Computational Fluid Dynamics
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Lecture – 31
Introduction to Finite Element Method

Welcome to model 7 on finite element method, in this module we will briefly discuss with third discretisation strategy would be called finite element method which is one of the most popular analysis tools developed in the second half of the last century and it has changed the commercial design and analysis completely in the last few decades. This is one of the most exciting tools is fairly involved compared to finite difference methods.

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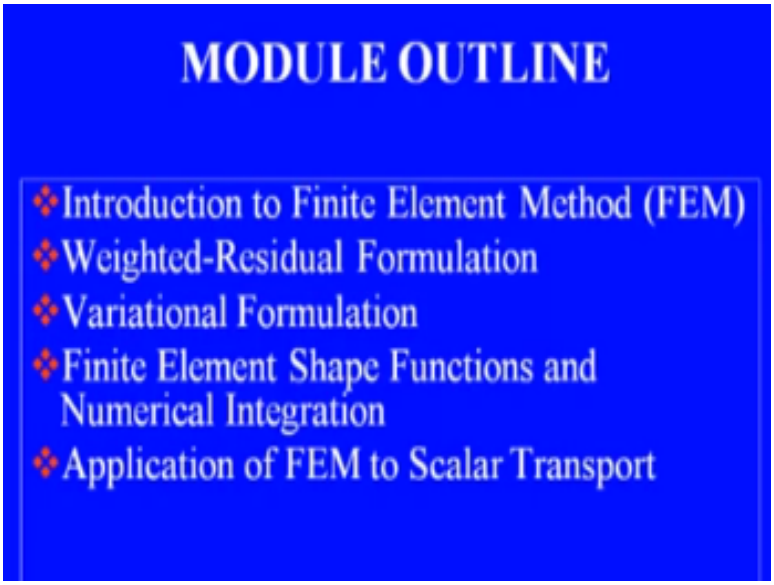

And in fact, we need a full course to understand this method fully but here in this course we are going to discuss particular method very briefly, keeping in view that there are certain schools of analyst which prefer finite element discretisation for solving a flow problems. So, module outline we will have a brief introduction to the finite element method, the way it developed and different types of it.

We will have a look at one particular type weighted residual formulation based finite element method and then we will have a brief look at the so called variational formulation, we will look

at what are finite element interpolation or shape functions and the numerical integration schemes which are used in finite element analysis and towards the end, we will apply this finite element method to one example problem that of heat conduction.

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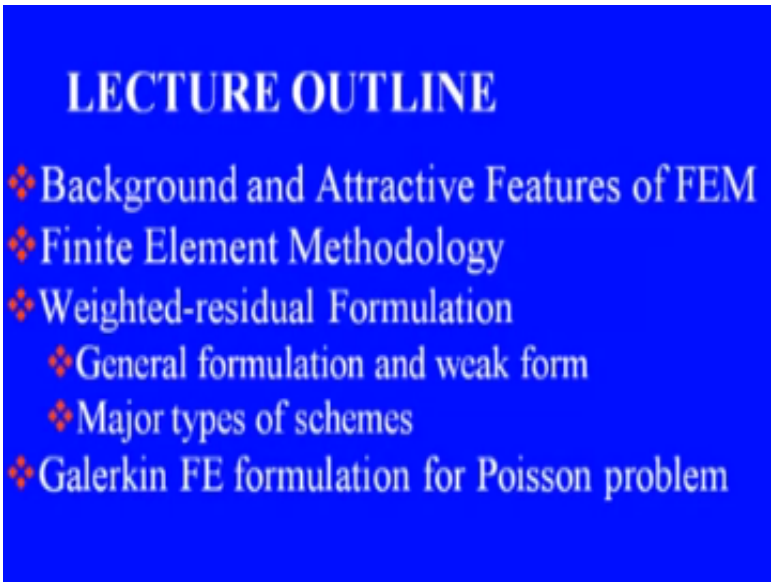

Now, in the first lecture, we will have a brief introduction to finite element method, we will have a look at the background of the finite element and its developments and for attractive features which have made this method so important. We will have a brief look at finite element methodology and then we will discuss weighted residual formulation, we will look at the general methodology of weighted residual formulation and what are known as strong and weak forms.

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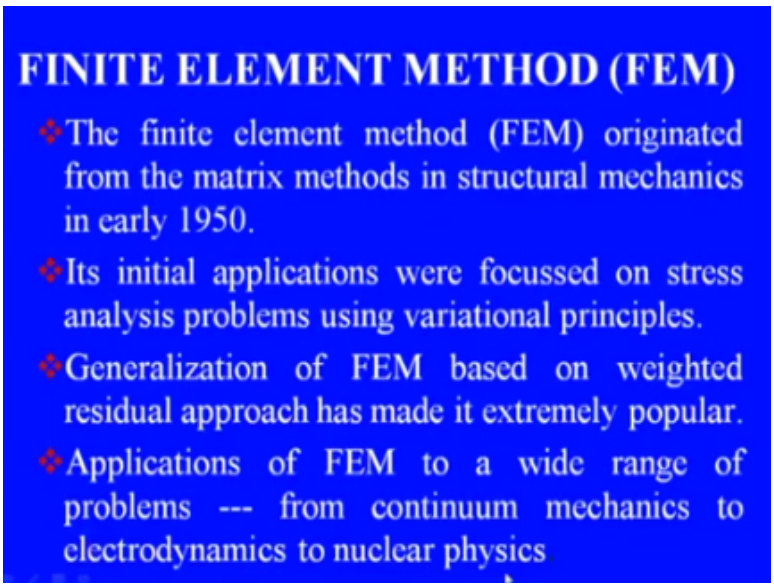

And what are major types of schemes which we can obtain using weighted residual formulation finite element is just one of them and then we would consider what we called Galerkin finite element formulation for Poisson problem to illustrate the application of weighted residual formulation in finite element analysis. Now, this finite element method, it originated from matrix methods in structural mechanics in early 1950s.

And it was popularized initially by civil engineers, the most significant contribution came from the two sides of the Atlantic Professor. Olgierd Zienkiewicz at Swansea and Professor (()) (3:28) they took the matrix method with new heights of what we call the finite element method and its initial applications were focused on a stress analysis using variational principles because thus were the major originated in civil engineering applications on analysis of the structural problems where this and this analysis of stress is what is important.

So, the initial applications were entirely focused on stress analysis. Now, later on this method was generalized based on weighted residual approach and this approach has made the finite element extremely popular in a broad spectrum of computational physics not just computational and mechanics and we will see very shortly for in, that so and it had made application of finite element possible to avoid range of problems from continuum mechanics to electrodynamics to nuclear physics.

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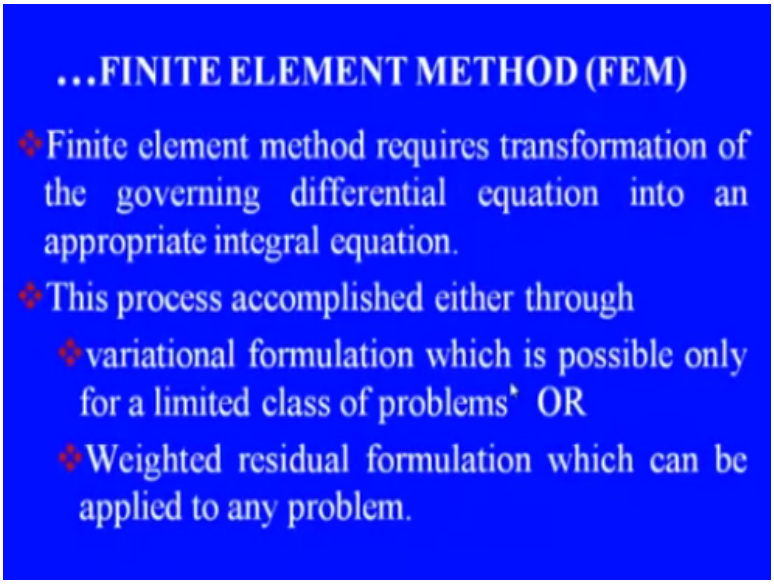

Because it gets a rid of the restrictions imposed by the requirement of variation principles in one type of finite element formulation, so this is the method of weighted residual which you are going to have a detailed look today and how does this method work, in fact what we require here is; in finite element formulation, we would start off from the governing differential equation and we would transform it into an appropriate integral equation.

And most of it converted into an integral equation either through the variation formulation which is only possible for a limited class of problems where we have something like energy and we can look into variation of the energy minimize in principle or we can use a more general approach of weighted residual formulation which can be applied to any differential equation and use this approach to convert our differential equation into an integral equation.

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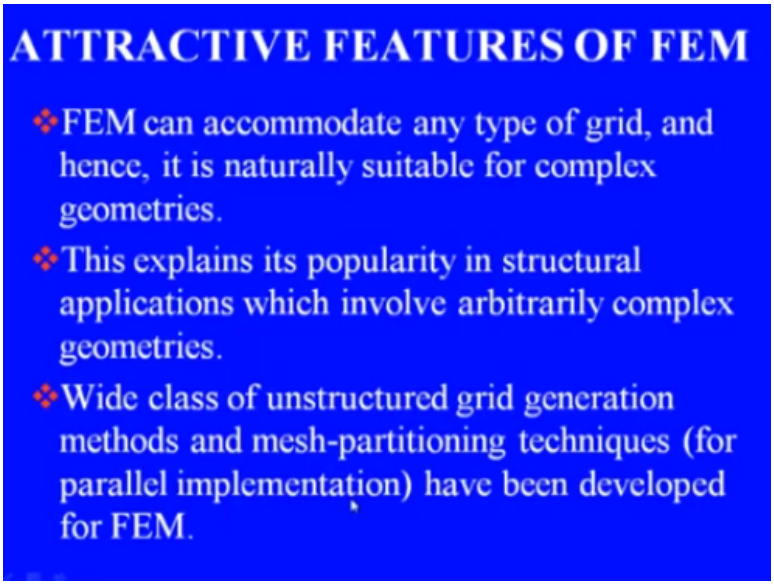

We required that this transmission simply because we would like to sum up, we would like to obtain this integral over the entire computational domain by summing up, the integrals over a small small finite elements which constitute of whose super set is our complete computational domain. What are the attractive features of this method? A finite element can accommodate any type of grid, we can have a structure grid, we can have unstructured grid.

We can have finite elements of any shape and size, we can have triangular elements, we can have quadrilaterals, we can have curved collateral elements and their analogues in 3 dimensions, so

this would makes this method extremely suitable for complex geometric and that is the reason why this method became extremely popular in the structural applications which involve arbitrarily complex structures.

So, in geometric complex, instructional applications this method can be easily applied there and it has become very very popular and to deal with complex geometrics wide class of unstructured grid generation techniques would have been developed both for 2D applications as well as for 3D problems and in the past 4 or 5 decades, when the size of the problem is grew, we were not able to solve them in a single machine.

And what was required is to work on massively parallel computers, the requirement came for partitioning work problem domain into smaller small subdomain, so in the case of unstructured finite element analysis mesh partitioning techniques were developed, you can find many of them available freely for download, you can download it, put your complete mesh and they will partitioning for the required number of parallel process structured.

So, we have got wide class of both grid generation methods and as well as mesh partitioning schemes which are now available for parallel implementation of finite element method. Now, these both of these categories if software they are available in public domain, that is freely available or they are available as commercial software. So, depending on your budget and availability you can have any of them and these are available on almost all conceivable platforms; windows machines, Linux based machines, Max or UNIX based machines.

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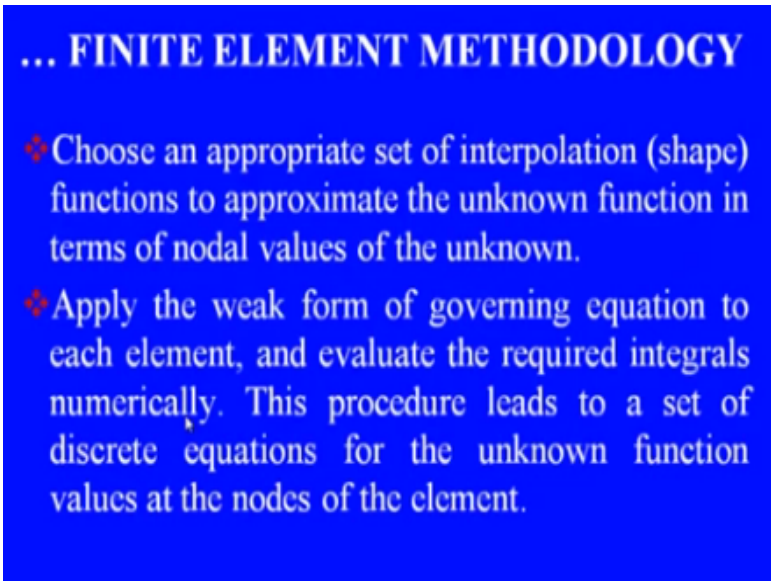

Now, let us have a brief look at the overall finite element methodology. So, as I mentioned earlier what we need to do first is to obtain an integral form of an equation, so we would imply either a variational formulation or a weighted residual formulation to transform the governing differential equation into an integral equation and we would get what is called a global weak form. What is weak form? We will have detail little later.

So, the starting point of finite element formulation is the global weak form, we call it global because it applies over the entire domain and then next we will now break our global domain or discreditable domain by a mesh that is to say we would divide our problem domain into a set of non overlapping finite elements and then once we have got our mesh discretisation depending on the element type which have chosen with that rectangular elements, quadrilateral elements.

And how many computational nodes, we want to keep on each of those elements, we have to choose an appropriate set of interpolation or shape functions to approximate the unknown function in terms of the nodal values of the unknown for a given element and that can be used in the weak form, so we will apply the weak form of governing equation to each element, evaluate the required integrals analytically possible which will happen if your geometric is very simple.

And we are dealing with regulars shape elements otherwise we will follow numerical integration. So, once you have done the integration analytically or numerically as the case might be, we will

obtain what we called discrete equation for the unknown function values at the nodes of the element, so for each element, we will get a set of equations for the unknown function at each of the computational node of the element. So, this is what we will call as element level equation.

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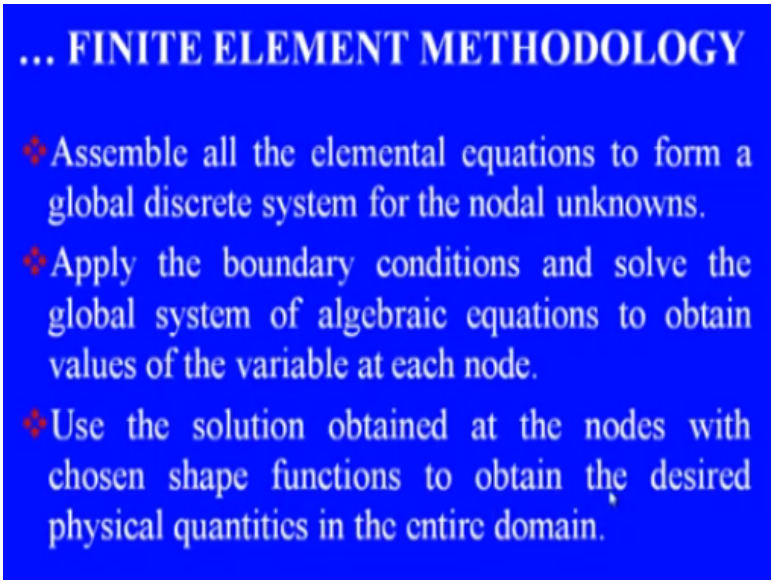

Next, what we do in finite element is; a symbol all elemental equations, so a small matrix equation corresponding to each finite element they would be assembled in the global discrete system. This global assembly process is very important and it will depend on the nodal connectivity from one element to another element, so this assembly process will lead us to a discrete system equation for all nodal unknowns.

And once they have got this discrete system, we would apply the boundary conditions and solve the global system of algebraic equations to obtain the value of the variable at each node. So, these 2 steps similar to what we have already done or these steps rather common to almost all discretisation is here be the finite element, finite difference or finite volume schemes that form the global system of equations apply boundary conditions.

And then solve the resulting system equations, once you have obtained the values of variable at each node in the case of finite element, we have got the shape functions and in the terms of shape functions, we can obtain value of the variable at any point in element, we can differentiate the

variable to obtain the gradients to say for instance, if you are interested in finding out the strains in structural analysis or strain rates in fluid analysis to get our stresses in post processing phase

Or if you are solving heat conduction problem you want to find out what heat flux are. So, we can differentiate let us say temperature obtained in the gradient of temperature in terms of shape functions and the nodal values of the temperature and we could obtain the physical quantities of interest to us in the entire domain. So, that is in nut shell, finite element methodology as well as very very similar to what we had in the case of finite difference on finite volume method.

Few steps were rather common in each case. Now, let us come to this most popular finite element methodology which we adopt which is known as weighted residual formulation. Now, as for this weighted residual formulation is concerned it predates finite elements, they were used long long ago to solve what we call global level approximate solutions but they have been popularized by the development of finite element method later on to generalise the finite elements to solve wide variety of problems.

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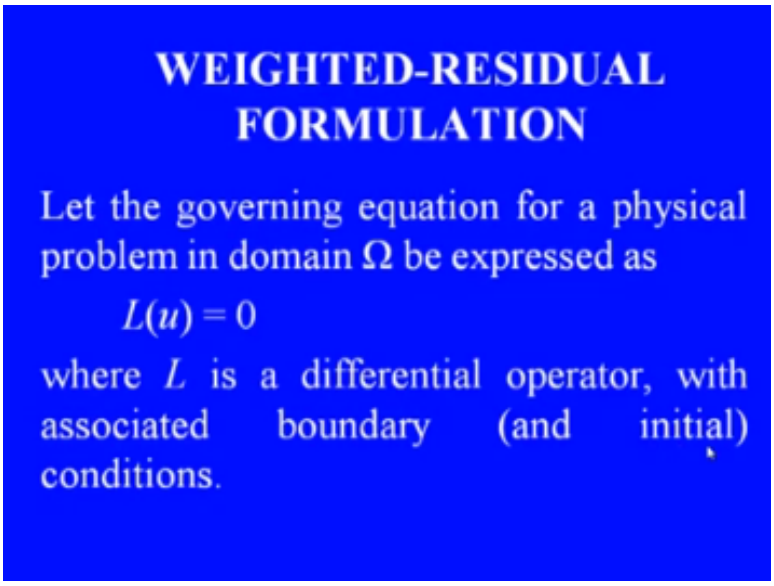

So, let us consider a genetic problem, okay let the governing equation for a physical problem in a domain omega; omega could be 2D or 3 dimensional domain and let us express it as Lu=0, where u is our unknown function and L is an operator, okay, L is the differential operator for

instance, if you are dealing with our heat; steady state heat conduction equation then this L would be our Laplace operator.

And in addition to the governing equation to define this problem fully we would have associated boundary and initial conditions. Initial condition would require, if you are dealing with time dependent problems. So, this is the problem which we want to solve, it is not possible for us to solve it analytically, we cannot get its close form active solutions, so we want to solve it approximately, we want to obtain an approximation to our problem variable or our function u, okay.

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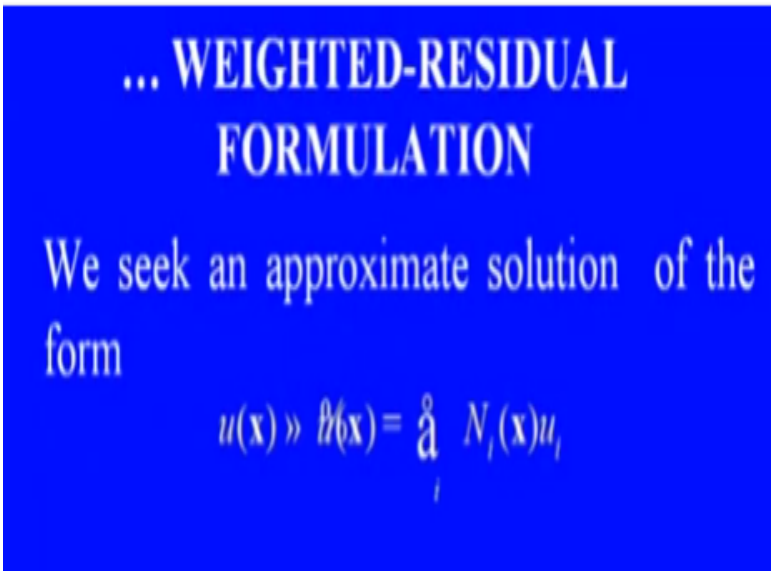

Now, let us seek an approximate solution of the form (()) (14:23) bit jumbled up.

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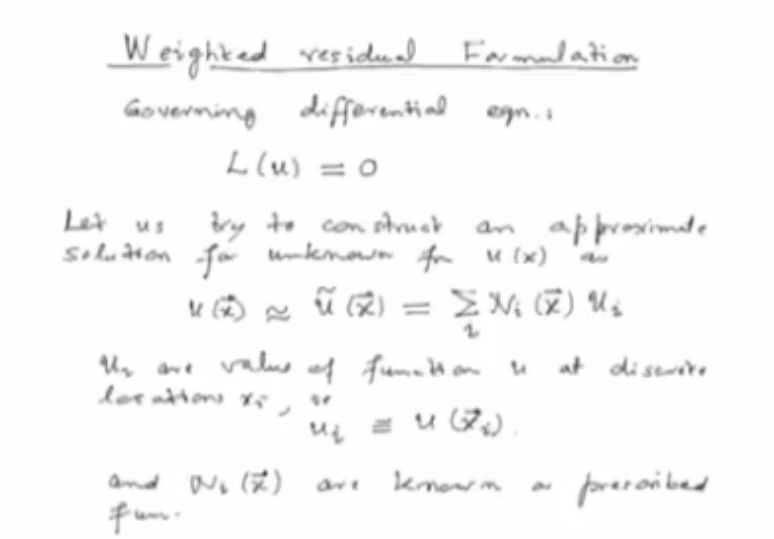

So, let us move on to our board, weighted residual formulation, so we had our governing equation and governing differential equation, by governing equation here, we mean the partial differential equation which governs as a certain conservation principle for a given problem, so we had expressed it as L of u =0. Now, what we want to do is; in the case of; we want to find out an approximate solution.

So, let us try to construct an approximate solution for unknown function, u of x that is ux is our exact solution which we do not know, so you want to approximate in terms of u ~x and this u~x is given by this sum; Sigma i Nix ui, where our ui is, are the value of function u at discrete locations xi, that this we have you the simplest symbol that ui is value of u at point xi and these Ni is; these are known functions; known or prescribed functions, okay.

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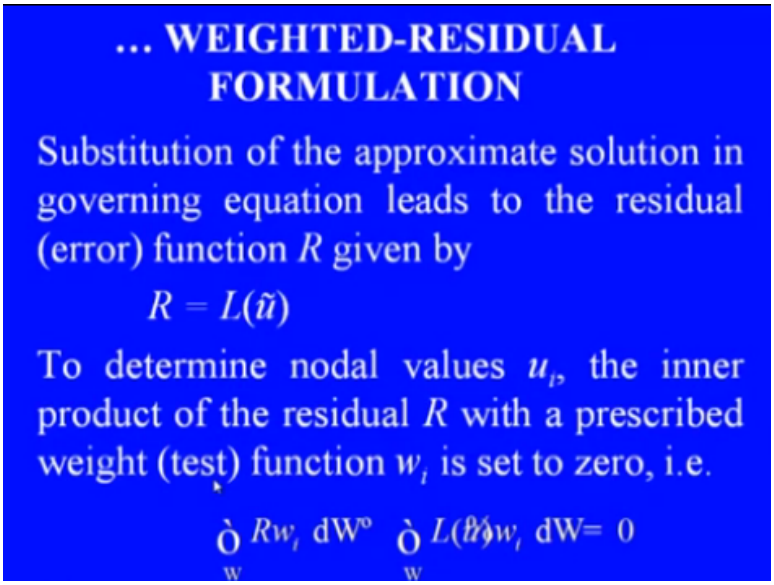

So, this is the starting for the weighted residual formulation okay, so next what we will do is; we will substitute the approximate solution in governing equation and that will lead us to a residual function which with denote by symbol R and to determine the nodal values ui, what we will do is; we will take the inner product of the residual with the prescribed weight function or what we called test function and will set it to 0.

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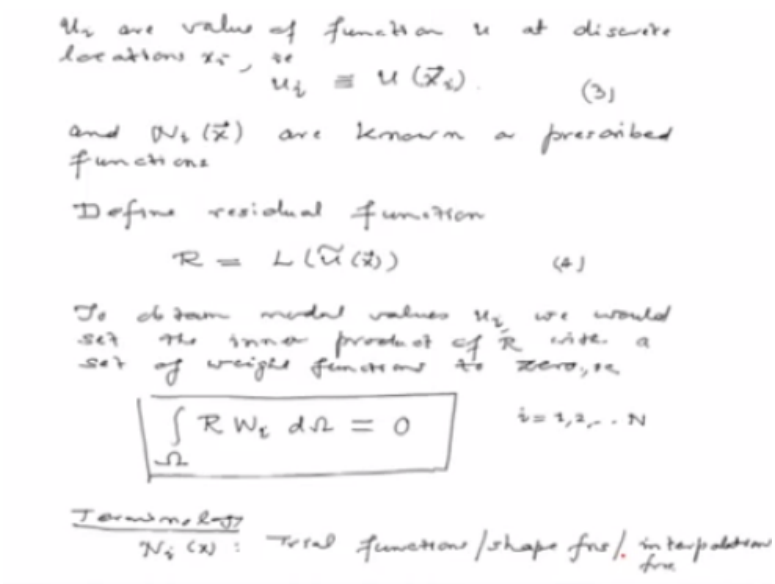

So this, what we would do you in our finite element formulation. So, let us get back to a board, we will define residual function R as L u~x, u~ is an approximate solution it is not in a exact solution, so when we substituted it in our governing eqaution, let us number it as 1, this 2, 3, this is 4, so once we have substitute it, the approximate solution we will get some residual, we will

get some error, so this also called as error function but since error function got a different connotation mathematics, we will use the term residual function.

Now, how do we obtain a solution? Thinking would be that look if u ~were very close to our exact solution then this value of R should be as small as possible. So, do that minimisation, what we will do is; to obtain nodal values ui is, we would set the inner product; while minor product we means the integral, inner product of R with a set of weight functions to zero, that is we will multiply R with a weight function integrate it over the problem domain and set it to zero.

And of course, we will have to choose as many number of weight functions as many number of discrete values are there, so I should go from 1, 2 so on N. Now, I would just like to recreate the terminology which we have used here; the function N ix, these we called a trial functions, shape functions or interpolation functions, so these are the 3 names, they are used interchangeably in computational mechanics.

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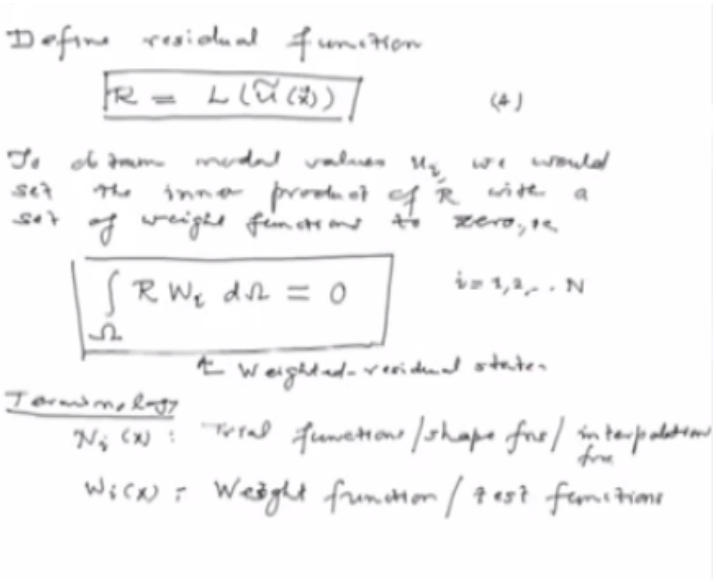

These weight functions what we called W,s that two widely used terms or names are weight function and other name which we uses test functions. So, now we can get hold of why we call this process as weighted residual, we had calculated the residual function R and then we had multiplied with a weight and integrated it over the domain and set it to 0. So, this particular

statement which we written RWi d omega integral omega = 0, this is our weighted residual statement.

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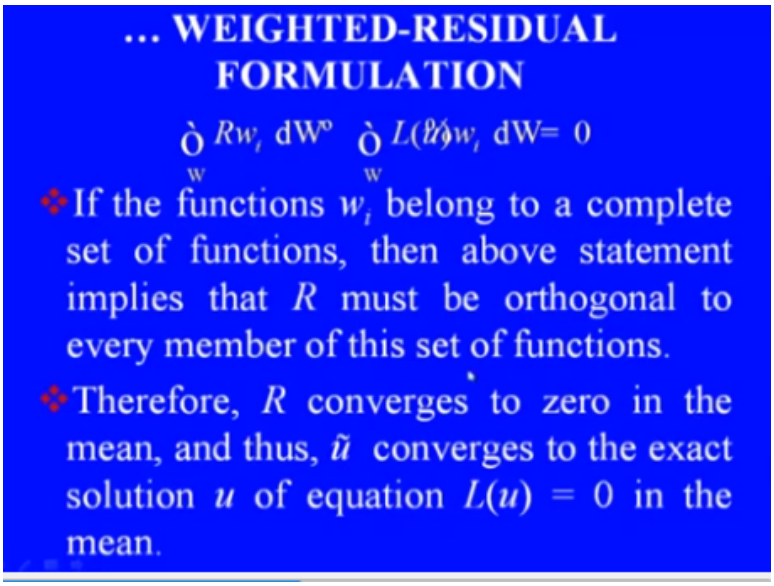

And this is what is going to help us solve or help us obtain an approximate solution to our problem and there is specific reason behind this weight residual formulation, what we say that if the weight functions Wi they belong to a complete set of functions, now this I would leave as exercise to you, look into your maths book and find out what do you mean by a complete set of functions that if Wi belong to complete set of functions than ever be statement which we had; R* wi integral over the domain = 0.

This implies that the residual R must be orthogonal to every member of the set of functions that is to say; our residual is orthogonal to each of the weight functions and therefore we use a specific terminology, we say that R converted to zero in the mean, that is what is integral size that we have made our residual function to almost 10 to 0 and there is specific theorems in mathematics, if you say that R converges to 0 in the mean then as a consequence, u~ that is your approximation du, which was expressed in terms of Nix * ui.

They are summation, that converts to an exact solution u of; the equation Lu=0 in the mean. So, this was the basic guiding principle of weighted residual method that if R is made to converse to zero by choosing a suitable set of weight functions then we can make our approximate solution u

~ to converge to exact solution in the mean. Now, depending on the choice of weight functions we can get a variety of solution methods; let us have a look at some of them.

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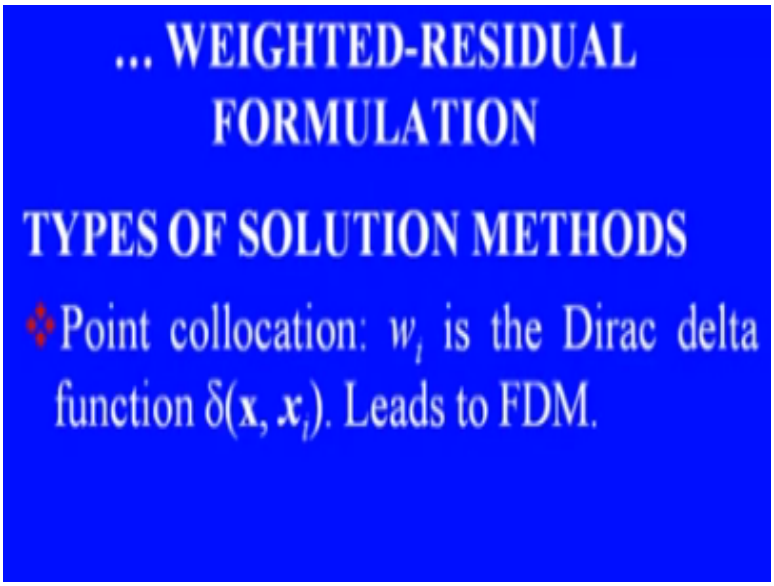

This one is what we call point collocation method; wherein our weight function is Dirac delta function and we will illustrate what does it mean? It leads to what we call finite difference method. Let us see there is in connection between this choice of our weight function and the finite difference method which we dealt with earlier.

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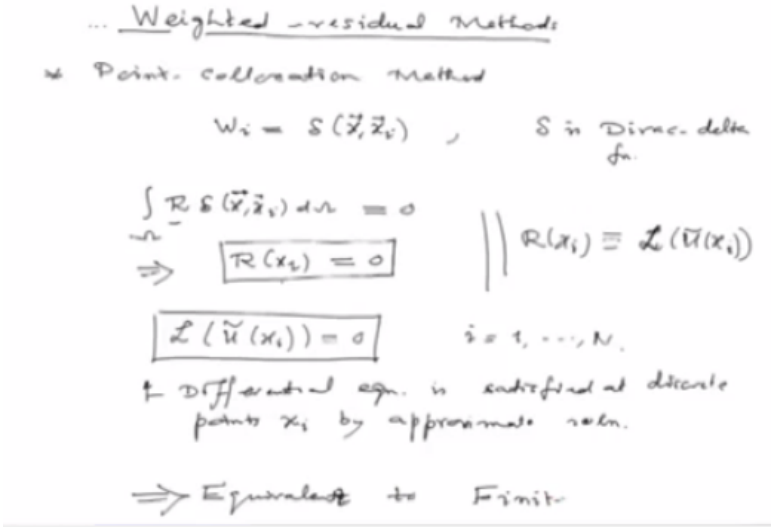

So, we are dealing still with our weighted residual methods and the first ones which are made in the case of point collocation method, why this method is called point collocation? That would

become obvious now that we have chosen wi to be delta xxi, okay this delta is the Diroc delta function and it has got a very beautiful property, this diroc delta function that it is nonzero only when x=xi and elsewhere takes a zero value.

So, once we put it our weighted residual statements, then this R times delta xxi d omega = 0, this leads to R at xi to be 0 this is referred to the substitution property, the operator, okay the moment we have this integral anything; with any function which you multiply with the delta function and integrated over a given domain that accumulate integral; or as the resulting integral adjust the value of the function at the point xi and what was this Rxi?

Our R at xi that was basically; so our operator L u ~xi, so effectively what we have done now is that we have just set at L u ~ xi =0 at a set of points that i=1, and so on N. So, we here our differential equation is satisfied at discrete points xi by approximate solution and if you recall that is what we did in the case of finite difference formulation that is to say we had chosen a set of grid points and at each point we said at look, we are going to set our differential equation to 0.

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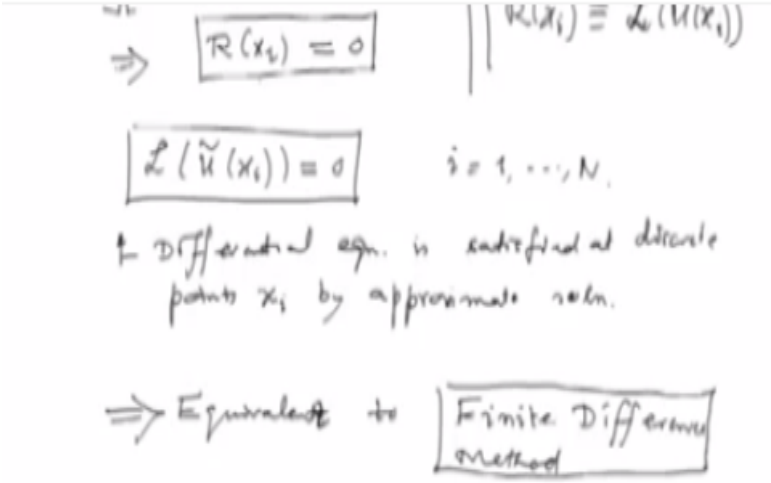

And all the differential operators would be replaced by the corresponding finite difference approximation. So this; why this point collocation method is equivalent to finite difference method. So, one feature of weighted residual method is obvious now that this represents a much

more general class of methods compared to a finite; volume finite difference or just finite element method.

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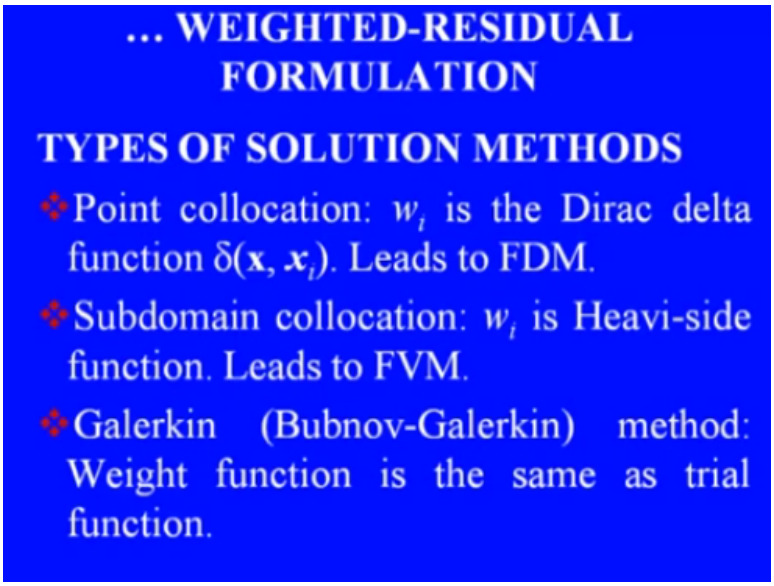

We can obtain a variety of methods based on our weighted residual formulation. The next one is what we called subdomain collocation, where in our weight functions wi would be would be what we called heavy side functions and this leads to a method which is analogous to the finite volume method which you have learned earlier and before we proceed to have a very detailed look at our subdomain collocation, just have a look at yet another method will just give its name, its detailed discussion will do later on.

Galerkin method or Bubnov Galerkin method, because this method was proposed by two gentlemen separately, they revoked on, they obtained this particular method. The weight function is the same as the trial function, a trial functions Ni, which we have used. If you the same functions of our weight functions in our weighted residual formulation, the resulting method is referred to as Galerkin method or Bubnov Galerkin method.

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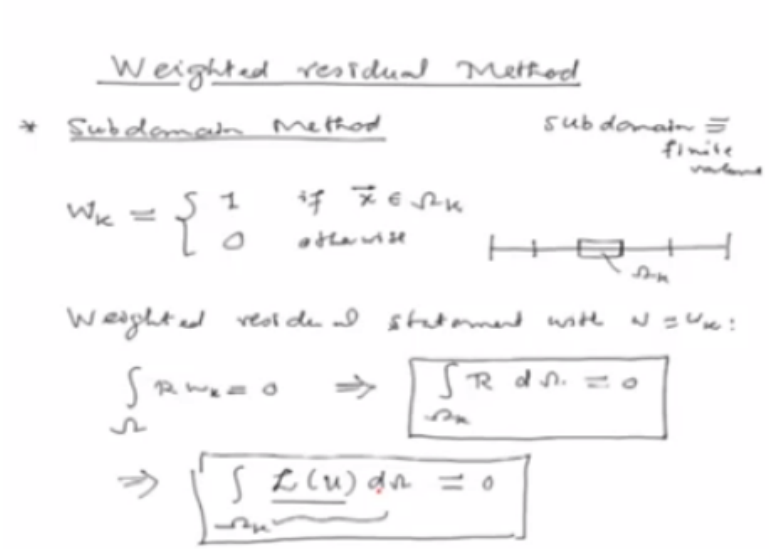

Now, let us see how this subdomain collocation method gives us something similar to the finite volume method. See it for what we do is, let us take a simple case of one dimension domain to state this problem sorry this methodology. So, the problem domain we can break it into smaller small subdomains that is what we did in finite volume method, we broke our big problem domain into small small finite volume.

So, this subdomains are basically similar to what where our finite volumes and if we pick up one specific subdomain, let us call it omega k, our weight function which we can; let us say this k is weight function, this would be defined at this wk=1, if variable x lies in the subdomain omega k, it is 0 otherwise, so now what will happened to K weighted residual statement, so weighted residual statement with w=wk twice.

So, integral over omega are wk, so what we had set to zero, wk is zero elsewhere accept that it is equal to 1 in omega k, so this is equal to omega k, R d omega=0. Or in other words, if we substitute for R, this is equal to Lu d omega =0 over omega k, so this is something fairly similar to what we did in one way of finite volume analysis, we said we can obtain the integral equations for finite volume analysis in 2 ways.

One is, by using the integral form of conservation equation, if that is not available what we said? That we will take our differential equation integrated it over the finite volume that is what we are

doing here, we are just integrating it over the finite volume and that will give us the desired integral equation, we will integrated further by parts to get was surface and volume integrals in terms of the appropriate quantities.

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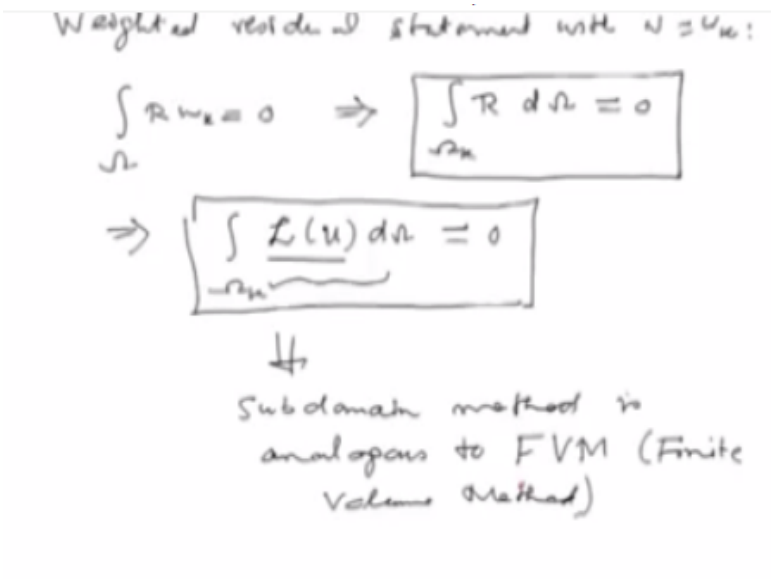

So, this tells us this subdomain method is analogous to FVM that is our finite volume method which we had already discussed in previous module. Since three of these, what is the next one?

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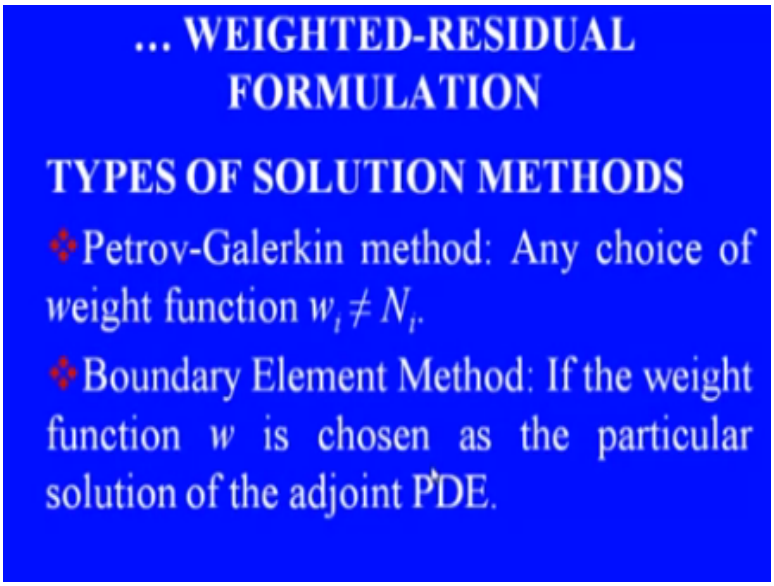

Now, this one terminology which is used for a set of methods called Petrov galerkin method and there could be wide variety of such methods, there is only one restriction have that the choice of weight function could be anything other than out trial functions or interpolation functions, for

this is a contrast here with Bubnov Galerkin method, in Galerkin method or what we call as other name as Bubnov Galerkin method.

We chose weight functions to be the same as interpolation functions, anychoice other than, that wi is not equal to Ni would theoretically lead to what we called Petrov Galerkin method, so in that way our point collocation method or subdomain collocation method all of them, they would fall in this wide category of methods called Petrov Galerkin methods and the next we will discuss next, there is also theoretically fall into this category.

Now, the next one is what we called boundary element method and here the choice of weight function is very explicit, it is chosen as the particular solution of adjoint PDE and that converts our integral form of the weight residual equation which was over the entire domain into a boundary integral and that is the reason, this method is referred to as boundary element method or boundary integral equation.

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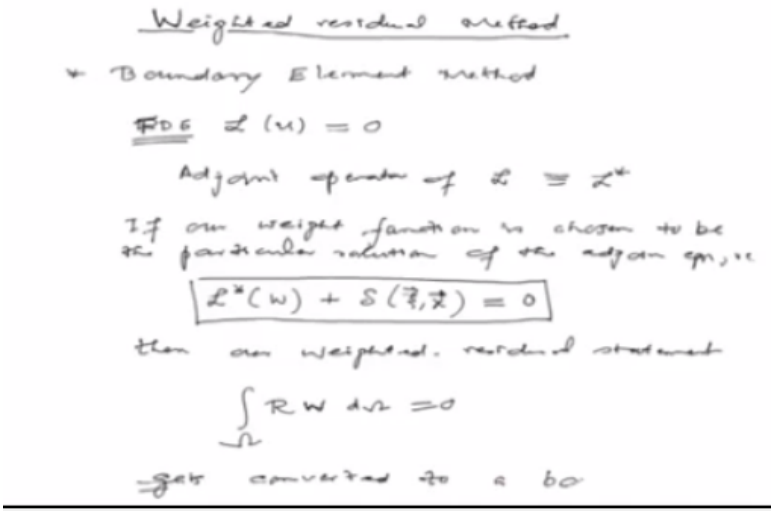

So, now let us have a detailed look at this method, so we have weighted residual method and we are looking at a specific choice which leads us to what we call boundary element method. So, the equation was L u=0, this was our PDE, L is a differential operator and we will denote by the symbol L*, at the adjoint operator of L, this we are going to denote by symbol L*, so in this case

if our weight function is chosen to be the particular solution of the adjoint equation that is; L*w + delta xi x=0, so our weight function is the solution of this equation.

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So, then our weighted residual statement which is a domain integral because we had this R*w d omega =0 this gets converted to a boundary integral equation, which we can write it as cxi uxi + integral over the domain boundary gamma, so this was our problem domain omega; gamma denotes the boundary and we will have a set of differential operators Fu Gw d gamma =0. Now, this operators F and G; differential operators F and G would depend on operator L.

Now, the beautiful feature about this method is that instead of dealing with a domain integral, we need to only evaluate or we needed to work with what we call the boundary integrals, so in 2D basic our problem would become equivalent to solving in one D problem around occur, if it dealing with 3D problem, instead of handling 3 dimensional integrals, we need to only have our surface integrals, we need to generate a mesh only on the boundary surface of the problem domain.

So, that is why some people claim that boundary element method leads to a dimensionality reduction by one, since some structural mechanics application this method is very very attractive to use. If you want to; if you are interested in further details of this method, you can look into

appropriate books on boundary element method; there are many books, textbooks available now on boundary element method.

There are certain limitations though that this particular solution is available only for the case where L is a linear operator. If it is a nonlinear operator, we cannot get its adjoint operator and its particular solution in that case will have to work with what we call the particular solutions for linearized equation and we will not actually get a boundary integral equation that equation will be in terms of boundary plus some domain terms.

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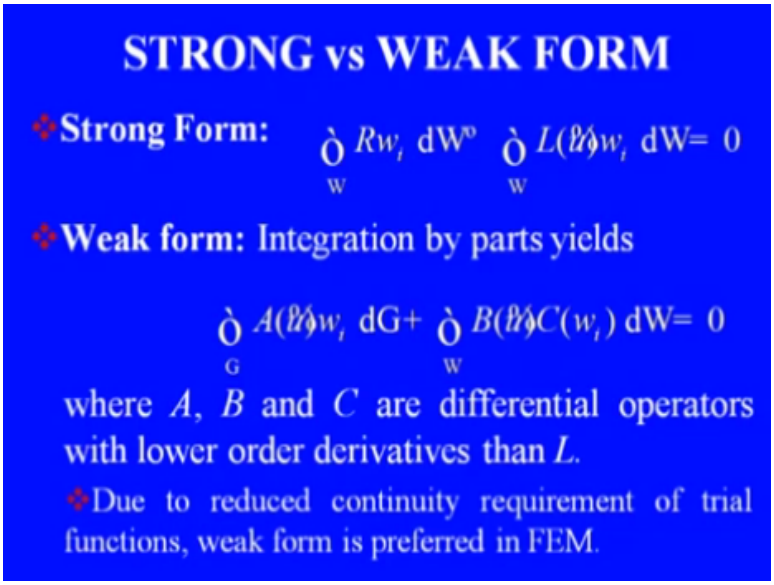

So, you now look into further details on this method, interested readers can pick up any book on boundary element to satisfy their curiosity. Now, the two terms used; strong form and weak form now, why do we call them is strong form and why do call them weak form. Let us have a detailed look at our board.

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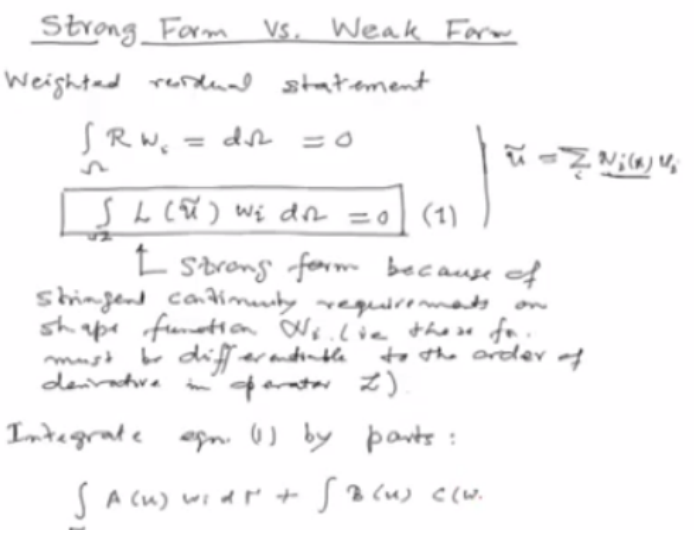

Let us get back to our weighted residual statement; R w d omega =0 or if you substitute for R in terms of the approximate solutions what we get, L of u ~wi d omega =0, so this particular form of integral equation this is referred to as strong form. Why we called as strong form because there are continuity requirements, so because of stringent continuity requirement on shape functions Ni.

Because remember that we have used R stands for sigma I Nix ui; ui are unknown coefficients Ni are specified functions, so that Ni should be as many times differentiable as the order of or operator L, so that is these functions must be differentiable to the order of derivatives in operator L, so for instance if L were our Laplace operator which involves second order derivative, okay our shape functions would also be differentiable twice.

So, that is a very strong requirement, so that is the reason this particular form it is called as strong form and there no such requirements are on continuity of wi, in fact we already seen few examples for instance; our point collocation method which was diroc delta function order, okay in subdomain method we chose wi=1, it was not a continuous function at all, so with a strong form there are no continuity requirements or save it on the weight function.

All the requirements on the interpolation function, in normal finite element analysis we would like to emulate the situation that is to say we would like to relax the continuity requirements on

the shape functions, so what we can do is for relaxing the continuity requirements that integrate our strong form by parts, so let us term this equation as 1, so integrate equation one by parts and that integration process will now lead to a different set of operators, okay.

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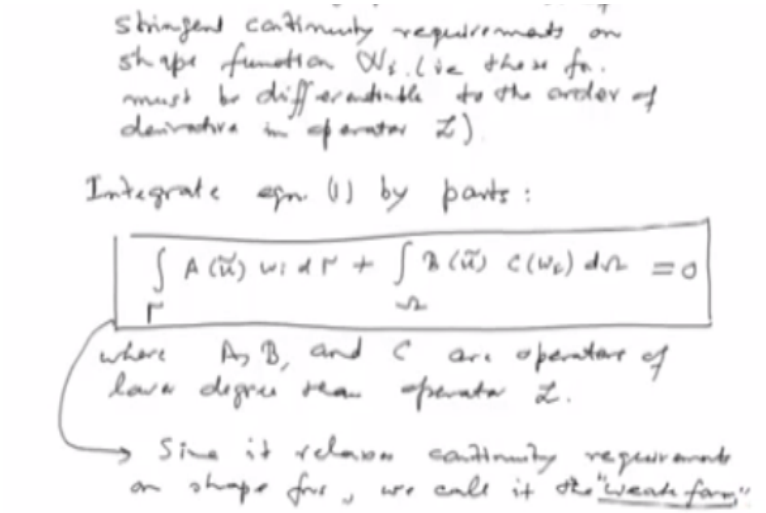

So, maybe we get some surface integrals plus Bu Cwi d omega, now here A, B and C are operators of lower degree than operator L, okay, so now what we have seen here that the requirements on a functional u ~ or that you say the differentiable requirements or continuity requirements for a shape functions have been reduced to a lower level and there are some continuity requirements which have been transferred onto a weight function, okay.

So, in this integration by parts relaxes, so since it relaxes continuity requirements on shape functions, we call it weak form and it is this weak form is what we use in our finite element formulation, okay that is what is there with a strong verses weak form that we will always integrate our strong form by parts and thereby would reduce the differentiability requirements on our shape functions.

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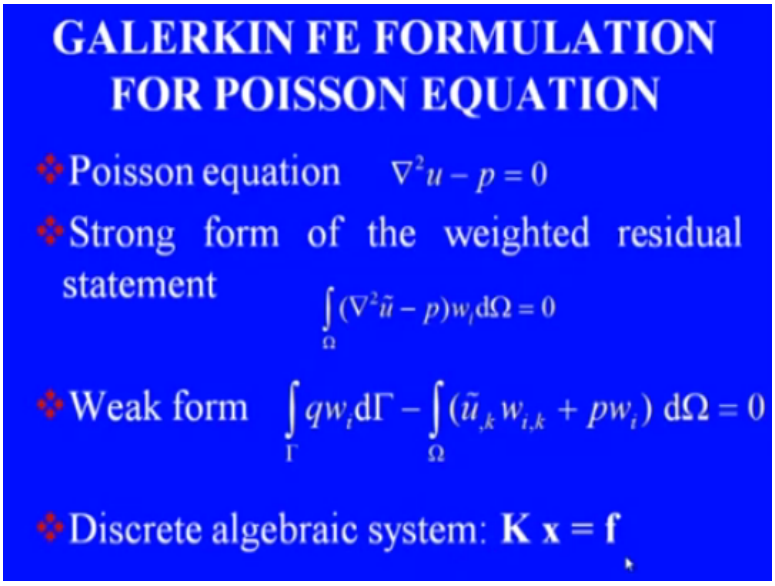

So, due to the reduced continuity requirement of trial functions, weak form is preferred in finite element method. Next, let us take a Galerkin approach that is we would come up with a finite element formulation based on Galerkin weighted residual formulation for one specific equation that is our Poisson equation, so we will have a brief look at the method, let us say the Poisson equation is given by del square u –p=0.

Now, this Poisson equation could represent our steady state heat conduction equation with volumetric heat generation is also encountered in electrostatics and in CFD, we encounter the secretion in solution of incompressible flow problems, where in we have to solve a Poisson equation for pressure, so this small difference between Laplace and Poisson equation; in Laplace equations, function P does not exist.

So, we will simply have del square u=0 that is why we call it at the Laplace equation and when we have got something similar to what we can call as source term, the equation is referred to as Poisson equation. Now let us try just this strong form with residual statement, the integral over omega del square u ~ -p wi d omega=0, so this is our strong form of weight residual statement, this is not what we are going to use for finite element analysis.

So, what do we do? Let us perform integration by parts, see we have got the second order derivative, once it perform the integration by parts, we get the first term which we are going to

get is a surface integral or boundary integral q wi d omega, let us q is the gradient of u that is in fact is delu/del L, so we have got del u/ del N wi times d omega - this domain integral of functions u ~ gamma k and this gamma k that stands for the derivative with respect to xk.

So, I think this notation, we have already seen when we discussed our considerations, we have just use that notation for the sake of simplicity, so u ~, k that represents du ~/dxk * wi, k which is a derivative of wi with respect to the xk plus p times wi d omega is equal to zero and now next what we will do is; we will introduce our finite element discretisation and perform or apply this weak form of our this element to obtain the elemental equations.

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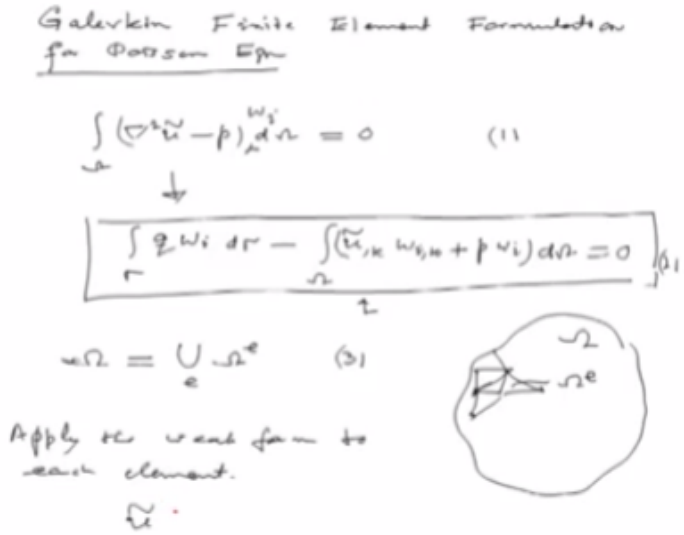

Collect all the elemental equations to obtain the global discrete algebraic system given by Kx=f. Now, let us have a brief bit further look at the something which we passed over a slide, so galerkin finite element formulation for Poisson equation, so we started off with a strong form; del square u ~- p d omega, this was required to 0, sorry multiplied by a weight function and we integrated it to get our weak form q times wi d gamma – integral over omega u ~ gamma k wi gamma k + p times wi d omega =0.

So, far we have just done the mathematical manipulations, we have been introduced our finite element discretisation, next what we will do is; this a problem domain omega, this we would represent by a union of these small small subdomains or finite elements, so our omega would be

represented as a union of element omega e, so any specific this layer this particular angle, one of this can be called out element omega e.

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And this weak form which we derived earlier, on this we are going to apply to each element, so apply the weak form to each element and we have to make use of few approximation that for each element how our u ~would be define? This would be defined in terms of the interpolation functions which are exclusive or which will depend on choice of our element referred to as 3 noded element.

So, we will have the submission going from 1 to 3, Nix ui at E, summation of I and our weight function wi they will be chosen to the same as is element shape functions, so with this bias substituted in our global weak form and then after integration supply the weak form for each element by substituting integrate for each choice of wi, you have to choose as many wi as there are nodes in the element.

So, we will sincerely get let us say we have got 3 noded triangular element, we will have a set of 3 questions, let us collect those 3 equations, so integrate for each choice of i to get what we call elemental equation which we can write in matrix form as Ke, ue =fe. In our next step, now how this elements of matrix ke, so kei sorry ke mn, this is given by integral over our element omega e Nm, k N of n,k d omega and fem this would be given by Nm p d omega over this element.

So, this was an equation for only one element, now you have got to collect algebraic equations for all the elements and that is why we have to collect all of them and perform a step what is called as global assembly. So, global assembly will give us the final system equations k * x=f, where x is our vector of unknowns u at each node and this global Kij, this would be obtained as Kij at each node, taking care of the nodal connectivities, we have to 2x, appropriately add the terms in the global matrix.

And similar this global load vector would be given by sigma e fe at fm component, so thus would we obtain our global system equations, once we have got global system equations, we can solve the system to obtain our nodal solutions, so this way we would put a stop to our discussions on the Galerkin finite element method. We will come back to it again and we will do each step in considerable detail for the exam; for one of the example problems which are going to take up in our section on the applications to scalar transport problems.

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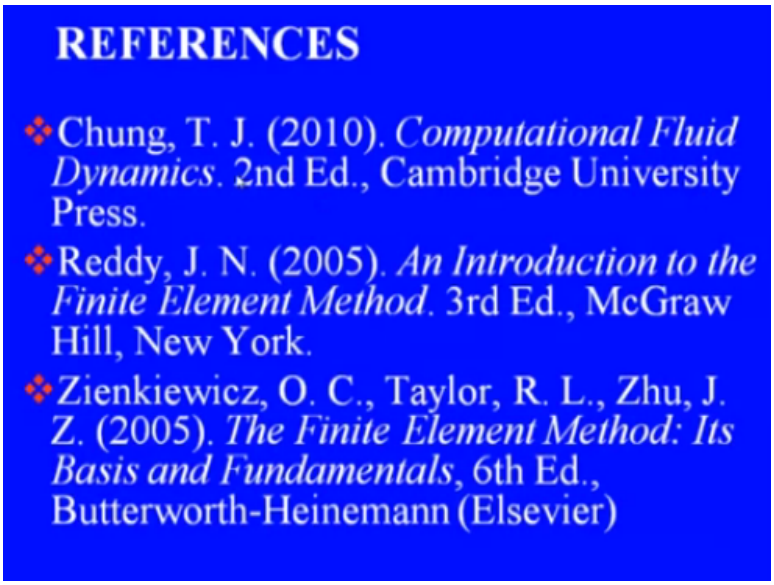

So, that will illustrate the steps involved clearly. For the time think I would like to refer you to few good books, at one of the most definitive books on finite elements is the one written by Zienkewicz, who is supposed to be the grandfather of the finite element method, so this book called the finite element method, it is basis in fundamentals. This is basically a set of 3 books which was originally written as one book in 1970s.

It is now going to a 3 set volume and this is the first volume of that set; the finite element method, its basis and fundamentals where in you can get the complete description of finite elements, the basics of finite elements. There are 2 other volumes, the one which is specific to structural applications and third one is specific to fluid applications, similarly this one book by J. N. Reddy on finite element method and even on this book on CFD by Chung computational fluid dynamics.

There is one part which is dedicated to application of finite elements to fluid problems. So, we will stop here for this lecture, in the next lecture, we will take a very strong methods and further description of shape functions and numerical integration.