Computational Fluid Dynamics
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Lecture - 29
Finite Volume Interpolation Schemes

Welcome to the next lecture in module 06 on Finite Volume Methods. In the previous lecture we had a brief introduction to finite volume method and we had a look at how do we approximate finite volume integrals. And in this lecture we are going to focus on Finite Volume Interpolation Schemes. And in the next lecture we will have a look at simplifications of Finite Volume Scalar to Transport Problems.

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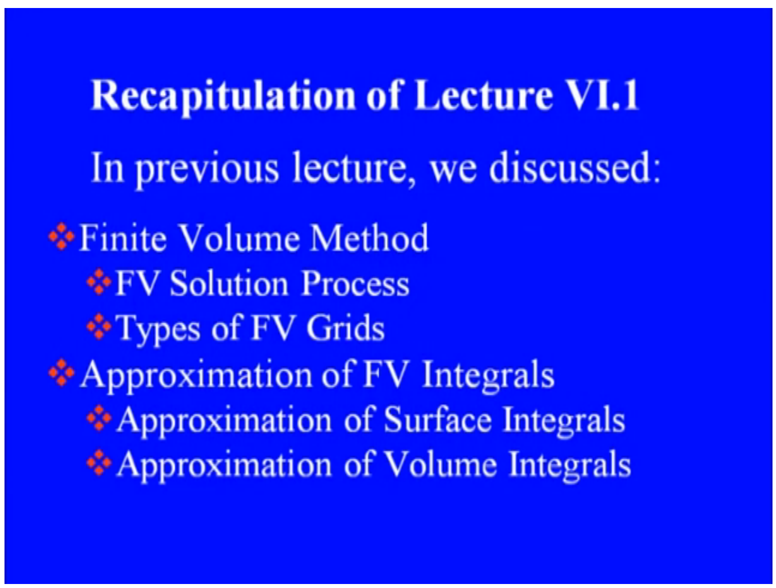

Now let us have a brief recap of what we covered in the previous lecture, we discussed the beginnings of finite volume method that is what is finite volume solution process, what types of finite volume grids are used. And we also had a look at approximation of finite volume integrals namely, the approximation of surface integrals and approximation of volume integrals. Now in current lecture we would focus on the interpolation schemes which are used in finite volume method.

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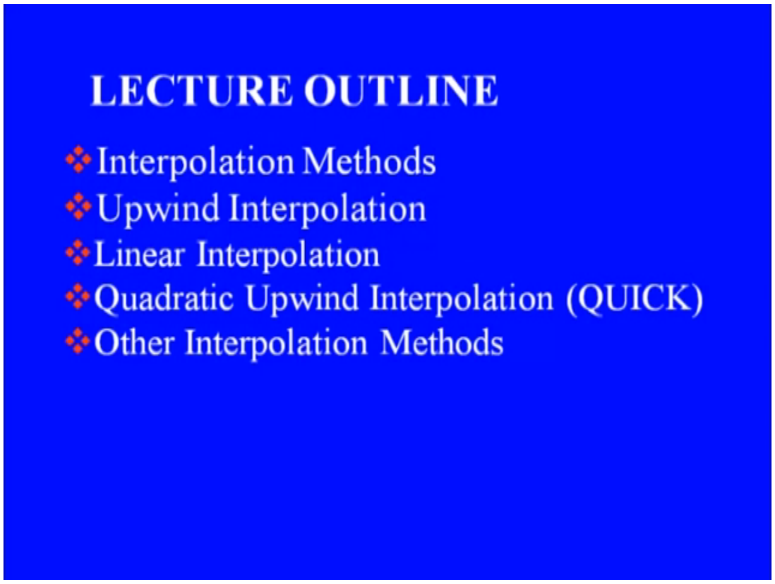

So brief outline of the lecture, we will have a look at why we need interpolation schemes and finite volume method. And then few most commonly used interpolation schemes namely, Upwind interpolation, Linear interpolation and Quadratic upwind interpolation. We will also have a very brief look at other recently used or developed interpolation methods, now why do we need interpolation in finite volume methods.

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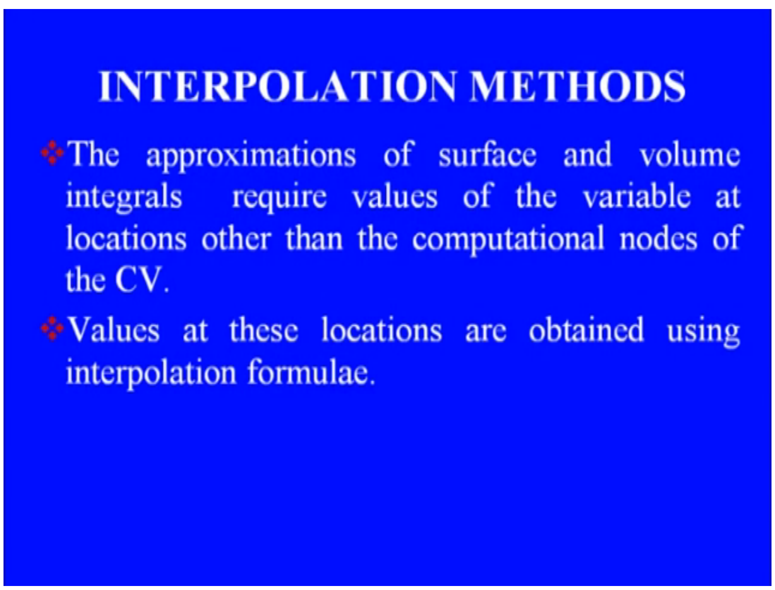

In the previous lecture we have seen that we need evaluation of surface and volume integrals in our finite volume formulation, but the surface as well as volume integrals they require the value of the unknown variable at locations other than the computational notes of the CV. So somehow we have to cover with some sort of schemes or mathematical approximation to compute the values at these locations, and that is why we need various types of interpolation formulae.

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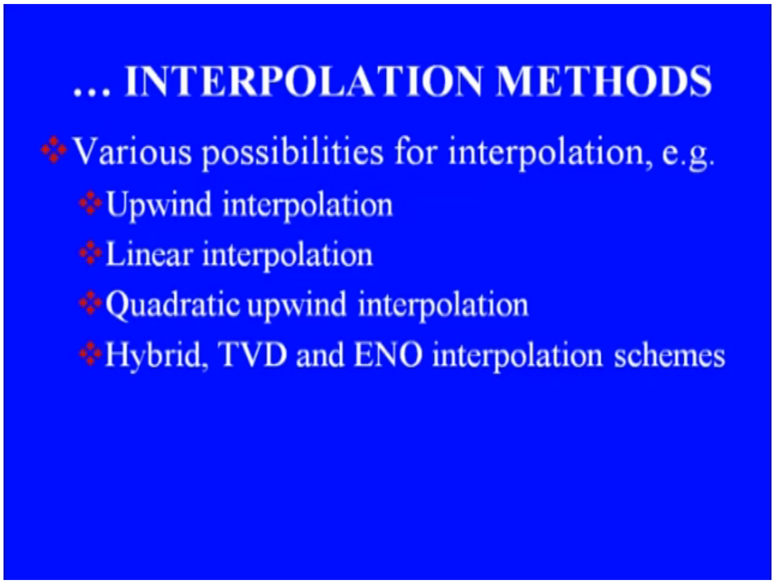

Now the various possibilities for interpolation includes upwind interpolation, linear interpolation, quadratic upwind interpolation and recently developed hybrid schemes, TVD that is Total Variation Dimensions schemes and ENO Essentially Non-Oscillatory interpolation schemes. Let us have a look at each of these schemes one by one in bit more detail. So first is our upwind interpolation scheme.

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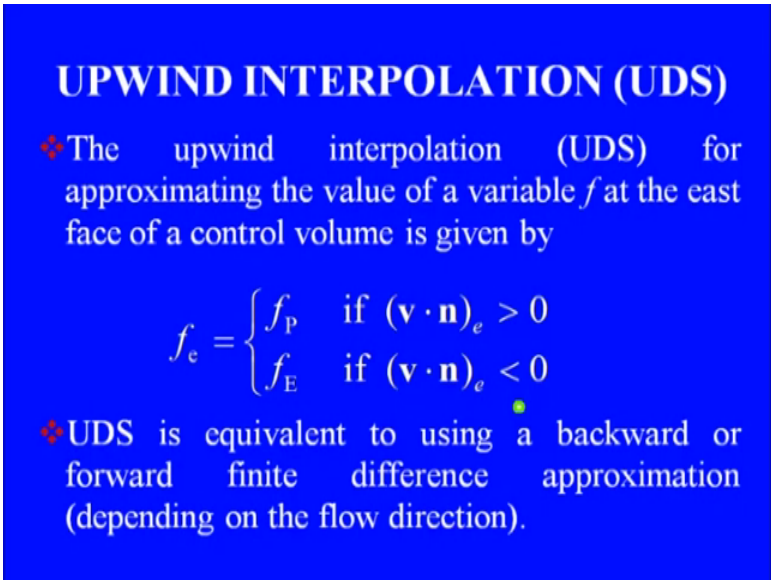

Here, we would require upwind interpolation scheme it is essentially used for what we called convection dominated problems, and here the velocity at a given point direction of that velocity that determines the variable values to be used in the interpolation process, the name upwind refers to that we are going to make use of the node which is upwind or upstream of the given node.

For instance, if we are at the east face of the control volume, now in that case for a function f we are going to approximated f is given by f at P where P is the point which is to the left of the face e if v dot n at point e that is the velocity component at point e in x direction that is pointing towards the positive x-axis that is when we get v dot n at e=0. And in case, the velocity is directed in negative x direction then our upwind node becomes the node capital E, so in that case f small e= f capital E if v dot n e<0.

Now this use of upwind interpolation we also call it UDS in short, it is equivalent using a backward or forward finite difference approximation depending on the flow direction, and before we look at the advantages and disadvantages of upwind interpolation scheme, let us have a look at why we say that it is equivalent to a forward or backward difference approximation. So let us move on to the bored and have a look at the characteristics of this upwind interpolation scheme.

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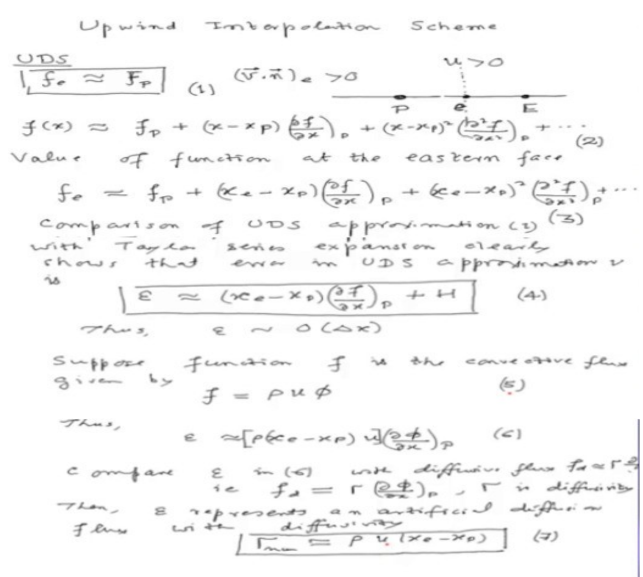

So let us first draw a schematic diagram, so it is our computational node capital P, node east to a capital E, and we want an approximation for the function f at the eastern face denoted by small e. Now upwind interpolation scheme suppose u velocity=0, so by definition what we had in UDS f at e would be approximated by value of f at point P why? Because now in this case our v dot n at e>0. Now in what way it is related to the first order difference approximation.

So once I have the difference approximation for that let us make use of the Taylor series expansion, so using Taylor series expansion the function value at any point near our computational node P this can be approximated as f at P+ x-xP del f/del x at P+ x-xP square del 2 f/del x square at P+ so on. So now what would we be the value of function at eastern node small e fe=f at P+xe-xP del f/del x at P+ xe-xP square del 2 f/del x square at P+ higher order terms.

So now let us have a look at the UDS approximation which we have shown earlier, this number equation as one the Taylor series expansion 2 and Taylor series expansion for the function value as equation 3. Now if you compare the equation 1 and 2, what do we say that error in our UDS approximation is comparison of UDS approximation 1 with Taylor series expansion, clearly shows that we have made an error, that error in UDS approximation is let us call this error is epsilon.

so this is xe-xP del f/del x at P+ higher order terms, now xe-xP that is half of our grid spacing, so essentially it is proportional to delta x where delta x is the grid spacing , so this error is proportional to delta x so this epsilon is of order delta x, and that is why we have that look UDS expansion is similar or UDS interpolation is very similar to our forward or backward difference approximation both of which have got an order of accuracy as one.

Now next, let us see what other things UDS approximation introduces, suppose f were a convective flux. So suppose function f is the convective flux given by it is f=rho phi, u is velocity component in x direction and phi is the scalar quantity whose flux f represents, so now in this case what happens to our error, that is our error e if u is assume to be constant this is xe-xP times rho times u del phi/del x at point P.

Now if you compare it with the definition of diffusive flux, so compare epsilon in 6 with diffusive flux normally diffusive flux let us call it fd this we would represent as gamma times del phi/del x, so you compare these 2 expressions our fd=gamma del phi/del x at P, gamma is diffusivity. So then what do we get? That epsilon essentially represents the sort of numerical flux numerical diffusion with numerical diffusivity.

So this epsilon represents an artificial diffusion flux with diffusivity gamma numerical, this given by rho u xe-xP. So in this sense use of this upwind interpolation leads to an scheme which is first order accurate that is what we got from the equation number 4 and in addition this scheme introduces numerical diffusivity or numerical diffusion in our solution process where numerical diffusion coefficient can be given by rho u*delta x/2.

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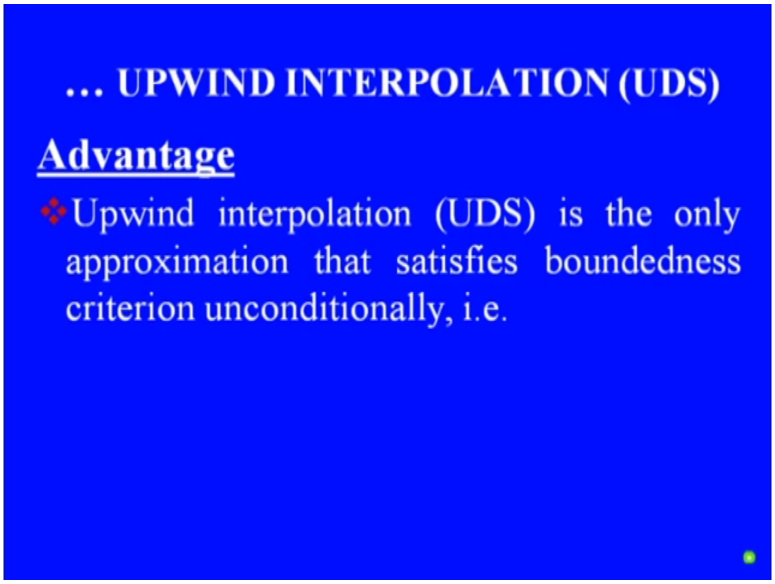

So now let us have a look at the further characteristic of the scheme, what is the advantages of this scheme? Now upwind interpolation is the only approximation that satisfies boundedness criterion unconditionally, unconditionally in the sense that respect irrespective of what the velocity values are and irrespective of what is the grid spacing the solution will always remain bounded that is this use of upwind interpolation will never yield oscillatory solutions.

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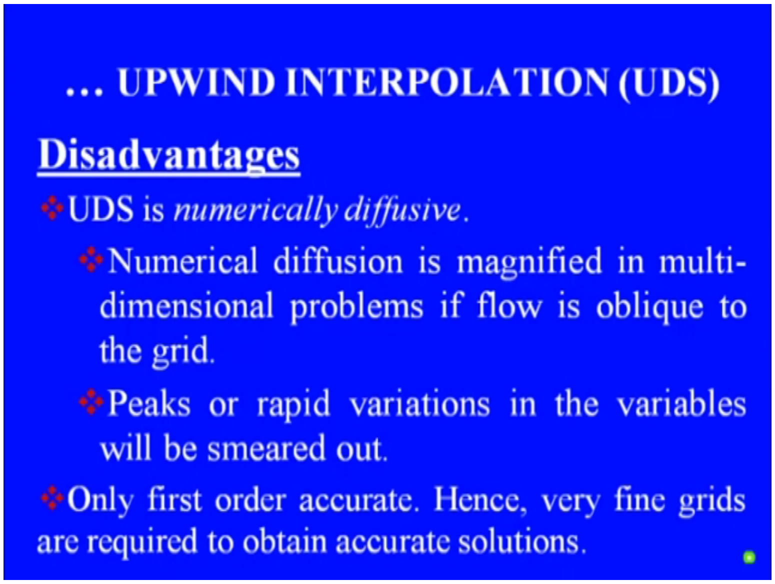

This advantages is coupled with disadvantages which we had already seen. The first one is numerical diffusive, if we have got sharp peaks or oscillatory values which are actually present in the solution those would be damped out because of the presence of numerical diffusion, so the numerical diffusion is magnified in multi-dimensional flows especially of the flow is oblique to the grid space.

And peaks or rapid variations in the variables will be smeared out because of the introduction of additional diffusive coefficient what we called numerical diffusive coefficient by the use of upwind interpolation schemes, and we have already seen that it is only the first order accurate. Hence, we require very fine grids to obtain accurate solution, but nevertheless the advantages with this scheme provides that is the use the unconditional stability specifically for advection dominated problems.

UDS scheme is very popular we can go for a very fine grids and we can rest assured that we will not get a solution which becomes unbounded.

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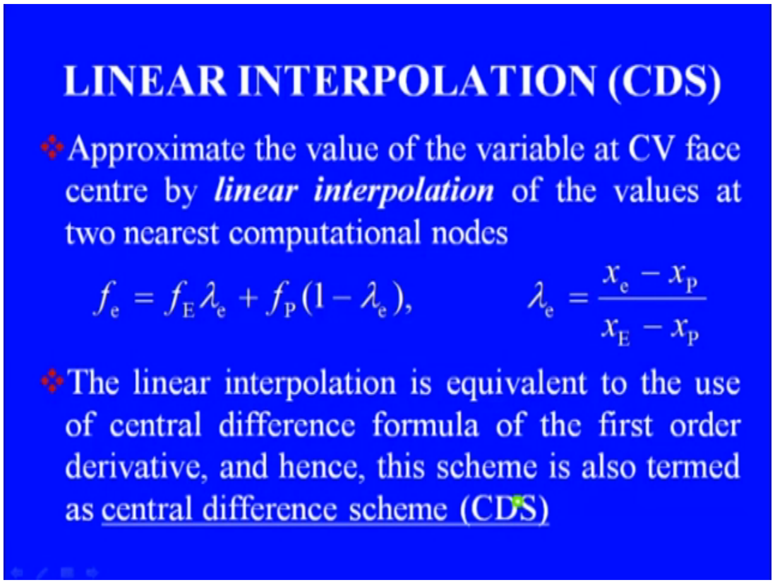

The next, interpolation schemes is also very commonly used is Linea interpolation and in short it is called CDS that is Central Difference Scheme, we will see very shortly what is the correlation between the central different scheme which we have learned earlier in finite difference module and the simple linear interpolation. So now here we have approximate the value of a variable at control volume face center by linear interpolation of the values at two nearest computational nodes.

So for instance if you are looking at the east face so function value at the east face that would be a linear interpolation for the function value at the capital E that is the eastern node and the current node P, and we use a waiting function a lambda e this lambda e would depend on the relative locations of the computational nodes capital E and capital P with respect to eastern face. So fe=f at E*lambda e+fP 1-lambda e where lambda is given by xe-x capital P/x capital E-xP.

Now this linear interpolation is equivalent to the use of central difference formula of the first order derivative. Hence, this scheme is also termed as central difference scheme or CDS, so before looking at the characteristics let us see mathematically let us derive what are the characteristics of linear interpolation scheme? Which make it identical to our central difference approximation.

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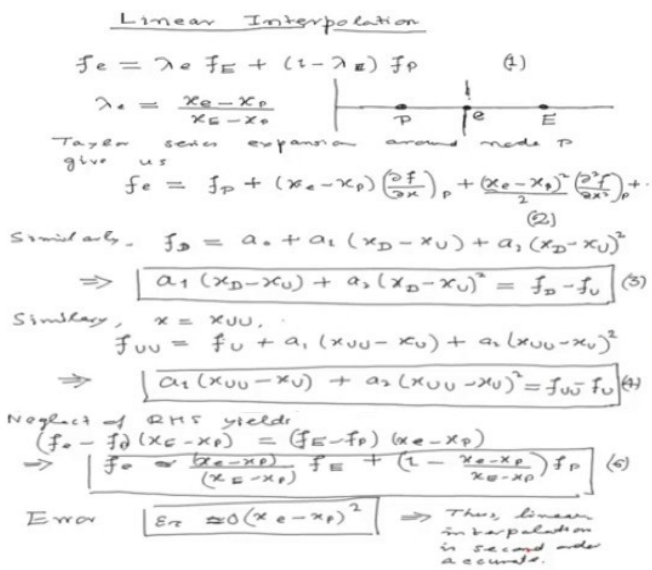

Now let us draw a schematic diagram, so x axis so computational node capital P computational node capital E these are boundaries of the control volumes, and suppose we are interested to finding out the values at the cell face small e and for small e we wrote the formula that f at e=lambda e times f at E+ 1-lambda E f at P, so this was our linear interpolation formula. And we had defined lambda e recapitalization which is note it down this was given by x at small e-xP/x capital E-x capital P okay.

Now to obtain the function value at node e first let us write down the Taylor series expansion in terms of the function value at our computational node capital P, so this Taylor series expansion around node P gives us f at e=f at capital P+ x small e-x capital E del f/del x at P+x small e-x capital E whole squared/2 del 2 f/del x square at P+ other higher order terms, so this is called as equation 2.

Similarly, we can also find out the function value at the function value at the eastern computational node fE, so function value at node capital E in terms of Taylor series expansion around node capital P, so this will give us f of E=f of capital P+ x capital E-x capital P del f/del x at P+x capital E-x capital P whole squared/2 del 2 f/del x square at P+ higher order terms, so this is called equation as equation 3.

Now let us eliminate the first order derivative terms, so that we can obtain an expression for fe in terms of the function values at equations 2 and 3, so now let us multiply equation 2 by x capital E-x capital P and equation 3 xe-xP and subtract these 2. So x capital E-x capital P times your equation 2- xe-xP times equation 3, this will yield let us see what we get fe-fP times x capital E-x capital P this is what we have from the second equation first term- f capital E-f capital P times xe-xP the first order derivatives is simply vanish.

And we would be left with the second order derivative terms so +1/2 del 2 f/del x square at P*xe-x capital P whole square*xe-xP from equation 2-x capital E-x capital P square*xe-x capital P sorry here we have got this is -, now these terms on the right hand side they represent an approximation in terms of these differences in mesh spacings, so now if you approximate or fitting or fit as an error and neglect it.

So neglect of RHS yields fe-fP xE-xP = f capital E-f capital P xe-xP and if you rearrange the terms here we precisely that our interpolation formula that is fe= xe-xP/ xE-xP times fE+ 1-xe-xP/x capital E-x capital P times fP okay. So now we compare the equation 6 with equation 1 these two are identical and what is error made in approximation error is what we have neglected, so let us called this as epsilon tau equal to rather this is of the order of your xe-xP whole squared this is our linear interpolation is second order accurate.

And we call it central difference approximations simply that if you want to find out what is the derivative of f at our eastern node e, if you work it out that derivative will be in terms of the value at 2 nodes on both the sides small e that is capital E and capital P, so the formula for del f/del x at e that would be essentially equivalent to the use of central difference approximation and that is the reason why? This linear interpolations is referred to be CDS scheme.

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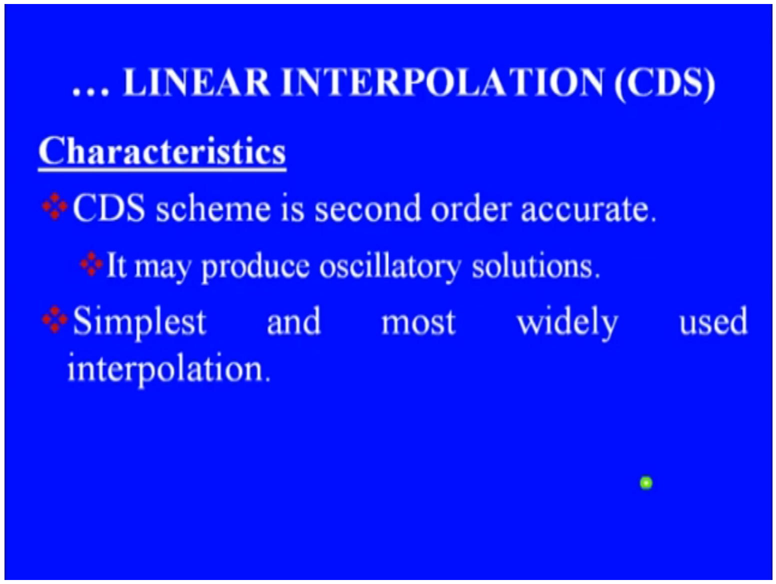

Let us get back to our presentation and look at the characteristics of scheme CDS is second order accurate, so from now onwards you refer linear interpolation by its short acronym that is CDS, it is second order accurate the accuracy is very good but we might have one small problem for advection dominated situations, that this scheme may produce oscillatory solutions, so that is the negative side. On the positive node it is simplest and the most widely used interpolation.

We need to worry about the oscillatory only if an the grid spacing is very large, with a small grid spacing we do not have to worry about the oscillation present and in that case CDS gives us a choice of a very good and very simple second order accurate scheme, so that is the reason why? It is used very commonly especially in the flow simulation which requires the use of finite spatial and temporal grid for instance, if you are looking at the simulations of turbulent flows using largely simulation or direct numerical simulation approach.

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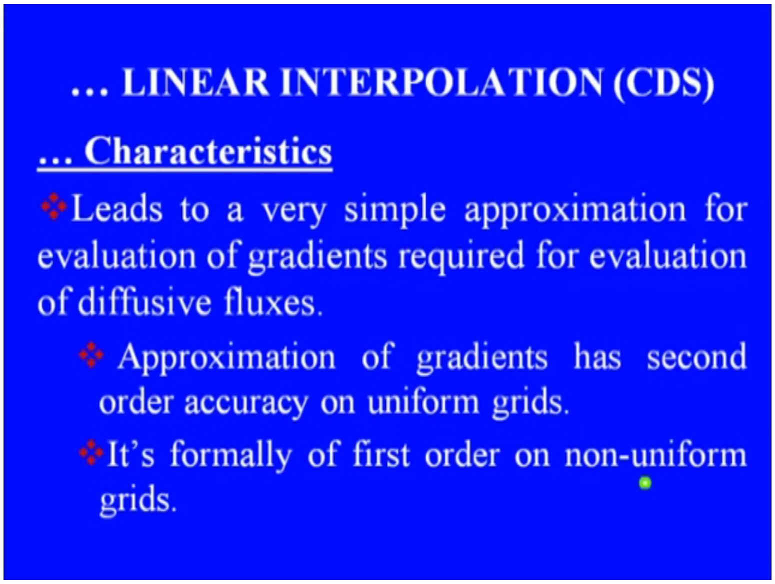

And not just that it leads to a very simple approximation for evaluation of gradients, and gradients are the ones which we require in the evaluation of the diffusive fluxes, so that is why? It is very popular okay, and this approximation of gradients are second order accuracy on uniform grids. And non-uniform grids we have already seen in the case of finite difference discussions that it is formally of first order but nevertheless as we refine the grid we get an accuracy which is close to second order.

So that tells us why? This CDS scheme is very popular in CFD community.

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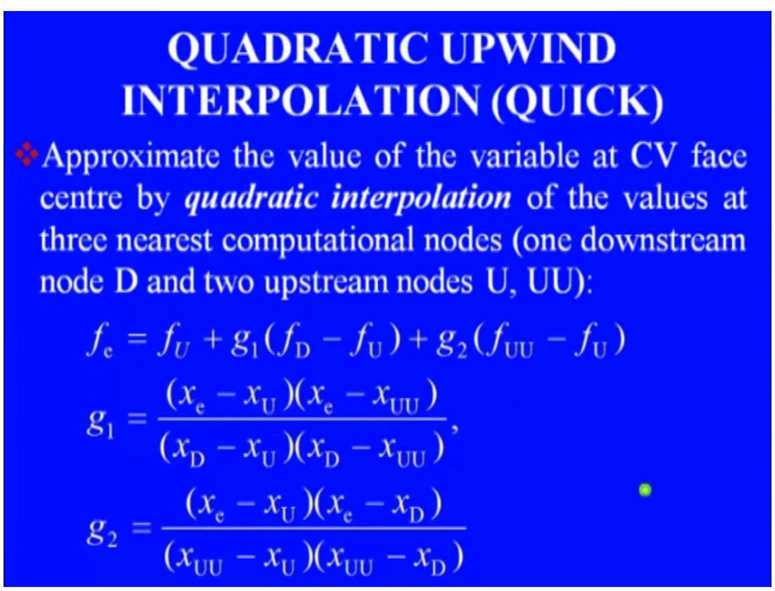

Our next, scheme is what we called quadratic upwind interpolation scheme, we had a look at the upwind scheme which was first order accurate, which we said it takes care of the presence of the high velocity or the advective fluxes, we do not have to worry about introducing oscillatory solutions, but downside is UDS is only first order accurate. Then we had a look at linear integration, linear interpolation is second order accurate but it is prone to producing oscillation solutions.

So the next thing was we can be get a second order accurate scheme, which has got better damping properties then CDS and it is got better accuracy then UDS, so this quick upwind interpolation schemes was devised for this particular purpose, the name upwind which is still here that tells us we are going to get more prominence or more weightage to the function values at the nodes which are on the upstream side okay.

So now let us have a look at eastern face, so approximate the value of the variable at CV face center by using quadratic interpolation of the values at 3 nearest computational nodes, and which will be these 3 nearest computational nodes? We are going to use one downstream node let us denote that by capital D and 2 upstream nodes capital U and capital UU, in this generalized description we are going to use capital P or E or capital W.

Which we normally used to denote our computational nodes on the eastern and western side, because it depends on the direction of the flow velocity, for instance if you are looking at a flow in positive x direction at this cell face e, then our upstream node will become the capital P and downstream nodes will become capital U is identical with capital E and UU becomes capital EE that is east to west.

So now first let us have a look at this generic formula fe=f capital U+g1 times fD-fU + g2 times fUU-fU, where these coefficients g1 and g2 they are based on the mesh spacings xe-xU xe-xUU/xD-xU times x capital D-xUU. And g2 similarly, given as xe-x capital U times xe-xD/ xUU-xU times x UU-xD. Now let us have a brief look at how do we obtain this formula and for that let us move back to our board.

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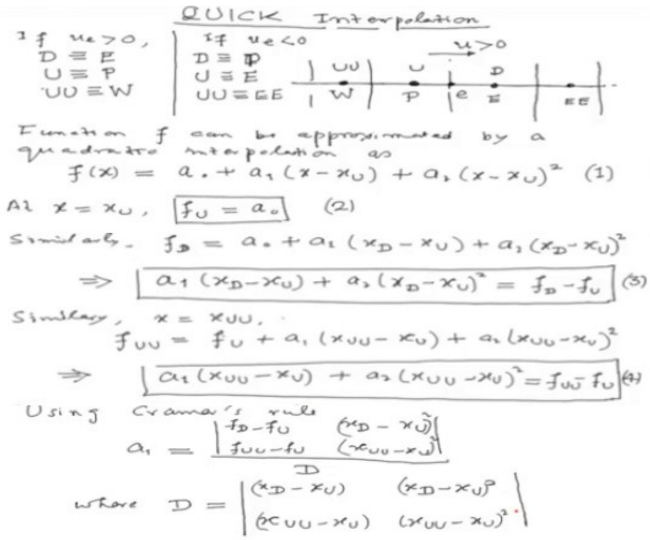

We have already seen one thing that we are going to use quadratic interpolation scheme and for that quadratic interpolation we need values that 3 mesh points, let us first to our schematic diagram, this is our eastern face this is one cell, this is next cell for computational node capital P capital E capital EE and the western node capital W. Now let us have a look at the face center small e here we want to obtain approximation of our function variable.

Now if the velocity at e flow velocity is in positive x direction, then in that case our downstream node will be eastern node, so if u>0 in that case downstream node that is the eastern computational node, the first upstream node that will become our computational node capital P and upstream to upstream that will become our node capital W, so UU is node capital W, in contrast if we had the other way around that is if the velocity at face<0 that say its point is negative x direction in that case our downstream node that will become the node P.

And the first upstream node will be eastern node capital P and the second upstream node will become EE, so we have two situations which we have might have, so to keep our discussions generic or general let us workout the formula or let us use the quadratic interpolation using the symbols D, U and UU. So we can approximate the function, function f can be approximated by a quadratic interpolation f(x) =a0+ a1 times x-xU +a2 times x-xU whole squared let us call this equation as 1.

Now here a0, a1, a2 are the coefficients which we need to work out, now what will happen if you choose x=xU, so at x=xU f at capital U=a0 so we will get a very simple expression for the coefficient a0. Similarly, now let us apply this or use this formula to get the functions values at the node D and UU similarly, your fD=a0+ a1 times xD-xU +a2 times xD-xU whole square or we can rearrange the terms.

We can write (()) (40:16) equation for a1 and a2 is a1 times xD-xU +a2 times xD-xU whole square=fD-fU where we substituted for a0, so this is our first equation in terms of the unknowns a1 and a2 let us number this as 3. Similarly, for x=xUU we will get that f at UU=f at capital U+ a1 times xUU-xU +a2 times xUU-xU whole square, rearrange the terms, so we get a1 times xUU-xU +a2 times xUU-xU whole square=fUU-fU, so let us number this equation as equation 4.

So now you have got 2 equations 3 and 4 in terms of two unknown coefficients a1 and a2 it is very simple linear system and you can easily solve for that I am going to leave that as exercise. For instance you can use Cramer's rule to solve the system, so using Cramer's rule your a1 will become the determinant fD-fU fUU-fU whole square and xD-xU xUU-xU whole square/D where capital D is second order determinant given by the coefficients in these two equations, that is xD-xU xD-xU whole square and xUU-xU xUU-xU whole square okay.

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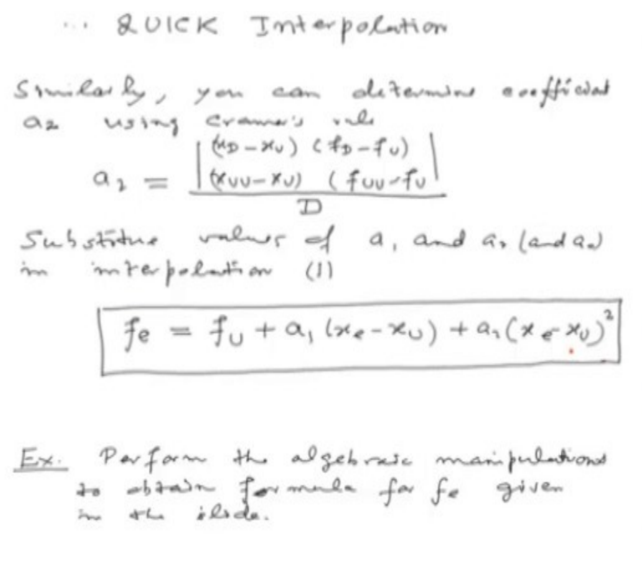

Similarly, we can write expression for a2 and thereafter you can combine the terms together. Similarly, you can determine coefficient a2 using this Cramer's rule, this a2 is given as determinant xD-xU times the second element fD-fU and the second row we have got xUU-xU the fourth element of this determinant is fUU-fU/D okay. So next, you can expand the determinants get the coefficient and see values a1 and a2.

And substitute values of a1 and a2 and of course a0 in interpolation which we had used that is quadratic interpolation and then you can find out what is the value of fe in terms of this fU+ a1 times x small e-xU +a2 times xe-xU whole squared okay. Now for further simplification I would leave as an exercise to you that perform the algebraic manipulations to obtain fe or rather the formula for fe given in the slide, so I would leave this as an simple exercise which you can work out.

So this is what we have that if you carry out the algebraic manipulation you can represent fe in terms of fU+ g1 times fD-fU + g2 times fUU-fU and for your reference the final form for these coefficients g1 and g2 are given here okay.

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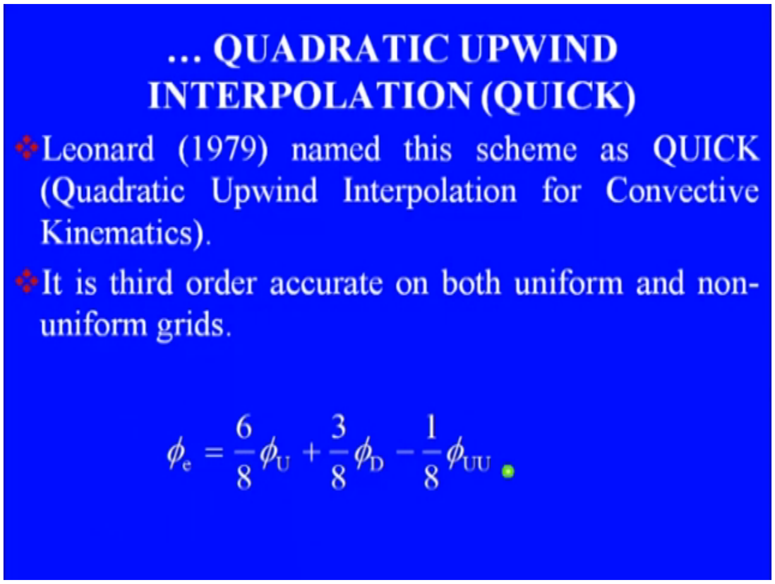

Now this scheme was derived by Leonard in 1979 and he named this scheme as QUICK, the QUICK is basically an acronym which is stands for Quadratic of Upwind Interpolation for Convective Kinematics, convective kinematics because this particular interpolation would be used for convection dominated problem, and quadratic because we have obtain this interpolation based on what quadratic interpolation formula.

And upwind because is there because we used 3 nodes 1 downstream and 2 upwind that is why? This named as quadratic upwind interpolation for convective kinematics. Now this particular scheme it is third order accurate in both uniform and non-uniform grids, and in case if we had uniform grid we get a very simplified form that phi at e=6/8 phi at upstream node+ 3/8 phi at downstream node-1/8 phi the function value at upstream to upstream node.

Now because of the third order accuracy this scheme is again very popular in modeling advection dominated problems, but note here that still there are some issues with oscillations which are parallel similar not as severe as central difference scheme, but nevertheless there are some possibilities of obtaining unwanted oscillations in the numerical solutions with use of quadratic upwind interpolation scheme.

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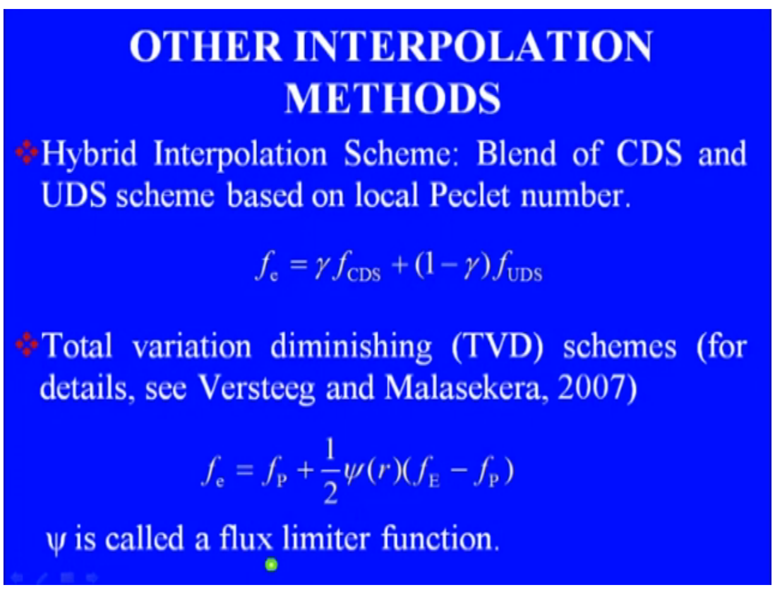

Now let us have a brief look at few other interpolation schemes. The first one is what we called Hybrid Interpolation Scheme which is a blend of CDS and UDS based on the local peclet number that is fe=gamma times fCDS+ 1+gamma times fUDS where this gamma would depend on the local peclet number. The local peclet number is very small okay then this gamma can be set to 1 fCDS should be used for the interpolation.

If the peclet number is large okay then in that case choose a small values for gamma so that we make use of upwind difference approximation, so this hybrid interpolation scheme it combines the desirable properties of both CDS and UDS. More recent developments include total variation diminishing scheme and for details you can see the book Versteeg and Malalasekera okay, for instance you will express fe=f at the computational node P+1/2psi r fE-fP where this psi r is a specified function which depend on the local flow conditions.

Now this UDS is a one form or like one of this TVD schemes, and formally psi is called as flux limiter function. We have many other interpolation schemes available in Literature.

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And for details please have a look at by the book Chung on Computational Fluid Dynamics. Ferziger and Peric's book. And you can also have a look at the book by Versteeg and Malalasekera which gives pretty detailed introduction to TVD and ENO that is Essentially Non Oscillatory schemes. As for this particular course is concerned or this lecture, we will stop here and in the next lecture we will have a look at some applications of the Finite Volume Method to Scalar Transport Problems.