

Introduction to Mechanical Vibration
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Lecture - 30
Undamped Forced Vibration with Harmonic Excitation

So welcome to the lecture on two degree of freedom systems. So - so far we have discussed the undamped cases, undamped free vibration case, the damped free vibration case we have discussed the coupling of the two modes we have discussed mode shapes and the natural frequencies. Now we are going to discuss the forced vibration of a two degree of freedom system so what is the forced vibration?

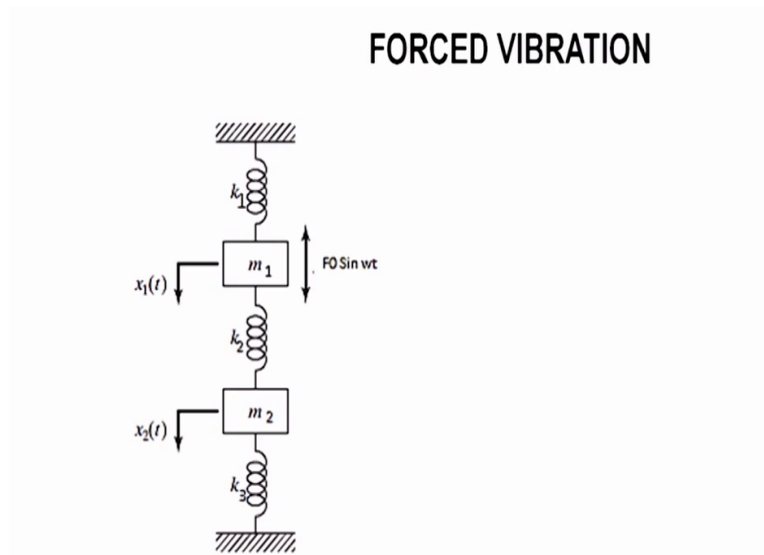
So in compared to the free vibration as free vibration we disturb the system and we release the system, so the system starts vibrating with its own natural frequency when it is undamped when it reaches damped it starts with the damped natural frequency, however when we have forced vibration and we apply some harmonic load here we will talk the forced vibration with harmonic load, so harmonic load is a load that where is like sinusoidal or cosine manner.

So when our system is subjected to a harmonic load so the - the response of the system contains two parts, that is transient part and the steady-state part, so the transient part that due to the effect of the damping it will die out very soon, however the steady-state part that is due to the external loads that will remain there due to the till the load is there, okay. And we know that in case of steady state vibration the steady state response the vibrations usually have the vibration of - of the system

So the system vibrates with the same frequency as the forcing frequency, so if I have a system that has some natural frequency 5 hertz but I have applied load that is 15 hertz, so the system steady state response the vibration will contain the 15 hertz frequency, when the forcing frequency is equal to the natural frequency of the system like if I have my system has 5 hertz frequency natural frequency.

And I apply a load harmonic load that has also the frequency of 5 hertz, then that is called the resonance condition and in that case the - the amplitude of vibrations are very high, okay. So these all things we have already discussed and here I just wanted to recall, because I am going to discuss the forced vibration of two degree of freedom system. So let us take a system so this is our system two degree of freedom system.

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So we have two masses m_1 and m_2 and two springs k_1 , k_2 and k_3 and the force harmonic forces applied on first mass and that is $F_0 \sin \omega t$, so F_0 is the amplitude of the force and ω is the frequency of that force. So let us study this system, okay.

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$x_1 = X_1 \sin \omega t \Rightarrow \ddot{x}_1 = -\omega^2 X_1 \sin \omega t$
 $x_2 = X_2 \sin \omega t \Rightarrow \ddot{x}_2 = -\omega^2 X_2 \sin \omega t$
 $[-m_1 \omega^2 X_1 + (k_1 + k_2) X_1 - k_2 X_2] \sin \omega t = F_0 \sin \omega t$
 $[m_1 \omega^2 + k_1 + k_2] X_1 - k_2 X_2 = F_0 \quad \text{--- (iii)}$
 $[-m_2 \omega^2 X_2 + (k_2 + k_3) X_2 - k_2 X_1] \sin \omega t = 0$
 $(-m_2 \omega^2 + k_2 + k_3) X_2 - k_2 X_1 = 0 \quad \text{--- (iv)}$
 $X_2 = \frac{k_2}{(-m_2 \omega^2 + k_2 + k_3)} X_1 \quad \text{--- (iv)}$
 $(-m_1 \omega^2 + k_1 + k_2) X_1 - \frac{k_2^2}{(-m_2 \omega^2 + k_2 + k_3)} X_1 = F_0$
 $(-m_1 \omega^2 + k_1 + k_2) (-m_2 \omega^2 + k_2 + k_3) - k_2^2 = \frac{F_0}{X_1} (-m_2 \omega^2 + k_2 + k_3)$
 $m_1 m_2 \omega^4 - m_1 (k_2 + k_3) \omega^2 - m_2 (k_1 + k_2) \omega^2 + (k_1 + k_2)(k_2 + k_3) - k_2^2 = 0$

$m_1 \ddot{x}_1 = -k_1 x_1 + k_2 (x_2 - x_1) + F_0 \sin \omega t$
 $m_1 \ddot{x}_1 + k_1 x_1 + k_2 x_1 - k_2 x_2 = F_0 \sin \omega t$
 $m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = F_0 \sin \omega t \quad \text{--- (i)}$
 $m_2 \ddot{x}_2 = -k_2 (x_2 - x_1) - k_3 x_2$
 $m_2 \ddot{x}_2 + (k_2 + k_3) x_2 - k_2 x_1 = 0 \quad \text{--- (ii)}$

So we have this system m_1 m_2 and k_1 k_2 k_3 here we have not for simplicity we have not considered the damping, now this we have applied the force on $F_0 \sin \omega t$ on mass m_1 , so to understand to study the vibration response of this system we have to write the equations of motion and then solve it, so equations of motion we need to write and so we make the free body diagram so let us have this m_1 mass and we make the free body diagram.

So if at that instant of time it is x_1 here and its expression is directed towards the same direction of x_1 , so we are going to pull this spring so this spring will apply force k_1 times x_1 , however this spring we assume $x_2 > x_1$ so we will this spring will also apply a force here $k_2 x_2 - x_1$, so net it will pull this mass however there is a one force $F_0 \sin \omega t$ also that is external force acting on the system.

For the mass m_2 we have this force in this direction due to the spring k_2 and because it is compressing this spring so this spring will apply a force in this direction that is k_3 times x_2 , okay. so we have to write the equations, so we can write for the first mass m_1 , so $m_1 \ddot{x}_1 = \text{summation of the forces}$ and so forces are $-k_1 x_1 + k_2 x_2 - x_1 + F_0 \sin \omega t$. So we can rewrite this equation $m_1 \ddot{x}_1 + k_1 x_1 + k_2 x_1 - k_2 x_2 = F_0 \sin \omega t$.

So we can write $m_1 \ddot{x}_1 + k_1 + k_2 x_1 - k_2 x_2 = F_0 \sin \omega t$, so this is an equation number one. Now we have to write the equation of motion for second mass, so we write $m_2 \ddot{x}_2 =$, so we have this is the direction of $\ddot{x}_2$ so we have $-k_3 x_2 - x_1 - k_3 x_2$. So we can write $m_2 \ddot{x}_2 +$ so it is $k_3 x_2$ sorry, this is I think k_2 so $+k_2 + k_3 x_2 - k_2 x_1 = 0$, this is equation number two.

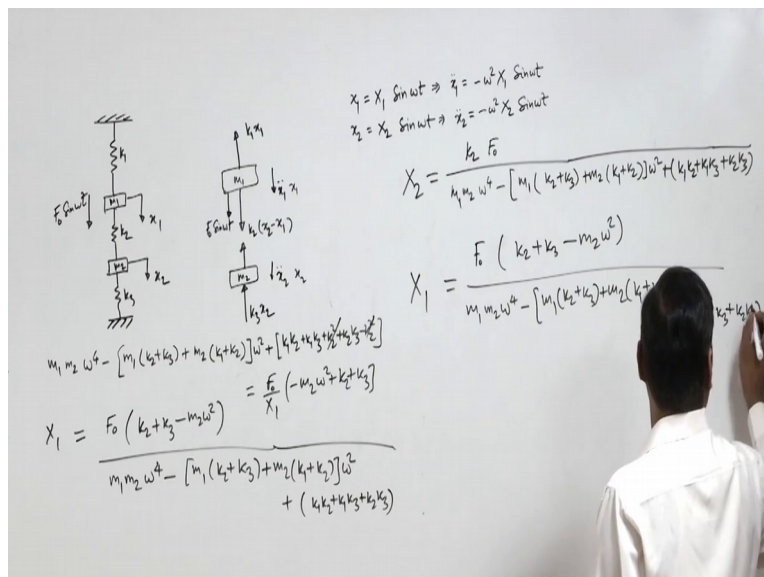
Now we assume the steady state condition we are going to interest be interested in the steady state condition, so steady state condition as we I said that the system will vibrate because system is subjected to some external force external harmonic force. So the system will vibrate with the same frequency as the forcing frequency, okay. so we can assume the solution so $x_1 = \hat{x}_1 \sin \omega t$ and $x_2 = \hat{x}_2 \sin \omega t$, where capital $\hat{x}_1$ and capital $\hat{x}_2$, they are the amplitude of the response.

So now we put these values in this equation, so here we have $x_1 \ddot{} = -\omega^2 x_1 \sin \omega t$ and here $x_2 \ddot{} = -\omega^2 x_2 \sin \omega t$. So we can put in this equation so $m_1 - \omega^2 x_1 + k_1 + k_2$ into $x_1 - k_2$ into x_2 and this is $\sin \omega t = F_0 \sin \omega t$, so we can so leave the case of $\sin \omega t = 0$. So we will have $m_1 \omega^2 + k_1 + k_2 x_1 - k_2 x_2 = F_0 \sin \omega t$, so this is equation number three.

Now the second equation if you put these solutions so $m_2 \ddot{x}_2 + k_2 + k_3 x_2 - k_2 x_1 \sin \omega t = 0$. So here we have -- here we have -- I sorry here this is like this, so here we can have $-m_2 \omega^2 x_2 + k_2 + k_3 x_2 - k_2 x_1 = 0$, so from here we can find x_2 by x_1 so we can write $x_2 = k_2$ by $-m_2 \omega^2 + k_2 + k_3$ in to X_1 , okay, so this equation number four. Now we put the value of x_2 in this equation, because we have to solve this equation.

So $-m_1 \omega^2 + k_1 + k_2 x_1 - k_2$ square so we put x_2 so it is k_2 square upon $-m_2 \omega^2 + k_2 + k_3$ in to x_1 we put here $x_1 = F_0$, so we solve this equation, so we can have we can multiply here so $-m_1 \omega^2 + k_1 + k_2$ and $-m_2 \omega^2 + k_2 + k_3 - k_2$ square = F_0 into $-m_2 \omega^2 + k_2 + k_3$ by X_1 , so X_1 . So we can multiply here so m_1 so this is $m_1 m_2 \omega^4 - m_1 k_2 + k_3 \omega^2 - m_2 k_1 + k_2 \omega^2 + k_1 + k_2$ into $k_2 + k_3 - k_2$ square that is equal to this same term, so we can rewrite this.

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So we will have $m_1 m_2 \omega^4 - m_1 k_2 + k_3 + m_2 k_1 + k_2 \omega^2 + k_1 k_3 + k_2^2 \omega^2 + k_2 k_3 - k_2^2 \omega^2$ and $k_2^2 \omega^2$ will cancel out and that is equal to F_0 by x_1 into $-m_2 \omega^2 + k_2 + k_3$, so here we can write $x_1 = \frac{F_0}{-m_2 \omega^2 + k_2 + k_3}$, so we can write like $k_2 + k_3 - m_2 \omega^2$ upon $m_1 m_2 \omega^4 - m_1 k_2 + k_3 + m_2 k_1 + k_2 \omega^2 + k_1 k_3 + k_2 k_3$.

So these are the this is the denominator term and so we can get also $k_2 X_2$, so $x_2 =$ so $x_2 = k_2 x_1$ by this term, so x_1 by this term we put here, so it is $k_2 F_0$ by so $m_1 m_2 \omega^4 - m_1 k_2 + k_3 + m_2 k_1 + k_2 \omega^2 + k_1 k_3 + k_2 k_3$, okay, so we have obtained the amplitude of x_2 and X_1 , so let me rewrite here because I want to discuss some concept. So $k_2 + k_3 - m_2 \omega^2$ upon $m_1 m_2 \omega^4 - m_1 k_2 + k_3 + m_2 k_1 + k_2 \omega^2 + k_1 k_3 + k_2 k_3$, okay - okay.

So we have these response amplitudes of x_1 and X_2 , so what we see that the we have this response of x_1 also I mean there is if force is 0 they are x_1 and x_2 they are 0 this denominator term is nothing but this is the frequency equation and so if you solve that equation it will give the natural frequency of the system.

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$x_1 = X_1 \sin \omega t \Rightarrow \ddot{x}_1 = -\omega^2 X_1 \sin \omega t$
 $x_2 = X_2 \sin \omega t \Rightarrow \ddot{x}_2 = -\omega^2 X_2 \sin \omega t$

$$X_2 = \frac{k_2 F_0}{m_1 m_2 \omega^4 - [m_1(k_2 + k_3) + m_2(k_1 + k_2)]\omega^2 + (k_1 k_3 + k_2 k_3)}$$

$$X_1 = \frac{F_0 (k_2 + k_3 - m_2 \omega^2)}{m_1 m_2 \omega^4 - [m_1(k_2 + k_3) + m_2(k_1 + k_2)]\omega^2 + (k_1 k_3 + k_2 k_3)}$$

$$k_2 + k_3 - m_2 \omega^2 = 0$$

$$\Rightarrow m_2 \omega^2 = k_2 + k_3$$

$$\omega = \sqrt{\frac{k_2 + k_3}{m_2}}$$

One more thing we can notice that x_1 is 0 when this term is 0 that is $k_2 + k_3 - m_2 \omega^2$ is 0, so it means that $m_2 \omega^2 = k_2 + k_3$ and so $\omega = \sqrt{k_2 + k_3 / m_2}$ under root, so

what it implies that if the excitation frequency $\omega = \sqrt{\frac{k_2 + k_3}{m_2}}$, the amplitude of x_1 is 0, so this mass m_1 will be at rest, so it means that if we select we can select the value of k_2 , k_3 and m_2 .

So if we can select this system such that $\sqrt{\frac{k_2 + k_3}{m_2}} = \omega$ then this mass will come at rest, so this gives the principle of vibration absorber that are used for bringing the main system to rest, if some force is coming to the main system we can select an auxiliary system such that the following this relation so that the vibration of the main system comes to zero and that vibration is transmitted to the vibration of the mass m_2 .

And here we see that vibration of x_2 is not 0, so vibration of mass m_2 is not 0 but vibration of m_1 is 0 here, so this gives the principle of vibration dynamic vibration absorber and that we will discuss in detail in the next - next class so we saw that in case of forced vibration the system vibrates with the same frequency as the forcing frequency the steady state response contains and that frequency.

Moreover, in case of two degree of freedom system we can select the parameters of the springs and secondary mass in such a way that we can bring the first mass to rest for any excitation frequency ω and this will give the - the theory for the design of the vibration absorber. So that we will discuss in detail in the next classes. So I stop here and I thank you for attending the lecture and see you in the next lecture.