exponential integral is -exponential integral one order less than the exponential integral $E_n(x)$ that $E_{n-1}(x)$.

Similarly, we can integrate this exponential integral we get an order higher exponential integral the first 0 order exponential integral is $\frac{e^{-x}}{x}$ then there is a property $E_{n+1}(x) = \frac{1}{n}[e^{-x} - xE_n(x)]$. So many times when we solve radiative transfer equation these type of integration will appear in the solution and to simplify this some of these properties can be called and we can find out the solution.

As I said these integrals are tabulated for different values of x. I have given a truncated table this table is taken from radiative heat transfer M. F. Modest. So up to fourth order exponential integral have been given. So E_1 is exponential integral of order one E_2 second order third order and fourth order and so on. So this table is available in the text book at the back of the textbook and can be used to solve the radiative transfer equation.

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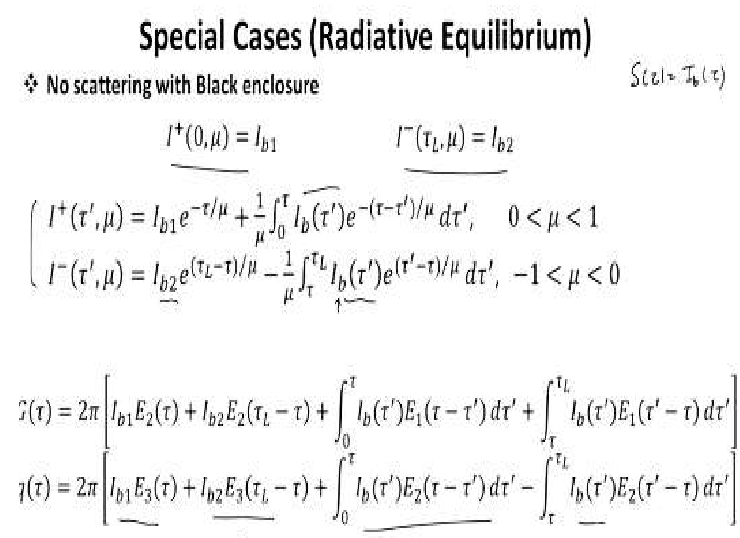

Now there are some special cases specially there are two types of special cases in this problem. One is radiative equilibrium problem and one is specified temperature. In many applications the temperature is known and you can solve radiation problem straight forward because you know the temperature, but in many applications the temperature is not known. For example, if you are trying to solve for a furnace a boiler where we do not know how much temperature exist at different location in the boiler.

So that problem can be solved using radiative equilibrium problem assuming conduction and convection are negligible and the temperature in the furnace is mainly governed by radiative energy balance. So the analysis basically remains the same in this simplified case special case we assume that there is no scattering that means the intensity the source function is simply I_b. So $S(\tau)$ is simply I_b (τ).

If we have no scattering the source function depends on only blackbody emissive power. The walls are black so I^+ is I_{b1} $I^-$is I_{b2}. When we assume that then I^+ and I^- solution which we have already calculated earlier can be simplified by substituting for I_{b1} and I_{b2} and I_b in terms of source term. So let me just show you the I^+ expression we derived earlier. So this was the I^+ expression here $I^+(0, \mu)$ was a boundary conditions.

So we have substituted the value of intensity at the wall at the boundary in terms of black body intensity I_{b1} and I_{b2} to simplified the solution and same thing we have done for the source term where we have substituted $S(\tau) = I_b$ (τ) and our expression for incident radiation and radiative heat flux have also simplified. This is the final solution where we have now replaced the integration with exponential integrals.

So the heat flux is simply=I_{b1} third order exponential integral at τ, I_{b2} third order exponential integral $(\tau_L - \tau)$ and then integration from 0 to τ of black body intensity multiplied by second order exponential integral - black body intensity and second order exponential integrals and this problem now is very simple. Of course if you do not know the quantities like temperature then it becomes difficult to solve by hand but you can easily program them.

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Radiative Equilibrium in Non scattering Black enclosure

$$\nabla.q = \kappa(4\pi I_b - G) = 0$$

$$G = 4\pi I_b = 4\sigma T^4$$

$$4\sigma T^4 = 2\left[\sigma T_1^4 E_2(\tau) + \sigma T_2^4 E_2(\tau_L - \tau) + \int_0^{\tau_L} T^4(\tau') E_1|\tau' - \tau|\, d\tau'\right]$$

- Solved using numerical techniques
- Closed form solution not posssible

Now radiative equilibrium means $\nabla. q$=0 or $\kappa(4\pi I_b - G) = 0$ and this gives you $G = 4\sigma T^4$. So if we have solved for *G* using this expression if you have solved for *G* using this expression then you can calculate the temperature unknown temperature *T*. So we will substitute $G = 4\sigma T^4$ and then I_{b1} is σT_1^4, I_{b2} is σT_2^4 and then this is the source term. Again this is an integral differential equation and you have to apply sorry this is an integral equation a Fredholm equation.

And you have apply the algorithms that we have discussed earlier to solve this equation. So if the temperature field is not known radiative equilibrium exist then the solution of the radiative transfer equation is basically an integral equation or the Fredholm equation which you have to solve using appropriate numerical procedure.

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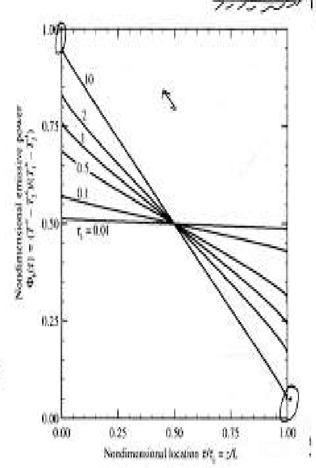
Non scattering Black enclosure

$$\Phi_b(\tau) = \frac{T^4(\tau) - T_2^4(\tau)}{T_1^4(\tau) - T_2^4(\tau)}$$

$$\Psi_b(\tau) = \frac{q/\sigma}{T_1^4(\tau) - T_2^4(\tau)}$$

- Temperature discontinuity at the wall
- Magnitude of discontinuity decrease with optical thickness



On the other hand, so here is a solution. So I have not solved this I have taken the solution from the book radiative heat transfer by M.F. Modest. We have written the non-dimensional temperature or emissive power $\frac{T^4-T_2^4}{T_1^4-T_2^4}$. This is non-dimensional emissive power at any location τ and this is non dimensional heat flux at any location tau and we see that almost the variation is linear.

So we see that the temperature variation between the plate. So we have a top plate and we have a bottom plate. This is at temperature T_2 this is at temperature T_1. So we are finding the temperature at some point in the middle or at location τ this is τ. So the emissive power or the fourth power temperature variation is linear okay. If we have this linear variation in emissive power, it is easy to solve for temperature.

We have to just find out power 1/4, but the emissive power is linearly varying and we see that there is a jump in temperature at the wall. Here non dimensional temperature at the wall is 1 here the non-dimensional temperature at the wall is 0 and there is a jump in that non dimensional emissive power. So there is a jump and this jump basically decreases when we increase the value of optical thickness τ_L.

So if you have more absorption that means kappa is large then the jump at the wall the temperature or the emissive power jump at the wall is going to decrease, but for optically thin medium the jump is going to be significantly large. So the wall is going to emit much more than the medium. So jump means there is more emissive power from the wall than the medium.

And this difference is going to decrease as we move towards optically thickness. In fact, if κ tends to infinity that is optically extreme case very large optical depth then the two will be basically same and the intensity or the heat flux q will basically tends to σT^4. There will be very little difference in the temperature in that case and the intensity or the flux leaving the enclosure will be = black body emissive power for extreme case of optical thickness.

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Non scattering Gray enclosure

$$G(\tau) = 2\pi\left[J_1E_2(\tau) + J_2E_2(\tau_L - \tau) + \int_0^{\tau} I_b(\tau')E_1(\tau - \tau')\,d\tau' + \int_{\tau}^{\tau_L} I_b(\tau')E_1(\tau' - \tau)\,d\tau'\right]$$

$$q(\tau) = 2\pi\left[J_1E_3(\tau) + J_2E_3(\tau_L - \tau) + \int_0^{\tau} I_b(\tau')E_2(\tau - \tau')\,d\tau' + \int_{\tau}^{\tau_L} I_b(\tau')E_2(\tau' - \tau)\,d\tau'\right]$$

$$\Phi(\tau) = \frac{T^4(\tau) - T_2^4(\tau)}{T_1^4(\tau) - T_2^4(\tau)} = \frac{\Phi_b(\tau) + \left(\frac{1}{\epsilon_2} - 1\right)\Psi_b(\tau)}{1 + \Psi_b(\tau)\left(\frac{1}{\epsilon_2} + \frac{1}{\epsilon_2} - 2\right)}$$

$$\Psi(\tau) = \frac{q/\sigma}{T_1^4(\tau) - T_2^4(\tau)} = \frac{\Psi_b(\tau)}{1 + \Psi_b(\tau)\left(\frac{1}{\epsilon_2} + \frac{1}{\epsilon_2} - 2\right)}$$

Same thing we can do for non-scattering gray enclosure. For non-scattering enclosure so for the black enclosure non scattering we have intensity I_{b1} and I_{b2}. If we have gray enclosure the I_{b1} and I_{b2} will be just replaced by the radiocity J_1, J_2 rest of the things will remain same and we can write down the emissive power non dimensional emissive power and non-dimensional heat flux. In terms of black surface emissive power Φ_b so Φ_b is emissive power or non-dimensional emissive power.

If the surface is black and Ψ_b is non-dimensional heat flux if the surfaces were black. So in these two variables we can find out the non-dimensional emissive power and non-dimensional heat flux.

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Special Cases (Known Temperature Distribution)

- No scattering with Gray enclosure

$$G(\tau) = 2\pi\left[J_1E_2(\tau) + J_2E_2(\tau_L - \tau) + \int_0^{\tau} I_b(\tau')E_1(\tau - \tau')\,d\tau' + \int_{\tau}^{\tau_L} I_b(\tau')E_1(\tau' - \tau)\,d\tau'\right]$$

$$q(\tau) = 2\pi\left[J_1E_3(\tau) + J_2E_3(\tau_L - \tau) + \int_0^{\tau} I_b(\tau')E_2(\tau - \tau')\,d\tau' - \int_{\tau}^{\tau_L} I_b(\tau')E_2(\tau' - \tau)\,d\tau'\right]$$

- Heat flux and divergence of radiative heat flux varies with space
- All relations equally valid for non-gray enclosure as well

If the temperature is known so that was the case when the temperature was not known and we assume radiative equilibrium like the example. So the solution becomes difficult due to the involvement of integral equation, but if the temperature is known then directly we can solve for heat flux using this equation and we can solve for incident radiation G directly using this integration and the solution is relatively simple.

We can directly solve for it and the good thing in many problems the solution in iterative even in furnace problem what we do is we assume that the temperature is known find out the heat flux and divergence of heat flux. Satisfy the overall energy balance is the overall energy balance is not satisfied we increase or decrease the temperature. So in practice or in numerical procedures the solutions is always iterative.

So we do not have to assume radiative equilibrium so that solution we do not have to do. In practice what we can do is we assume that the temperature is known. Although the temperature may be a guess temperature it may not be actual temperature. You satisfy the overall energy balance equation if the energy balance equation is not satisfied you correct your temperature and then again solve the radiative problem using the equations given on this slide that is you do a forward calculation with temperature know all this terms.

You can solve for q and G and then you can find out $\nabla . q$, satisfy the overall energy equation and if not satisfied correct the temperature. So thank you. In this lecture we have discussed the solution of plane parallel medium. In the next lecture I will take one example of this type of problem the plane parallel slab and try to solve for a simple case.

We will also develop solution for one dimensional cylindrical media the exact method for one dimensional cylindrical media by simplifying the radiative transfer equation in cylindrical coordinates. Thank you.