

**Introduction to Mechanical Vibration**  
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**Lecture - 37**  
**Eigen Values and Eigen Vectors**

So welcome to the lecture on multi-degree of freedom systems. So we discussed in the previous class that when we have a system that has three masses and stiffness elements and how to write the equation of motions and finally we can assemble these equations to a matrix forms.

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n-dof system

$$[M] \{\ddot{x}\} + [k] \{x\} = \{0\} \quad \text{--- (i)}$$

$[M]_{n \times n}$  = mass matrix  
 $[k]_{n \times n}$  = stiffness matrix

$$[M]^{-1}$$
$$[M]^{-1}[M] \{\ddot{x}\} + [M]^{-1}[k] \{x\} = \{0\}$$
$$[I] \{\ddot{x}\} + [D] \{x\} = \{0\} \quad \text{--- (ii)}$$
$$[D] = [M]^{-1}[k] \rightarrow \text{dynamic matrix}$$

And therefore the matrix equation that we derived was I can write so this is the equation that for multi-degree of freedom on damped system so  $M \ddot{x} + kx = 0$ . Here  $M$  is a mass matrix and if it is  $n$  degree of freedom, multi-degree of freedom system, so  $m$  is  $n$  into  $n$  square matrix. Similarly, here  $k$  is also  $n$  into  $n$  matrix and that is stiffness matrix.

Now here we can do this matrix, we can do certain operations, so this is equation number one. Now we know that inverse matrix, so if we multiply this equation with  $m$  inverse that is the inverse mass matrix. So if we multiply this matrix into this equation, so here we have now  $I$  is the identity matrix, so when we multiply with the inverse of the mass matrix to the mass matrix it gives an identity matrix.

**(Refer Slide Time: 04:03)**

$$\begin{aligned}
 \{x\} &= \{X\} \sin \omega t \\
 \{\ddot{x}\} &= -\omega^2 \{X\} \sin \omega t \\
 \lambda &= \omega^2 \rightarrow \text{Eigenvalue} \\
 \{\ddot{x}\} &= -\lambda \{X\} \sin \omega t \\
 &= -\lambda \{x\} \\
 -\lambda [I] \{x\} + [D] \{x\} &= \{0\} \\
 [D - \lambda [I]] \{x\} &= \{0\} \\
 [D - \lambda [I]] \{X\} &= \{0\} \\
 \{X\} &\rightarrow \text{Eigenvector}
 \end{aligned}$$

Now here D is the dynamic matrix so  $D = M^{-1}k$  that is dynamic matrix. So this is equation number two. We can think for a solution of this equation as  $x = X \sin \omega t$  where  $x$  is this vector, the displacement vector and  $X$  is the amplitude of the displacement, so amplitude vector of the displacement and  $\omega$  is the frequency. So now we will have  $\ddot{x}$  equal to  $-\omega^2 x$ , so we will differentiate  $t$ .

So  $-\omega^2$  into here this  $x$  vector and then  $\sin \omega t$  into  $t$ . Now we define one term this  $\lambda$  that is equal to  $\omega^2$  and this is called Eigen value. And therefore under root of  $\lambda$  will give the natural frequencies. So we can write  $\ddot{x} = -\lambda x$  and that is equal to  $-\lambda x$ .

So now if we put these terms into this equation, so we will have here  $-\lambda I x + D x = 0$ . So here we can write so this will be  $D - \lambda I = 0$  and we can put  $x = X \sin \omega t$  so we can have  $D - \lambda I X = 0$ . So we have these equations. So this is Eigen value and this  $X$  is Eigen vector. So therefore if we want to find the Eigen values, we have to do these determinant  $= 0$ .

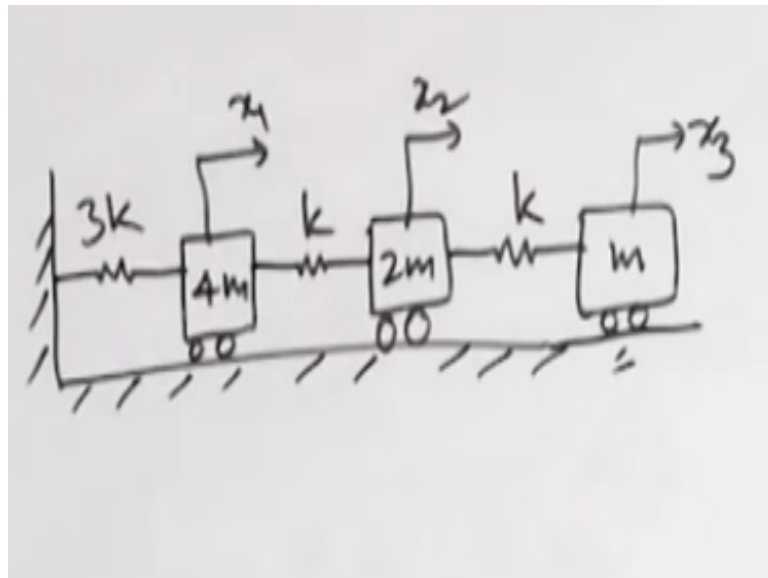
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$$|[D] - \lambda [I]| = 0$$

$\Downarrow$   
Eigenvalues  
 $\lambda_i$

So therefore we have to make  $D - \lambda I \text{ mod} = 0$ . This will give the Eigen values. Because all these lambdas is Eigen value and corresponding to each lambda if we put here, we will get this X corresponding to the lambda and that will be Eigen vector. So now we will take one example and we will try to apply what we discussed.

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So we have this three degree of freedom system that is comprises three masses and three stiffness elements. So these three masses that are  $4m$ ,  $2m$  and  $m$  and the stiffness are  $3k$ ,  $k$  and  $k$ . Now we want to know the Eigen values and Eigen vectors of this system.

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$$\begin{aligned}
 | [D] - \lambda [I] | &= 0 \\
 [D] &= [M]^{-1} [K] \\
 [M] &= \begin{bmatrix} 4m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix}_{3 \times 3} \\
 [K] &= \begin{bmatrix} 3k+k & -k & 0 \\ -k & k+k & -k \\ 0 & -k & k \end{bmatrix} \\
 &= \begin{bmatrix} 4k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}
 \end{aligned}$$

So as we find that we have to first find the Eigen values and for Eigen values we have  $D - \lambda I = 0$  this modulus. So we have to find the D matrix so this D matrix is M inverse into K matrix. So what is the M and K matrix? So M matrix as we have already derived in the previous lecture, we wrote equations of motion and we finally assembled into a matrix so that mass element they were diagonal.

So here the mass matrix will be we have this  $4m \ 0 \ 0$ , it is three degree of freedom system so it will be 3 into 3 matrix as I said n into n square matrix then this diagonal element  $2m$  then  $m$  and this is  $0 \ 0 \ 0$ . So this is the mass matrix. Now we have to know the stiffness matrix, so if we remember the K matrix that we collected was the term, first term was  $k_1 + k_2$  so here is  $k_1 + k_2$  so  $3k + k$  and then this was  $-k_2 - k_2$  so  $-k$  and  $-k$ .

Then here  $k_2 + k_3$  so  $k + k$  and this was  $0$  and then these terms  $-k_n - k_n$ . So the last one  $-k$  and  $-k$  and this is  $k_n$  so that is  $k$ . so this is our stiffness matrix we can simplify so that is  $4k$ , this is  $-k \ 0 \ -k \ 0 \ 2k \ -k \ -k \ k$ , so now we have mass matrix, stiffness matrix, the inverse of the mass matrix.

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$$\begin{aligned}
 [M]^{-1} &= \begin{bmatrix} \frac{1}{4m} & 0 & 0 \\ 0 & \frac{1}{2m} & 0 \\ 0 & 0 & \frac{1}{m} \end{bmatrix} \\
 [D] &= [M]^{-1} [K] \\
 &= \begin{bmatrix} \frac{1}{4m} & 0 & 0 \\ 0 & \frac{1}{2m} & 0 \\ 0 & 0 & \frac{1}{m} \end{bmatrix} \begin{bmatrix} 4k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \\
 &= \begin{bmatrix} \frac{k}{m} & -\frac{k}{4m} & 0 \\ -\frac{k}{2m} & \frac{k}{m} & -\frac{k}{2m} \\ 0 & -\frac{k}{m} & \frac{k}{m} \end{bmatrix} \quad \lambda[I] = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \\
 [D] - \lambda[I] &= \begin{bmatrix} \frac{k}{m} - \lambda & -\frac{k}{4m} & 0 \\ -\frac{k}{2m} & \frac{k}{m} - \lambda & -\frac{k}{2m} \\ 0 & -\frac{k}{m} & \frac{k}{m} - \lambda \end{bmatrix}
 \end{aligned}$$

So if we find the inverse of this matrix. So M inverse will be because it is diagonal element, so it will be simply the inverse 1 by 4m 0 0 0 1 by 2 m and 0 and then 0 0 1 by m. So this is our inverse mass matrix. Now we have to find the D matrix, so D matrix = m inverse into k, so we will have 1 by 4m 0 0 0 1 by 2m 0 0 0 1 by m, so this is M inverse and k matrix. So we have 4k - k 0 - k 2k - k 0 - k and k.

So now we can have this. So we have to multiply this. So if we want to find this element, we have to multiply this first row and first column. So it is 4k by 4m, other terms are 0, so this is 4k by 4m so it is k by m. Then this second term, so 1 this term into this column so - k by 4m and this one is 0 0 0 so it is 0 where this term is 0 and this multiply with zeroes. Now this term we have second row and first column so 0 and - k by 2m so it is 0 - k by 2m and 0.

Then this term is second row and second column so 0 and this is 1 by 2m into 2k so it is k by m and then 0. Then here we have this element so we have two (( )) (14:32). So these with these multiply so 0 and then - k by 2m here and the last one, so we multiply third row with first column so it is 0. Then here so it is - k by m and then here so it is k by m. So we have got this D matrix.

And we have to now perform this Eigen value analysis and we have to now do the determinant of this matrix = 0. So this if we now want D - lambda I, so D - lambda I is = so it simply get here - lambda to the diagonal terms because here D is this and lambda I, I is unity matrix so lambda I is a matrix with lambda 0 0 0 lambda 0 0 0 lambda. So this is lambda I, so if we do D - lambda I so this other terms are 0, only the diagonal terms are non-zero.

So this lambda will be - at the diagonal terms. So we will have k by m - lambda and here k by m - lambda, then k by m - lambda. Ratio of the terms will remain as such, so here - k by 4m and then 0, here - k by 2m and then 0, - k by m and - k by 2m.

(Refer Slide Time: 17:09)

$$([D] - \lambda [I]) = 0$$

$$\begin{vmatrix} \frac{k}{m} - \lambda & -\frac{k}{4m} & 0 \\ -\frac{k}{2m} & \frac{k}{m} - \lambda & -\frac{k}{2m} \\ 0 & -\frac{k}{m} & \frac{k}{m} - \lambda \end{vmatrix} = 0$$

So this is D - lambda I, now we have to do mod of this 0. So we will have D - lambda I mod 0. So the determinant is 0, so we will have k by m - lambda and - k by 4m and 0 and - k by 2m k by m - lambda - k by 2m 0 - k by m and k by m - lambda. So this mod = 0.

(Refer Slide Time: 17:09)

$$\left( \left( \frac{k}{m} - \lambda \right) \left[ \left( \frac{k}{m} - \lambda \right)^2 - \frac{k^2}{2m^2} \right] + \frac{k}{4m} \left[ \left( \frac{k}{m} - \lambda \right) \left( -\frac{k}{2m} \right) \right] \right) = 0$$

$$\left( \frac{k}{m} - \lambda \right) \left[ \left( \frac{k}{m} - \lambda \right)^2 - \frac{k^2}{2m^2} - \frac{k^2}{8m^2} \right] = 0$$

$$\Downarrow$$

$$\frac{k^2}{m^2} + \lambda^2 - 2 \frac{k}{m} \lambda - \frac{5k^2}{8m^2} = 0$$

$$\left( \frac{k}{m} - \lambda \right) \left( \lambda^2 - 2 \frac{k}{m} \lambda + \frac{3}{8} \frac{k^2}{m^2} \right) = 0$$

So now we have to solve this determinant. So k by m - lambda so k by m - lambda we take first element and then this is k by m - lambda is square - so it is k by m - lambda square -. So this is - k by 2m - k by m so - - it is multiplied with +, so k square by 2m square, so this is k

square by  $2m$  square. Then this one term so -, -  $k$  by  $4m$  so it is  $k$  by  $4m$  and we have this element so we take this into this and this is 0 so there will be no this term.

So  $k$  by  $m - \lambda$  into -  $k$  by  $2m$ , so now we have to solve this. So  $k$  by  $m - \lambda$  here and here and this is = 0 so we can take out this  $k$  by  $m - \lambda$  and then inside we have  $k$  by  $m - \lambda$  square, so it is  $k$  by  $m - \lambda$  square -  $k$  square by  $2m$  square. Now we have taken this out and this is -  $k$  square by  $8m$  square -  $k$  square by  $8m$  square and that is equal to 0.

So now we take this term so this is  $k$  square by  $m$  square +  $\lambda$  square -  $2k$  by  $m$   $\lambda$  and this is  $5k$  square by  $8m$  square so this is 4 by  $8m$  square and this is 1 by 8 so it is 5 by  $8k$  square by  $m$  square that is = 0. So this is  $\lambda$  square -  $2k$  by  $m$   $\lambda$  and this is  $k$  square by  $m$  square so it is = + so because it is  $8k$  square by  $8m$  square so  $8 - 5$  it is 3 by  $8k$  square by  $m$  square and that is  $k$  by  $m - \lambda$  so that is = 0.

So finally the solution of this equation will give us the Eigen values so we see that already  $k$  by  $m - \lambda = 0$  is giving one Eigen value so this is  $\lambda = k$  by  $m$ . So one Eigen value we have already obtained.

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$$\begin{aligned}
 & \left( \frac{k}{m} - \lambda \right) \left[ \left( \frac{k}{m} - \lambda \right)^2 - \frac{k^2}{2m^2} \right] + \frac{k}{4m} \left[ \left( \frac{k}{m} - \lambda \right) \left( -\frac{k}{2m} \right) \right] = 0 \\
 & \left( \frac{k}{m} - \lambda \right) \left[ \left( \frac{k}{m} - \lambda \right)^2 - \frac{k^2}{2m^2} - \frac{k^2}{8m^2} \right] = 0 \\
 & \quad \downarrow \\
 & \frac{k^2}{m^2} + \lambda^2 - 2\frac{k}{m}\lambda - \frac{5k^2}{8m^2} = 0 \\
 & \left( \frac{k}{m} - \lambda \right) \left( \lambda^2 - 2\frac{k}{m}\lambda + \frac{3}{8}\frac{k^2}{m^2} \right) = 0 \\
 & \quad \downarrow \\
 & \frac{k}{m} - \lambda = 0 \Rightarrow \lambda = \frac{k}{m} \\
 & \lambda = \frac{2\frac{k}{m} \pm \sqrt{4\frac{k^2}{m^2} - 4 \cdot 1 \cdot \frac{3}{8}\frac{k^2}{m^2}}}{2} \\
 & = \frac{k}{m} \pm \frac{k}{m} \sqrt{\frac{5}{8}} = \left( 1 \pm \sqrt{\frac{5}{8}} \right) \frac{k}{m} = \left( 1 \pm 0.791 \right) \frac{k}{m} \\
 & \quad \quad \quad = 0.209 \frac{k}{m}, \quad 1.791 \frac{k}{m}
 \end{aligned}$$

Now we have to solve this quadratic equation to get other two Eigen values because this is a three degree of freedom system and therefore we will have three Eigen values and therefore if we solve this equation, so  $\lambda = -b$  that is  $2k$  by  $m + -$  under root  $b$  square so this is

square means this is  $4k$  square by  $m$  square -  $4$  into  $1$  into  $c$  so  $c$  is  $3$  by  $8k$  square by  $m$  square and by  $2a$  so  $2$  into  $1$  so it is  $2$ .

So if we have here  $k$  by  $m + -$  because we have this. Now  $4k$  square by  $m$  square we take outside so it is  $2k$  by  $m$  and  $2$  and  $2$  will cancel out so we will have  $k$  by  $m$  under root, we have  $1 -$  so this is  $3$  by  $8$  so  $1 - 3$  by  $8$  and this we can write  $1 - 3$  by  $9$  as  $5$  by  $8$ . So this we can have  $= 1 + - \text{root } 5$  by  $8$   $k$  by  $m$  and that is equal to, So root  $5$  by  $8$  is  $0.791$ . So it is  $1 + - 0.791$   $k$  by  $m$  and so we are getting  $0.209$   $k$  by  $m$  and second one is  $1.791$   $k$  by  $m$ . So these are the two other Eigen values that we have got.

(Refer Slide Time: 25:07)

The diagram shows a three-degree-of-freedom mass-spring system. A fixed wall on the left is connected to a mass of  $4m$  by a spring with stiffness  $3k$ . This mass is connected to a second mass of  $2m$  by a spring with stiffness  $k$ . The second mass is connected to a third mass of  $m$  by a spring with stiffness  $k$ . Displacements  $x_1$ ,  $x_2$ , and  $x_3$  are indicated for the three masses respectively.

Below the diagram, the eigenvalues are listed:

$$\lambda = 0.209 \frac{k}{m}, \frac{k}{m}, 1.791 \frac{k}{m}$$

These are labeled  $\lambda_1$ ,  $\lambda_2$ , and  $\lambda_3$  respectively.

The eigenvalue equation is given as:

$$[D] - \lambda_i [I] \{x\}_i = \{0\}$$

The matrix equation for the eigenvalues is shown as:

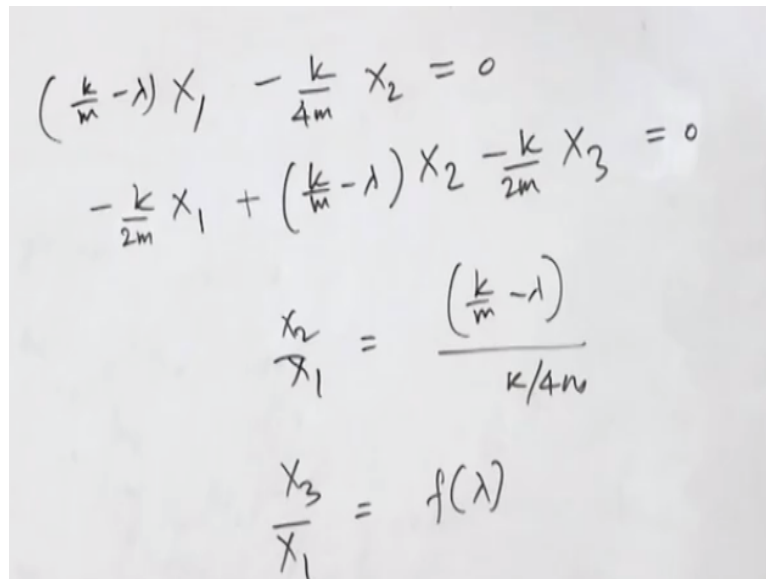
$$\begin{bmatrix} \frac{k}{m} - \lambda & -\frac{k}{4m} & 0 \\ -\frac{k}{2m} & \frac{k}{m} - \lambda & -\frac{k}{2m} \\ 0 & -\frac{k}{m} & \frac{k}{m} - \lambda \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

So finally our Eigen values so  $\lambda$  we write these Eigen values in terms of lower to higher values. So the lowest value is here  $0.209$   $k$  by  $m$  then we have  $k$  by  $m$  and then we have the last one  $1.791$   $k$  by  $m$ . So these are the Eigen values so we have Eigen values and so this is  $\lambda_1$ , this is  $\lambda_2$ , this is  $\lambda_3$ . So as we have obtained these Eigen values now our next task is to get the Eigen vectors.

So to get the Eigen vector, we have to use this equation  $D - \lambda I X = 0$ . So here we can have  $\lambda I$  and this is  $X_i$  so we have to put here, for each  $\lambda$  we will get 1 this Eigen vector so it means that corresponding to each Eigen value we have one Eigen vector. So we have this equation. So  $D - \lambda I$  we know here this  $k$  by  $m - \lambda$  -  $k$  by  $4m$   $0 - k$  by  $2m$   $k$  by  $m - \lambda$  and  $-k$  by  $2m$   $0 - k$  by  $m$  and  $k$  by  $m - \lambda$ .

So this is my  $D - \lambda I$  and this is now we have  $X_1, X_2, X_3$  and that is equal to 0. So we have this equation.

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$$\begin{aligned} \left(\frac{k}{m} - \lambda\right) X_1 - \frac{k}{4m} X_2 &= 0 \\ -\frac{k}{2m} X_1 + \left(\frac{k}{m} - \lambda\right) X_2 - \frac{k}{2m} X_3 &= 0 \\ \frac{X_2}{X_1} &= \frac{\left(\frac{k}{m} - \lambda\right)}{k/4m} \\ \frac{X_3}{X_1} &= f(\lambda) \end{aligned}$$

And now we can write these equations like  $k \text{ by } m - \lambda X_1 - k \text{ by } 4m X_2 = 0$ . Now from this we can write  $-k \text{ by } 2m X_1 - k \text{ by } m - \lambda X_2 - k \text{ by } 2m X_3 =$ . So we can solve these equations, so from here  $X_2 \text{ by } X_1$  we get and that is  $k \text{ by } m - \lambda \text{ upon } k \text{ by } 4m$ . Similarly, we put the value of  $X_2$  from here and we can get  $X_3 \text{ by } X_1$  in terms of function of  $\lambda$ .

So now if we put one frequency the natural frequency, we will get the first mode and if we put the second frequency, we will get the second mode and so on we will get  $X_3 \text{ by } X_1$ . So now by solving these equations and using these Eigen values, we can find the mode shape that is the Eigen vector of the system and that was the objective of this lecture. So I thank you for attending this lecture and see you in the next lecture.