Modeling & Simulation of Discrete Event Systems
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Lecture - 13
Analysis of a Single Server Queueing System

Welcome to the lecture on analysis of a single server queueing system. So, in this class we will discuss about the theoretical aspects, how we try to find the derivation for the steady state probability having n customers at any time t. Or a steady state probability of you know other parameters. So, or average number of customers in the q or so. So in this lecture we will try to see how it is derived in case of the poisson queues where you have poisson arrival and service and then you have one server. And you have the first in first out or general discipline kind of discipline.

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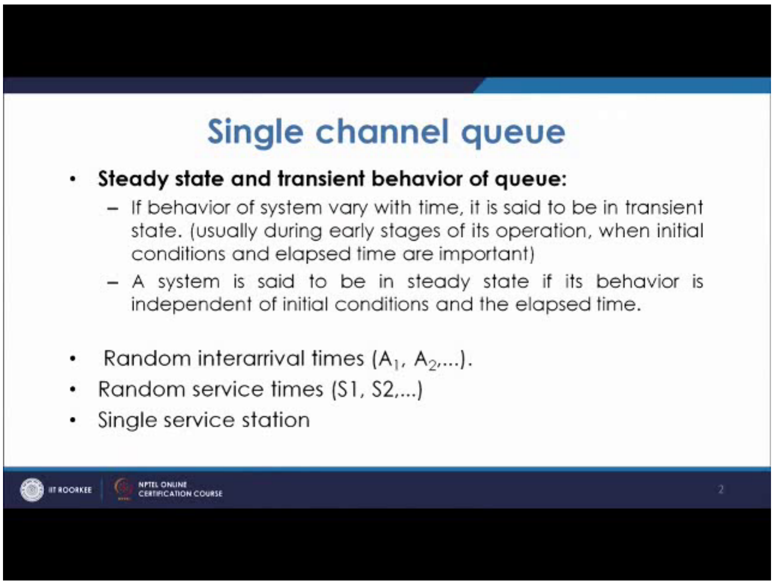

So, as before proceeding that we must be able to understand that you have the steady state value as well as the transient value. So, that that way the steady state and transient behavior of the queue is characterized or it is specified. So, as we know steady state means at that point of time things do not change with time and transient means things change. So, the if the behavior of the system will vary with time. Then it is said to be either transient state that is the normal meaning. So, when the queue starts at that time thinks or the queue parameters or the measures of performances, they are basically dependent upon the initial conditions or the time elapsed.

So, that is why that state is the transient state. Whereas, when it runs for a large number of time then it reaches in the steady state. So, that time the behavior will be independent of the initial conditions and the elapsed time. So, this way the steady state and transient behavior of queue is you know recognized. We also discussed that the queueing system has one of the input parameter that is inter arrival time. So, that that is random and that follow certain distribution. So, normally it is the poisson distribution or poisson arrival process we call it. Similarly, you have you know service. So, that is also random these are basically identically distributed and independent.

So, that is why they are known as I I d random variables. And in this case we have a single service station. So, we will discuss about this case. Now how to find these you know queueing parameters, mainly we are interested to find the steady state you know value in that case the probability of having n customers in the system. So, if it is at any time t will be p and t, and if it is the probability I mean you know steady state value it is P n. So, that is, you know having n customers at you know at that steady state value.

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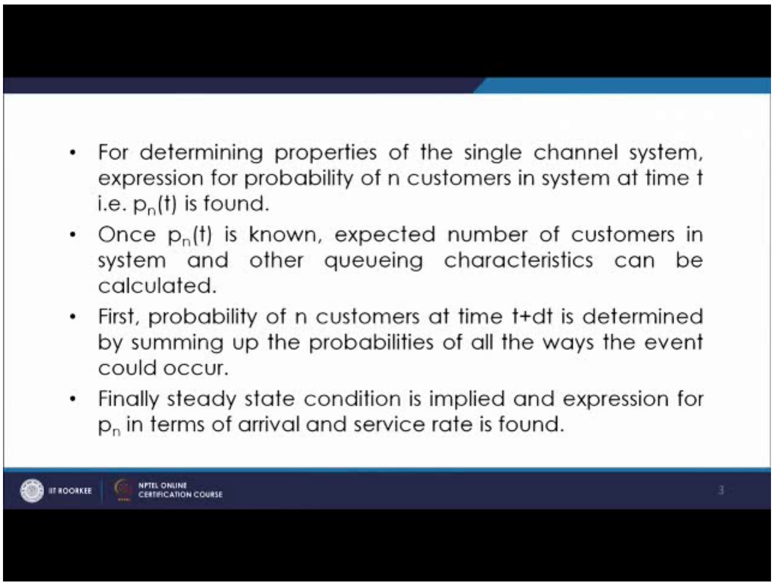

So, for determining that basically what we do is we normally express the probability of n customers in system at time t that is a special for P n t. And once we know the P n t then from there you can have other queueing characteristics.

So, what we do is first of all, we try to find the probability of t you know n customers at t plus d t time. So, we know this you know queueing in the queueing different terms are used like lambda is the arrival rate mu is the service rate, d t is the time that is elapse between t n t plus d t. So, similarly you have other terms. And then we go to the steady state condition where basically the derivative is equated to 0. So, d by d t of p n t or whatever it be that is equated to 0. So, from there we get certain expressions and we get these expressions in terms of this lambda and mu normally. So, we will have the expression for them.

Before that let us have the introduction to certain terminologies which we normally discuss. So, the terminologies are what we do normally the terminologies. We deal with terminologies like lambda.

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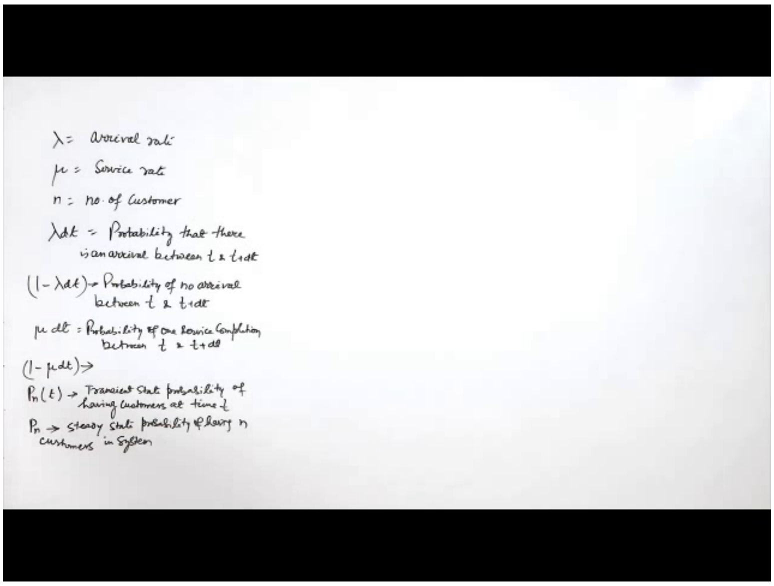

That is service rate, your arrival rate sorry this is arrival rate. So, that is number of arrivals per unit time. Similarly, you have another term that is mu that is service rate. So, this lambda and mu are the 2 you know parameters. Then n will be number of customers. So, in the system at time t. So, that way we are assuming that at time t you have n customers in the system.

So, lambda d t it will be. So, lambda d t basically will be probability that there is one arrival between time t to t plus d t. So, probability that there is an arrival between I mean in the interval d t. So, we are analyzes the situation at t and t plus d t. So, certainly the interval is d t. So, between time t and t plus d t. So, lambda d t tells that there is one arrival between the time t and time t plus d t. It means the probability that there is no arrival between time t and t plus d t will be 1 minus lambda d t.

So, this will be having the probability that probability of no arrival between time t n time t plus d t. Similarly, you have mu d t mu d t is again the probability that there is one service completion, one service being done in the time t plus d t. So, probability of one service completion, between time t plus d t, t and t plus d t. So, similarly 1 minus mu d t will be as usual it will be that there is no service completion between time t plus d t.

Now, another terminologies are like P n t, P n t is when we use this term t it means it is the transient state probability. So, it is transient state probability of having n customers at time t. So, that is P n t. And then if we talk only p n it means this is the steady state probability of having n customers in system. So, these are the terminologies which will be required in between when we try to you know find the expression for p n. Now let us assume the case that you are at a situation where at time t the prob you have n customers in the system. So, we will deal with the cases in which at time t plus d t also there will be n customers in the system. So, it is like this.

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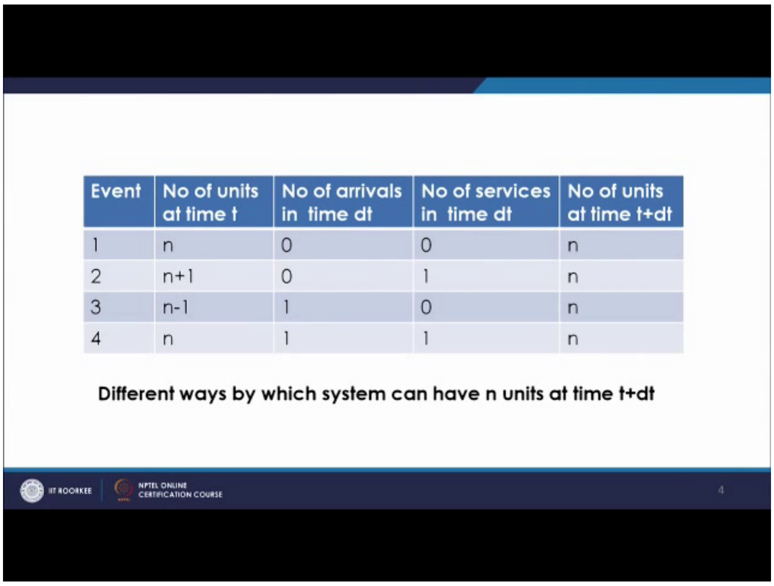

Event	No of units at time t	No of arrivals in time dt	No of services in time dt	No of units at time t+dt
1	n	0	0	n
2	n+1	0	1	n
3	n-1	1	0	n
4	n	1	1	n

Different ways by which system can have n units at time t+dt

So, if you have n customers at time t, then there are 4 ways in which there will be n customers in the system at time t plus d t. So, this table shows that you have n customers in the system at time t. And then in the first case if there is n units you know at time t you know we will basically discuss those cases in which you will have n customers in the system at time t plus d t. So, if you have n customers in the system at t, and if there is no arrival and no service in that case you are going to have number of units and itself.

Similarly, the second cases that you have n plus 1 customers at time t. So, this will this probability will be having p n plus 1 into t, p n plus 1 t. And then in that case if there is no arrival and one service. So, again you will have n customers remaining third case will be you have n minus 1 customers in the system at time t you have on arrival and no departure or no service in that case you have n customers in the system. And the fourth cases that you have n customers in the system you have one arrival and one departure. So, you get n itself at time t. Now let us see that how we have to proceed.

So, what we see is that in this case you have these 4 you know cases. And only one of them has to happen others will be you know not happening. So, anyway the addition of all these cases. So, you will have provide probability for event 1 probability for even 2 probability for event 3 and probability of event 4 all added. It will give you that you will have the n customers at time t plus d t.

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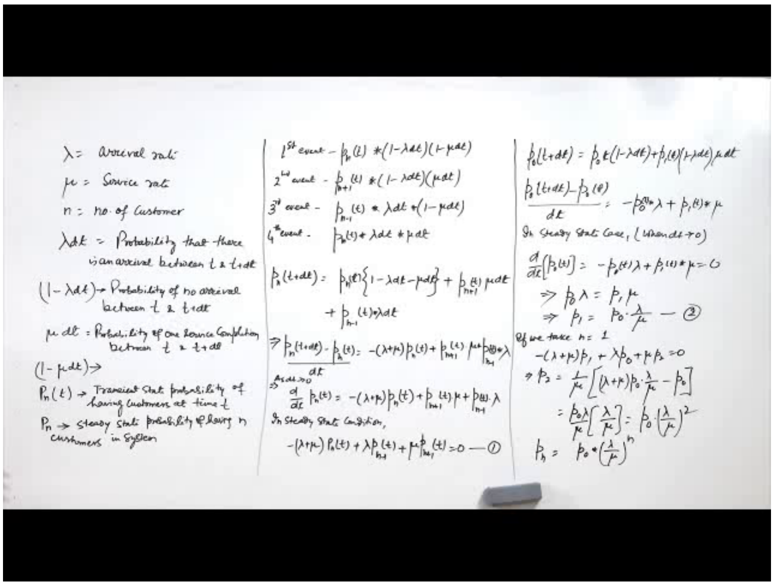

So, now, let us see probability of the occurring of the first event. In the first event you have probability of having n customers at time t that is p n t multiplied by, so, it will be. So, first event you have p n t multiplied by there you have number of arrivals is 0. So, for number of arrivals 0 as we see you have probability as 1 minus lambda d t. So, it will be

1 minus lambda d t. And you have number of services also 0. So, it means 1 minus mu d t. So, this is the probability of the first event. Similarly, probability of the second event second event will be it will be probability of having n plus 1 customer at time t. So, it will be p n plus 1 t multiplied by 0 arrival and one departure.

So, 0 arrival is 1 minus lambda d t and you have one departure. So, it will be mu d t. Similarly, in the third case you have probability of having n plus 1 customers at time t. So, p n minus 1 t multiplied by there is one arrival and 0 departure. So, one arrival means lambda d t and multiplied by 0 departure 1 minus mu d t. And the fourth is having n customers. So, it will b p n t multiplied by one arrival and one departure.

So, that is lambda d t multiplied by mu d t. So, this together it gives you the probability that there are n customers in the system at time t plus d t. So, p n t plus d t will be the summation of all these 4. So, you have now in this what we will do is the d t square terms will be basically neglected. In that case you will have p n t into this will be 1 minus lambda d t minus mu d t. So, because the last term is lambda mu into d t square that is this basically neglected. Then you will have p n plus 1 t multiplied by again mu d t then p n minus 1 t multiplied by lambda d t.

Further this term is basically neglected because it is lambda mu d t square. So, you want write and it further. So, what we see is from here, we can get p n t, plus d t minus p n t this p n t will be taking out. So, you will have minus of lambda plus mu. And we will also take d t from to this side. So, it will be minus of lambda plus mu p n t. Then they will have we will have plus p n plus 1 t into mu and plus p n minus 1 t into lambda.

So, as d t will be approaching towards 0. Then this term can be said to be as the derivative that will be d by d t. So, as d t approaches towards 0, it will be d by d t of p n t. So, d by d t, p n t will be minus of lambda plus mu p n t. Plus p n plus 1 t into mu plus p n minus 1 t into lambda. So, this is basically now in the case of steady state this rate of change will be equal to 0.

Now, we have to find other things. So, that you can get the expression for that. So, that will be basically in the steady state condition your this condition will hold. So, in steady state condition, you will have minus of lambda plus mu p n t plus lambda p n minus 1 t plus mu p n plus 1 t will be equal to 0. Now what we have to do? We have to find this expression. So, first of all we will try to find p 0 or p naught probability of having 0 customers in the system. So, from there we will have the expression and then finally, we can get the expression for p n t.

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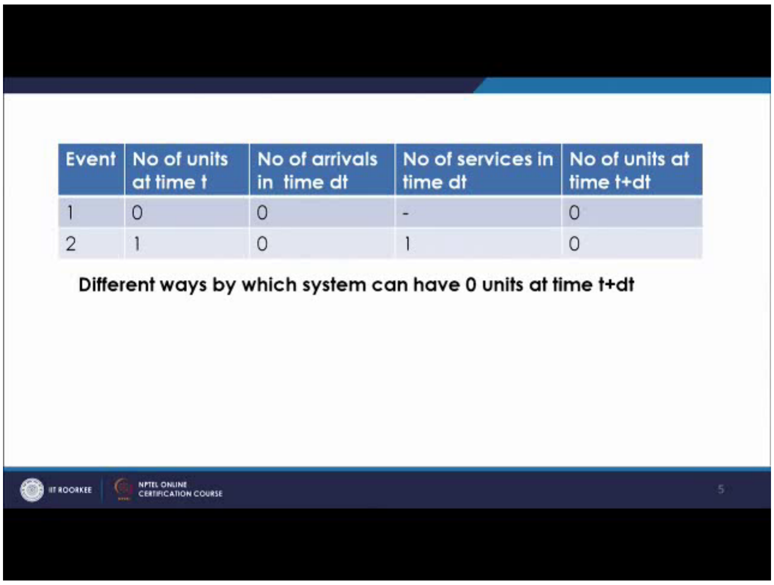

Now, for that if we try to see that how you have you can have the 0 units in the time t plus d t. So, basically in that case. So, that will be p 0 t. Now for that at time t plus d t p 0 t plus d t. So, for that if you have 0 customer. So, no arrival. So, anyway there is no meaning of server service. So, you have 0 customer and again you have one customer 0 arrival and one server one service. So, that will also give you the 0 number of units at time t plus d t. So, we can further write p 0 t plus d t. We will have these 2 mutually exclusive cases there is another case in which you may have one here and one both.

So, that case anyway will be cancelled. So, you have. So, it will be p 0 t and in that case you have number of 0 arrival. So, it will be 1 minus lambda d t and the probability of having no service certainly will be one itself. So, that that is left the next is plus p 1 t. So, we have one customer you have basically 0 arrival and one service. So, 0 arrival means 1 minus lambda d t and service. So, that is mu d t. So, that is the case. So, in this case in this situation you can have the number of units 0 at time t plus d t.

So, in that case again p 0 t plus d t minus p 0 t and again by d d t. So, that will be minus of. So, that will come as minus of p 0 times lambda, plus p 0 t basically plus p 1 times p 1 t times mu. So, this because this term lambda mu d t square this term anyway will be cancelled. So, what we see is, now in the steady state case when d t is standing towards

0. So, when d t this d t will tend towards 0 this will be the expression for d by d t of p 0 t. This will be minus p naught t lambda plus p 1 t into mu.

So, it will be p 0 plus p 1 we can write that is 0. So, this implies that we can write p 0 into lambda equal to 1 into mu. So, p 1 will be written as p 0 into lambda by mu. We get p 1 as p 0 into lambda y mu. Now if we use this expression further by putting t equal to 1 I mean n equal to 1. So, if we take n equal to 1, and we take this expressional one and this has 2. So, if we take n equal to 1 in that case this expression comes minus lambda plus mu into p n t. So, n is 1. So, p 1 plus lambda into n is 1.

So, it will be p 0 plus mu into p 2, will be equal to 0. So, what we see is p 2 will be one by mu lambda plus mu into p 1. So, p 1 is again p naught or p 0 into lambda by mu and minus p 0. So, it will be again p 0 times p 0 by mu. And on this side you will have lambda will come here. And then you will have lambda plus p minus lambda. So, it will be lambda by mu. So, it will be p 0 times lambda by mu 2.

So, what we see is p 2 is coming as p 0 dot lambda by mu square. So, if we again take n equal to 2. In that case we can get p 3 as p 0 into lambda by mu raise to the power 3. So, this way we can get p n that is steady state, you know in the steady state case the probability of having n customers in the system, it will be p 0 multiplied by lambda by mu raise to the power n. So, this is how this steady state probability of having n customers in the system can be found out. Now we have to find the value of p 0.

Now, what we have seen is that you have the expression for p n. And p n is coming as lambda by mu raise to the power you know n and multiplied by p 0. Now for calculating the p 0 what is the thing that the probability that the channel is busy that will be lambda by mu. So, basically for calculating.

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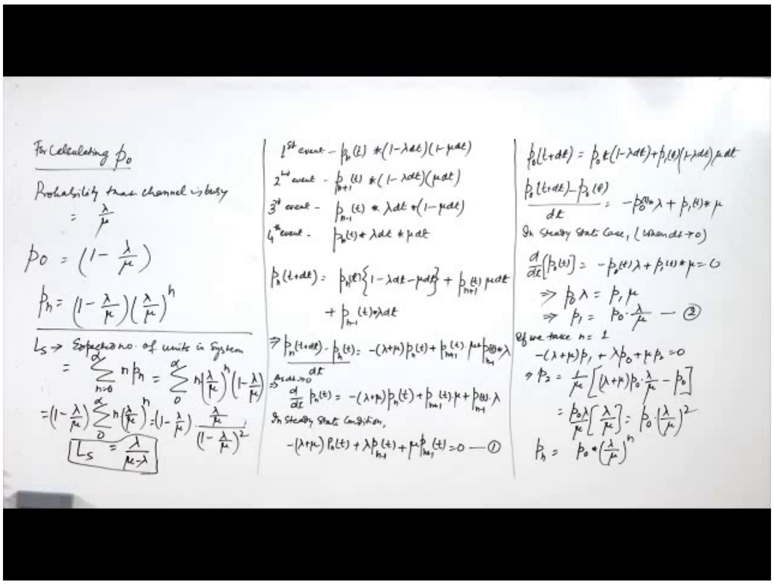

Probability of having 0 customer in the system. Now what is there in that case probability that channel is busy it is the ratio of the arrival rate to service rate. So, it will be lambda by mu.

So, the probability that you have the server is not busy that is idle or you have 0 customer in the system that will be. So, p be 0 will be 1 minus lambda by mu. So, once we get this probability value then further you can find. So, from there you can find p n as. So, it will be 1 minus lambda by mu, raise to the and then lambda by mu raise to the power n. So, this is how the you know probability of having n customers in the system can be found out if you know the arrival rate and the service rate. So, while solving the different type of queueing problems, if you know the arrival rate and the service rate then you can find the steady state probability value for having n customers in the system at time t.

Now, let us see if you have to calculate the. So, if you have to calculate suppose the LS, L S is nothing, but expected number of units in system. So, it is the expected number or average number of customers in the system. Now that will be nothing, but it will be summation of n equal 0, to alpha it will be n p n. If n is the customer at anytime now having n customers having a probability p n. So, having 0 customer having probability g p 0 1 customer having p 1 like that and it is addition.

So, that will tell you the expected number of units in the system. So, we can have p n from here. So, you will have summation u to infinity, n into lambda by mu raise to the power n into 1 minus lambda by mu. So, you can further write it as 1 minus lambda by mu will be a constants it will come out and summation of 0 to infinity n into lambda by mu raise to the power n. So, it is a series in infinite series which will go and it is you know once you do this computation for this arithmetic computation of the series it will be basically if a is something because in that case it is a by 1 minus a square a is lambda by mu. So, it will be a by 1 minus s a whole square.

So, that way it comes out to be 1 minus lambda by mu into lambda by mu divided by 1 minus lambda by mu whole square. So, this is how for this infinite series if you try to derive it will come like that. And ultimately you are going to get it will be lambda by mu minus lambda. So, this is expression which you get for LS. So, this way you can get the expression I mean the expression for the number of units expected number of units in the system. Now similarly you can have the LQ expected number of persons in the queue. So, that will be this much minus expected number of persons in the system itself, if the person getting service. So, that will be lambda by mu.

So, this divided by I mean subtracted with lambda when you will tell you the expression for l q. So, this way you can get the expression for all the different parameters like you may be interested to calculate the expected time per unit in the system. So, that will be L S upon lambda. So, it will be 1 by mu minus lambda. So, like that you can see a standard book on queueing and you can have the idea, how the values of these different parameters are calculated. So, because this will be used when we discuss I mean, when we analyze different type of queueing problems this concept must be very much clear.

Thank you.