Micro and Nanoscale Energy Transport
Dr. Arvind Pattamatta
Department of Mechanical Engineering
Indian Institute of Technology, Madras

Lecture - 31
Gas Flow and Heat Transport in Microcahnnels Part 2

Good morning. We will continue our discussion on the Slip Boundary Condition for Gas Flows in Microchannels. Originally it was you know Maxwell who proposed an idea to account for the cause of slip into the microscopic continuum models. One of the easiest ideas is you can draw an analogy from our reflection or scattering of phonons or electrons from the boundary either it could be a specular or defuse and of a scattering.

And we have all ready solved in the nanoscale transport, we solve the Boltzmann transport equation applying the boundary conditions and the first consolation was a defuse scattering. So, in that what we assumed is whatever energy carrier that is coming out of that boundary is having the value of equilibrium. So that is based on for example; if you talk about the Bose Einstein, the non equilibrium function of the phonon or electron which is coming out in the f plus coming from the bottom wall will be equal to f equilibrium corresponding to that temperature and frequency. So, temperature whatever the boundaries maintained at.

Similarly, you can imagine this gas molecules at the microsacle also go through the same kind of phenomena. So, they encounter the boundary and a part of this a fraction of this you can say gas molecules can be kind of absorbed or thermalized by the boundary, absorb or thermalized means they come out they remitted with the distribution function equal to the Maxwell Boltzmann distribution function. So, that is the fraction sigma and the other fraction 1 minus sigma is simply reflected.

So, I will perfectly reflect in the sense they are like a specular reflection like a mirror image reflection. And we know all ready from our knowledge of nanoscale transport that whatever is emitted diffusely, scattered diffusely so that is not going to conserve momentum; the momentum is actually destroyed in the process and because whatever incoming and gas molecule and whatever is coming out of the boundary have different values of the distribution function, and therefore different momentum.
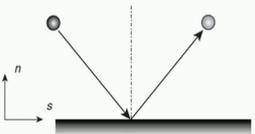
Therefore, the emitted gas molecule will be thermalized based on the temperature of the boundary. So, based on the Maxwell Boltzmann distribution function you have to calculate what is the value of momentum for this particular emitted gas molecule. Whereas, in the specular scattering it is momentum is perfectly conserved, so only the direction is changed; so the outgoing will be equal to the incoming the same momentum it possesses.

And therefore, in that case it does not really understand what is the condition of the boundary; the gas molecule which is coming out of the boundary does not really have feeling of what is the condition prevailing on the boundary.
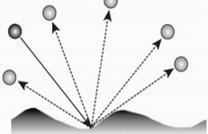
(Refer Slide Time: 04:02)

Therefore, these are two distinct and the different ends of the spectrum two different you know extreme cases. One in which it completely forgets it is, so this is like you know loses its memory. It is emitted equally in all the directions based on the condition at the boundary, so that is the diffused scattering phenomena. See the other is completely ignorant of what is the boundary condition. It is only emitted with the momentum of the incoming a gas molecule falling on the boundary.

So therefore, Maxwell proposed that you can assume of the total number of molecules fraction sigma will be emitted like the way diffused scattering takes place and the rest 1 minus sigma will be emitted the way specular scattering takes place.

Therefore, this sigma he called this with the name called the momentum accommodation coefficient. That means, it tells you whether you conserve momentum or not.

(Refer Slide Time: 05:28)

The expression for this; you can say that I mean you can either write components in the tangential or in the normal direction, but the general philosophy is you defined the momentum accommodation coefficient as the ratio of the difference between the incoming momentum minus whatever is coming out of the boundary, the momentum of the reflected are the energy carrier or gas molecule which is living the boundary divided by the difference between the incoming momentum and the momentum according to the Maxwell Boltzmann distribution function.

So that is your Pw, so that is your thermalized value based on the Maxwell Boltzmann distribution. So, what does it mean? In the case of diffuse scattering what should be this value? What should be the value of the momentum accommodation coefficient? What do you think will be in the value of Pr? Pr will be equal to Pw in the case of diffuse scattering, because everything is absorb, thermalized and emitted from based on the conditions on the boundary. So, it is emitted like your condition we put f plus is equal to f equilibrium that is your diffused boundary conditions. Therefore, in the case of diffuse scattering your sigma will be equal to, what? 1.

Whereas, if you talk about specular scattering, what happens to sigma? 0, because Pr will be equal to Pi; the momentum is perfectly conserved and it does not have any idea about

what is the momentum depending on the Maxwell Boltzmann statistics. In that case your sigma will be equal to 0, it is a perfectly reflecting surface and the other diffuse scattering will be sigma equal to 1.

So therefore, depending on in the kind of scattering either diffuse r specular, and this all ready you know from the knowledge of nanoscale transport; you know that these two are that possible kinds of boundary conditions you apply and solve from the Boltzmann transport equation. However, what happens, how do you incorporate this information into the continuum equation? Now this is where it is becoming very tricky.

(Refer Slide Time: 08:23)

If you solve the equation is through the approximation made by Kundt and Warbury. So, they have said that this is equivalent to giving a slip boundary condition at the wall provided you know your slip boundary condition is related to the slip length, this slip length value is zeta times the velocity grading at the wall. Therefore, somehow the slip length should be related to this accommodation coefficient.

So, the accommodation coefficient has the knowledge of the microsacle scattering at the boundaries and somehow in ad hoc empirical manner that has to be accounted through the formulation of slip length. And therefore, you assume you know that the slip length can be related to the accommodation coefficient through an expression like this.

(Refer Slide Time: 09:20)

Velocity Slip Boundary Condition

- From momentum balance at wall, velocity slip is calculates as:

$$u_{slip} = u_s - u_{wall} = \frac{2-\sigma}{\sigma}\lambda\frac{\partial u_s}{\partial n}\bigg|_w + \frac{3}{4}\frac{\mu}{\rho T}\frac{\partial T}{\partial s}\bigg|_w$$

- Non-dimensional form of the above equation is:

$$u_s^* - u_{wall}^* = \frac{2-\sigma}{\sigma}Kn\frac{\partial u_s^*}{\partial n^*}\bigg|_w + \frac{3}{4\gamma k_2^2}\frac{Kn^2 Re_0}{Ma_0^2}\frac{\partial T^*}{\partial s^*}\bigg|_w$$

Where,

$u_s^* = u_s/u_0, T^* = T/T_0, n^* = n/L$ and $s^* = s/L$

second term in the above equation is called as thermal creep or transpiration phenomenon.

NPTEL

That means for example, in this case you can write this as 2 minus sigma by sigma times the mean free path; mean free path of the gas molecules. This is one ad hoc way of expressing the relation between slip length and the accommodation coefficient. In the case of diffuse scattering your sigma equal to 1, and therefore your slip length will be equal to what lambda, mean free path.

In the case of specular scattering, what will be the value? I mean if you are approaching value of sigma going to 0, this value will go to infinite; so this slip length is approaching infinity. So, these are two extreme limits. And between these two limits you can assume some values, something like sigma equal to 0.2 or 0.3 or 0.5 are you know, if you know exactly the value for particular fluid.

(Refer Slide Time: 10:42)

Momentum accommodation coefficient

- For tangential components: $\sigma = \dfrac{p_i - p_r}{p_i - p_w}$

- For normal components: $\sigma' = \dfrac{p_i - p_r}{p_i - p_w}$

where, $P = mv$ (momentum) and subscripts, 'i' and 'r' represent the incident and the reflected and 'w' refers to the Maxwell velocity distribution corresponding to the surface temperature T_w

- For specular reflection: $\sigma = \sigma' = 0$
- For diffuse reflection: $\sigma = \sigma' = 1$
- $\sigma = 0.87$ to 1 for air
- $\sigma = 0.75$ and 0.85 for N_2, Ar and CO_2 in silicon microchannels

10/7/2016   Dr. Arvind Pattamatta   22

For example; if you take the case of air, generally values of sigma are quite high it towards the diffuse side, so it is between 0.87 and 1. So, if you take some other invert gases like argon, nitrogen and also depend on the surface material; if you look at silicon material as a microchannel and if you use nitrogen or argon or carbon dioxide so the values of this sigma can vary between 0.75 and 0.85. They are always closed to the diffuse scattering. In that case you can assume also if you have a rougher rough interface more likely this value of sigma will be closer to 1. And most of these are practical surfaces have certain roughness. So therefore, always the value of sigma is biased towards the diffuse scattering.

Student: So, Maxwell velocity distribution (Refer Time: 11:48) or is there any (Refer Time: 11:50).

No, but when, I mean that is correct, but the value of sigma theoretically cannot be calculated. Although, we measure, we define sigma in this way I mean use a theory to calculate sigma; it is only coming from experimental conditions. Where, as I said yesterday how do you determine experimentally, you just extra polluted make a virtual argent and determine what is your slip length. And then now you have related your slip length with sigma and therefore you can reverse calculate; back calculate from experiment what will be the value of sigma. So, that is the standard procedure.

Therefore, this is a 1 component of the velocity slip. And in many experiments they have absorbed that only theoretical if you add this, the theoretical value of the velocity slip is not matching the experimental value. So, there is also something extra component which is at causing an additional slip at the wall which could not be explained by this particular term alone. So, later on then they understood that this was due to a phenomena call thermal creep in which you also produce additional slip due to a temperature gradient.

In addition to having uniform temperature, you know incase where you have a temperature gradient the gas molecules tent to move from the lower temperature side to higher temperature side; that is in the direction of the applied temperature gradient. So, in addition to the normal Maxwell slip you also have an additional slip due to the thermal creep. So, the second term here represents the thermal creep slip. So, that is related to your temperature gradient which is tangential, which is along the wall applied in the tangential direction, whereas the velocity gradient is in the normal direction to the wall.

So therefore, these two components put together would adequately explain the slip velocity that are measured experimental. In general you know this kind of a slip boundary condition is become popularly referred to as Maxwell slip boundary. Some people also refer this formulation the formulation that we are referring here, to navier slip bound. So, in the navier slip boundary you just give some values of slip length arbitrarily it could be some 0.5 or 0.7 or 0.2 whatever it is, but if you want to relate in terms of accommodation coefficients then it becomes Maxwell slip boundary condition.

(Refer Slide Time: 15:14)

And, apart from that you can also write boundary conditions or temperature we have seen that you can have slip of both - momentum as well as energy or temperature. Therefore, this we have all ready seen nanoscale transport. There is a temperature discontinuity at the boundary and high Knudsen numbers; you will find this discontinuity will increase.

(Refer Slide Time: 15:46)

Therefore, to account for this similar analogy has been drawn only that in place of the momentum accommodation coefficient sigma we replace this with what we called as the

thermal accommodation coefficient sigma subscript T. So, same analogy expect that we assume that these gas molecules with some energy is approaching the boundary and fraction of that is absorbed and reemitted based on the Maxwell Boltzmann statistics, the other is you know reflected perfect.

But in this case to distinguish this fraction from the momentum accommodation coefficient we use notation sigma subscript T. And we once again rewrite same way to the velocity slip we introduce parameter call the temperature slip which is the temperature of the fluid at the boundary minus the wall temperature. So, that is equal to a different slip length. Now this slip length is called temperature slip length. The other is the momentum slip length or velocity slip length this is the temperature slip length and this has to be multiplied to the temperature gradient. So if this temperature slip length is 0, then there is no slip.

That is your conventional no slip boundary condition. And similar to the Maxwell velocity slip boundary condition you can express the temperature slip in terms of the energy or temperature accommodation coefficients.

(Refer Slide Time: 17:24)

Temperature Jump Boundary Condition

- From energy balance Kennard(1938) temperature slip is given by:

$$T - T_{\text{wall}} = \frac{2 - \sigma_T}{\sigma_T} \frac{2\gamma}{\gamma + 1} \frac{k}{\mu c_v} \lambda \frac{\partial T}{\partial n}\Big|_w$$

- Non-dimensional form of the above equation is:

$$T^* - T^*_{\text{wall}} = \frac{2 - \sigma_T}{\sigma_T} \frac{2\gamma}{\gamma + 1} \frac{\text{Kn}}{\text{Pr}} \frac{\partial T^*}{\partial n^*}\Big|_w$$

where, 'Pr' is Prandlt number given by,

$$\text{Pr} = \frac{\nu \rho c_p}{k} = \frac{\mu c_p}{k}$$

10/7/2016　　　　Dr. Arvind Pattamatta　　　　26

So, that is we write now the temperature slip to be 2 minus sigma T by sigma T times parameters which are functions of thermo physical properties, thermal conductivity, heat capacity and so on.

This is actually popularly referred to as the Smoluchowski boundary condition, because Smoluchowski experimentally confirmed that this kind of equation can be used to match the theoretical value of temperature slip with the experiment very well; using this parameter and knowing assuming values of sigma T. Therefore, this boundary condition is referred to as Smoluchowski slip boundary. So, the velocity 1 is referred to as Maxwell slip and the temperature slip is referred to as Smoluchowski.

So therefore, we can non-dimensionalize whether it is the velocity slip or temperature slip. As you can see that you have a mean free path here so if you non-dimensionalize your coordinate with respect to the characteristic dimension, characteristic length scale. For example, you introduce a new a normal vector n star which is equal to n by l this is your normal coordinate and similarly tangential coordinate as s by l, so you can write this in terms of Knudsen number. And again the thermal creep slip due to thermal creep also can be written in terms of Knudsen number square Reynolds number by mark number square. Therefore, sometime these boundary conditions are also expressed in terms of only purely non-dimensional numbers. So these are non-dimensional values of slip.

Similarly, you can do the same thing for the temperature slip; you can write this purely non-dimensional fashion you get a ratio of Knudsen number by Prandtl number. Therefore, once you now understand how this microscale non-equilibrium at the wall can be incorporated into the continuum equation using these slip approximations for velocity and temperature now it becomes quite easy to solve these governing equations for some standard cases; like flow between two parallel plates, flow in adduct. These are some simple cases for which analytical solutions can be obtained. Only that now in the case of slip regime you replace your no slip boundary condition with the appropriate Maxwell and Smoluchowski slip conditions.

(Refer Slide Time: 20:30)

Thermal Accommodation Factor

- Thermal accommodation coefficient in terms of energy:

$$\sigma_T = \frac{\varepsilon_i - \varepsilon_r}{\varepsilon_i - \varepsilon_w}$$

where, ε – average energy of a molecule and ε_w – energy when the molecules are in thermal equilibrium with the wall.

- In term of temperature it is given by,

$$\sigma_T = \frac{T_i - T_r}{T_i - T_w}$$

  - For specular reflection $\varepsilon_r = \varepsilon_i$, $\sigma_T = 0$
  - For diffuse reflection $\varepsilon_r = \varepsilon_w$, $\sigma_T = 1$
  - $\sigma_T = 0.87$ to 0.97 for air – aluminium and air – steel systems
  - $\sigma_T < 0.02$ for pure He gas and clean metallic surfaces

10/7/2016   Dr. Arvind Pattamatta   27

Just to give some additional data for sigma T again for combination of air, aluminum air, steel you have different values also which are closed to 1. Whereas, if you take clean surfaces without any deposition or which is quite smooth, without much roughness. You use helium gas for example; this can be very small, close to 0, so is 0.02 is the value of the thermal accommodation coefficient.

Therefore, similar to how we define the momentum accommodation coefficient we can define this based on the difference between the incoming energy of the gas molecules and what is coming out divided by incoming minus what is emitted based on the Maxwell Boltzmann distribution. I hope this is clear. This is the foundation of the slip flow regime.

(Refer Slide Time: 21:39)

Basically, now that we have define the boundary conditions so we also know the conservation equations which are the compressible, Newtonian and isotropic case and also the Knudsen number should be less than 0.1. This is where the continuum equations can be applied, so this is call the slip flow regime.

In this case so you solve the compressible form of the Navier-Stokes equation along with these appropriate slip boundary conditions. And these can quite accurately predict most of the experimental results in this particular slip flow regime.

(Refer Slide Time: 22:20)
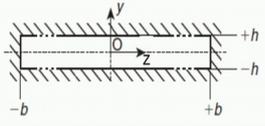
And we will take the case where we can develop analytical solution that is let us take a microchannel; microchannel here means it is a simply two dimensional case of flow between two parallel plates. So, just like you do your fluid mechanics of flow between two parallel plates is the same analysis is done here. So, we have to take the separation between the plates as distance 2 h. Therefore, the origins start from the center, so it is extent to plus h on to the top plate and minus h to the bottom plate.

And you also assume the following things. Number one; you are looking at only steady state, the steady flow system. So therefore, all the derivative with the respect to time can be assumed to be 0. So, all of the d rho by d T d u by d T d T by d T all of these are gone. And, the other important condition to simplify this is fully developed. So, assume the profiles to be fully developed meaning you do not have any gradients of velocity in the x direction, in this case you have z coordinate so in the z direction. And also this is a 2 D systems so you do not have any gradient in the third direction.

So, based on this you will be able to see that you can reduce the and also you can assume that there are no volumetric forces. And, to first start the problem the hydrodynamic problem that is no heat transfer so it is all isothermal.

(Refer Slide Time: 24:09)

So, with this you can reduce the compressible Navier-Stokes equation to the following forms. So, you only have the viscous forces, the shear stresses, and the other is your pressure gradient. So, the viscous force is balanced by the pressure gradient. The viscous

force in this case will be the diffusion only in the y direction that is your d square u by d y square. And the pressure gradient will be in the z direction so your d P by d z. That means, if you draw control volume you will have a viscous force on the top on the bottom plates and this is to be balanced by the pressure gradient. And this viscous force is nothing but due to the diffusion of velocity in the vertical direction.

So therefore, the entire compressible equations reduce to a very simple form which is similar to your incompressible equation for channel flow. Therefore, there will not be any big difference between compressible and incompressible now, that you have assumed the steady state and fully developed flows.

And then we go on we apply the boundary conditions. To solve this now the condition at y equal to 0 will be symmetry. So, we have taken the origin to be at the a center of the channel, and therefore at y equal to 0 you can say that the velocity profiles will be symmetry between the top half and the bottom half, and you can simply apply the symmetry condition which is d u by d y equal to 0.

And, on the other side so y equal to plus h, in the conventional channel case you will give in the no slip condition which is u equal to 0 but now you give the Maxwell slip boundary condition.

Therefore, you have both of both boundary conditions; one is the slip boundary condition on one side, the other is the symmetry boundary condition. And you substitute these two. Now this is an o d e which you can directly integrated to couple of times to get the velocity profile, so this is a very simple case. And if you do that you get your solution general solution for a velocity profile like this, it is a quadratic equation. So, in order to get the constants a 1 and a 2 you apply the two boundary conditions at y equal to 0 and y equal to h. So, can you try to do this now?

You apply these two conditions, find out the constants and calculate how the velocity profile looks. It should be, let me check, I think in this case the derivative is a negative, so your velocity profile is maximum and the center and at y equal to h it will be negative. So, in order to get a positive value you put I think the negative sign here, your d u by d y will be negative. In order to make sure your velocity is positive; therefore you put a minus sign there. Otherwise you get a negative value of velocity which is not physical.

In the case of no slip you will get a 2 equal to 0, but here you have a value at y equal to 0.

Student: (Refer Time: 29:43).

Yeah, a 1 is 0.

(Refer Slide Time: 31:31)

So, you should be getting something of this form this is non-dimensionalized here where I write u star as u z by u z naught. Where, u z naught is value of u z at y equal to 0 and Knudsen number equal to 0. So, that is your central line profile corresponding to Knudsen number of 0. If you put Knudsen number of 0 so you get 1 minus y star square and at y equal to 0 that is the maximum value; maximum value at the central line corresponding to Knudsen number of 0 y equal to 0.

Therefore, now the difference between the slip and the no slip velocity profile is if you simply substitute Knudsen number of 0 do you get your profile for the no slip boundary condition case 1 minus y star by 0. So, y star is nothing but y by h. Therefore, you get your conventional parabolic profile with the maximum at the central line. Now, apart from that you have the additional factor which is dependent on the accommodation coefficient, coming from the boundary condition.

From this you can evaluate what is your mean velocity at a given cross section, you can integrate the velocity profile from minus 1 to 1 that is our minus h to h and you can

calculate the value of the magnitude of the mean velocity and that comes out to be 2 by 3 plus 4 times 2 minus sigma by sigma Knudsen number.

Now, the way we have defined Knudsen number prime I think there is small that is basically 2 times root 2 times your Knudsen number. You can just leave it as a Knudsen number there is no problem, you do not have to write in terms of Knudsen number prime. So, essentially if your sigma is 0. For example; for the specular case you can say that the value goes to very large values of u star. And similarly sigma equal to 1, so what will be the value. So, it will be simply 4 times your Knudsen number prime or 8 times Knudsen number. If your Knudsen number is going to 0 then this becomes equal to 2 by 3.

Therefore, depending on the value of Knudsen number now the higher the value of Knudsen number the larger will be your value of the mean velocity. So, what you are getting seeing is that for a given refer for a fixed value of pressure drop as your Knudsen number keeps increasing the average velocity also keeps increasing, because of the higher slip. That means, you are getting more mass flow rate for the same value of pressure drop.

Student: (Refer Time: 34:57).

Instead of 1 minus y star square you are getting y star square 1 minus 1. If had applied this minus sign is it.

Student: (Refer Time: 35:20) minus n.

Minus n, if you do not apply the minus sign.

Student: we will get this one.

This one, probably there is an error because this is your correct profile; because Knudsen number is equal to 0 it gives back your macro channel velocity profile. So, probably then this minus sign is not needed here. You please check that, if you get an opposite sign then you remove the negative sign there.

Student: If you (Refer Time: 35:54).

So, then you are getting correctly is it.

Student: Y plus y star square.

One plus y star square. But you if you use the negative are you getting 1 minus.

Student: Y star square minus 1 we getting.

Minus 1.

Student: Yeah.

There should be the momentum equation should have a minus sign. The momentum equation itself should have a minus sign there because the viscous force and pressure gradient are in the same direction right now. See in the, these are both are in the right hand side right, so this is said minus d P by I think then it is ok. If you have a negative sign in the momentum equation will that be fine, it will make it 1 minus y star square. Or the other way of evaluating it apply this at y equal to minus h, y equal to 0 and y equal to minus h with the positive sign. And see if you are getting the same profile. If you get that then you use the bottom half that is the top half. So, in the bottom half you are slip at the wall should be 2 minus sigma by sigma lambda into d u z by d y.

But this should also work out I do not know; but the final equation is correct, you just please check where you are going sign.

Student: (Refer Time: 38:20).

So, this is your final equation in terms of Knudsen number. Therefore, depending on the value of Knudsen number you get additional average velocity or additional fluoride because of the slip.

Therefore, you can go head and you can find out the other integral quantities like skin friction coefficient or you can calculate your wall shear stress based on the gradient of the velocity at the wall, and therefore you can calculate your skin friction coefficient and so on. And you can also calculate the pressure or the Poiseuille number, so that Poiseuille number is nothing but basically friction factor times your Reynolds number.

Therefore, the friction factor that you have defined here this is called the fanning friction factor; this is in terms of wall shear stress by half row u square. Since this is similar to your skin friction coefficient for external flows. To the other friction factor which you can define is in terms of pressure, pressure drop that is called the Darcy friction factor. Therefore, if you write down the relationship for Darcy friction factor an in terms of Reynolds number for a circular duct, so for the case of macro ducts you get f is equal to 64 by r a. Or in terms of the c f that you have written here it will be 16 by r a. So, the product of the friction factor times Reynolds number is called the Poiseuille number. In the case of macro channel circular ducts your Poiseuille number is equal to 64.

Now, in the case of micro channel ducts with slip boundary condition this will become a function of the Knudsen number. Therefore, the product of friction factor times Reynolds number will not be a constant anymore, it will be a function of Knudsen number. For the case of Knudsen number going to 0 this will go to the value of 24 which is the case for channel flow. For duct flow you have 64 and for channel flow you have 24.

So, you can write, therefore the Poiseuille number as a function of Knudsen number through this particular expression. So, what it tells you that if you are increasing your Knudsen number what happens to the Poiseuille number, it will reduce. That means, for a given flow rate your pressure drop is actually lower or for given pressure drop you get higher flow rate. So, this is the effect of slip at the walls.

(Refer Slide Time: 41:21)

Pressure Driven Gas flows through Microchannels continued ….

- Considering long microchannel with inlet p_i and outlet p_o and neglecting entrance effects, mass flow rate is given by:

$$\dot{m} = \rho \overline{u_z} A = p \overline{u_z} A / RT$$

- Therefore, integrating mean velocity equation gives:

$$\int_{p_i}^{p_o} \left(\frac{2}{3} p + 8 \frac{2-\sigma}{\sigma} Kn_o p_o \right) dp = -\frac{2\mu \dot{m} RT l}{h^2 A}$$

- Therefore,

$$\dot{m}_{NS1,plan} = \frac{2bh^3 p_o^2}{\mu RT l} \left[\frac{\Pi^2 - 1}{3} + 8 \frac{2-\sigma}{\sigma} Kn_o (\Pi - 1) \right]$$

- Where, $\Pi = p_i/p_o$, K_o is outlet Knudsen number and subscript NS1 is Navier-Stokes equation with first-order slip flow boundary conditions.

10/7/2016   Dr. Arvind Pattamatta   33

So, this is good as far as the pressure drop is concerned. So, if you are pumping certain flow rate, so the pressure drop require to drive this in the micro channel will be lower than in the macro channel case. Or if you apply the same pumping pressure gradient to pump the flow you get higher flow rate. So, this is a beneficial effect of the gas flows in the slip regime of the micro channel.

(Refer Slide Time: 41:53)

Therefore, we can also calculate what is called a non-dimensional mass flow rate which gives you the actual mass flow rate calculated from the slip boundary condition divided by the mass flow rate calculated from the no slip boundary condition. Obviously, the numerator is larger than the denominator. So, this have comes out to be again a function of the accommodation coefficient, the pressure gradient, pressure ratio.

So, the denominator here has nothing but the pressure ratio between that is the ratio of inlet and outlet pressures and the Knudsen number. If you are Knudsen number is equal to 0 the non-dimensional mass flow rate will be equal to 1, and for larger Knudsen number this will be greater than 1. That means, as your Knudsen number keeps increasing this non-dimensional mass flow rate will also subsequently increase.

(Refer Slide Time: 42:58)
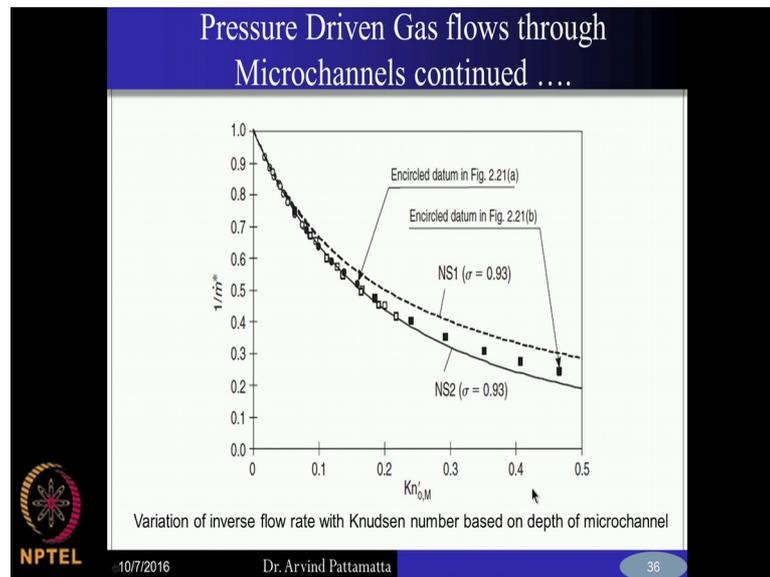
Therefore, when you plot the non-dimensional mass flow rate as a function of Knudsen number; so the reciprocal of this should keep decreasing with respect to increasing Knudsen number, as your Knudsen number keeps going up higher will be the flow rate or the inverse of this should be lower. So, depending on the value of the accommodation coefficients you can get different values. And also if you are going to slightly larger Knudsen number; for example, here Knudsen numbers have crossed 0.2, 0.3, and 0.4 and as we said Knudsen number of less than 0.1 is where we use our slip modal.

Now, what happens when you cross 0.1, suddenly the slip modals fails and continue fails not exactly to be more accurate we have to go to what is called the (Refer Time: 43:55) equation, but sometimes people still use the continuum equation, but they modify the slip from first order to a second order slip to the incorporate higher order slip boundary condition which in includes terms d square u by d y square. So, with that you see that if you use just a first order slip beyond Knudsen number of 0.1 these symbols are actually the experimental values, and whatever is computed is basically theses lines.

So, if you use the first order slip boundary condition you get a good agreement till Knudsen number of 0.1; for higher Knudsen numbers you do not get a good agreement. Whereas, if you use a second order slips even for Knudsen numbers of up to say 0.2, 0.3 you are able to get a good agreement. So, beyond that even the second order slip will not be able to match with the experiment well.

Heat transfer in plane microchannel

- Consider same plane channel limited by parallel plates(b>>h), the axial co-ordinate is 'z' and the cross-section is in the (x, y) plane.
- Let the lower as well upper wall be at constant heat flux.
- Flow is assumed to be hydro-dynamically and thermally fully developed and fluid properties are assumed to be uniform.
- Momentum equation is given by: $\dfrac{d^2 u_z}{dy^{*2}} = \dfrac{h^2}{\mu}\dfrac{dp}{dz}$
- Energy equation is given by: $\dfrac{k}{h^2}\dfrac{\partial^2 T}{\partial y^{*2}} = \rho u_z c_p \dfrac{\partial T}{\partial z}$

10/7/2016   Dr. Arvind Pattamatta   37

Therefore, coming to the heat transfer problem; we solved the fluid mechanics problem for velocity profile, now we can also solve the energy equation for fully developed both hydro dynamically and thermally. So, the energy equation will be similar to the macro scale parallel plate energy equation.

Heat transfer in plane microchannel continued ….

- Non-dimensional temperature is written as:
$\theta^* = \dfrac{T - T_{\text{wall}}}{q_h h/k} = \dfrac{4}{\text{Nu}}\dfrac{T - T_{\text{wall}}}{\overline{T} - T_{\text{wall}}}$
- For fully developed flows we have:
$\partial T/\partial z = \partial \overline{T}/\partial z = \partial T_{\text{wall}}/\partial z$
- Therefore, energy equation is written as:
$\dfrac{q_h}{h}\dfrac{d^2\theta^*}{dy^{*2}} = \rho u_z c_p \dfrac{d\overline{T}}{dz}$
- By conservation of energy on a control volume:
$2h\,\rho\,c_p\,\overline{u_z}\,d\overline{T} = -q_h\,dz$
- Therefore, energy equation reduces to
$\dfrac{d^2\theta^*}{dy^{*2}} = -\dfrac{u_z}{2\,\overline{u_z}}$

10/7/2016   Dr. Arvind Pattamatta   38

Now, only difference is we will be applying the slip boundary condition at the other wall. So, at y equal to 0 you still have the symmetry boundary condition and at y equal to h you have the slip boundary condition.

Just summarize that, if you I am just skipping all the details these are available in your text book, any text book related to micro channel close. The book that I am following is by Sathis Kandliker and Srinivas (Refer Time: 45:49). So, this flow a micro and mini channels, heat transfer in micro and mini channels. So, by Sathish Kandliker and Srinivas (Refer Time: 46:02).

(Refer Slide Time: 46:0)

So, I think this a pretty straight forward analysis similar to the macro channel. Now only difference comes in terms of the boundary conditions that you apply slip boundary condition at the wall.

(Refer Slide Time: 46:20)

And similar to your velocity profile you get your temperature profile, now this zeta star here is nothing but a function of your accommodation coefficient and Knudsen number and Prandtl number. This is coming from the slip boundary condition. So, all this are written in non-dimensional form. And therefore this is your expression for your temperature profile.

And from this if you calculate the Nusselt number at the wall, then Nusselt number also transfer to be a function of the accommodation coefficients and the Knudsen number at the wall. If your Knudsen number is equal to 0 this value of zeta star will be also 0 at the wall. In that case for the limiting case where your Knudsen number is equal to 0 this Nusselt number will reach the value of 140 by 17 which is the value of Nusselt number for macro channels, for heat transfer in macro channels.

For the larger Knudsen number case you use this particular expression to calculate the actual heat transfer co efficient. So, what happens now due to the temperature jump, what happens to the temperature gradient at the wall?

(Refer Slide Time: 48:05)

Originally your temperature profile, if you see look at the profiles that I have plotted. Here, so your original temperature profile was 1 without any slip. Now your temperature profile is shifting to towards 2 with slip. So, what happens to the temperature gradient at the wall from 1 to 2? Say your main stream value is assumed that this is constant. What happens from one to two, decreases? So, if your temperature gradient decreases what happens to a Nusselt number, heat transfer coefficient also decreases.

Therefore, all though the pressure drop decreases consequently the heat transferate convective heat transfer coefficient on Nusselt number also drops. From the hydro dynamic point of view it is very convenient to operate with slip flows, but from heat transfer point of view it is not so beneficial. Therefore, if you are looking at larger Knudsen numbers the value of the Nusselt number will be reducing as given by this particular expression. Whereas, for the macro scale case it comes back to the classical value of 140 by 17.

(Refer Slide Time: 49:49)

Therefore, I will just summarize scenario where you have now circular micro tubes. In the micro tubes you have to rewrite your conduction term slightly different, your diffusion term will be 1 by r d by d r of r d u by d r. Whereas, in the channel case it is just d square u by d y square. Similarly, for the heat transfer also. In this case the velocity profile, the mean velocity will be described in this particular fashion that is the mean velocity profile will be half of your central line value, in the macro channel case. Now that is going to be slightly different with higher Knudsen numbers.

Similarly, the Nusselt number was also described this equation. So, you see that for the case of Knudsen number going to 0 the Nusselt number will be now 48 by 11, so that is 4.36. That is the case of constant heat flux laminar fully developed hydro dynamically and thermally in a circular duct that is your macro scale case. Now in the micro duct, micro tube case these values are smaller than 48 by 11. This expression can be used similar to the channel case. Channel case you have different expression and circular micro tubes you have this expression.

So, this is the summery if you want to calculate heat transfer for gas flows in micro channels, so for 2 D micro channel you use expression on the top.

(Refer Slide Time: 51:28)

And for the 3 D micro channel for a circular tube q is the expression on the bottom. And if you plot this know you can basically see that the parallel plate case has a higher value of Nusselt number compare to the circular tube. And again for different values of you know momentum and the thermal accommodation coefficients you can produce different values of Nusselt numbers.

So, these are functions of accommodation coefficient as well as the Knudsen number. As your Knudsen number keeps increasing your Nusselt number reduces.

(Refer Slide Time: 52:23)

Just last topic I want to complete this portion. As we said all these analysis is good if you are looking at Knudsen numbers of up to 0.1, if you are going to higher Knudsen number with the use of second order slip to some extent you can still solve the continuum equation, but we are going away and away from the assumption of continuum. And therefore, we have to resort to finally going back to the Boltzmann transport equation for larger Knudsen numbers.

And for Knudsen numbers greater than 1 definitely continuum equation cannot be used, but what can be done is solving the Boltzmann equation is also computationally very expensive especially if you have three dimensional flows. So, one alternative is people from the Boltzmann equation use what is called as the Chapman-Enskog expansion. So, this is very classic way of deriving all the continuum equation starting from the Boltzmann equation.

So, this Chapman and Enskog expansion is done from the Chapman and Enskog expansion for example, for the viscous stress tensor and the heat flux vector you get several terms starting from the lowest order approximation to if you include all the terms together so that will be you are equal lent to solving your d T. Say if you remove all the higher order terms and use only the lowest order term that is your lowest order approximation.

Therefore, if you expand your viscous stress tensor and your heat flux vector based on the Chapman and Enskog you end up in getting a series solution with starting from the lowest order with to the highest order terms, and if you are taking only the lowest order term, so that will give you the Euler equation. For example, if you are retaining only the zeroth order term here.

(Refer Slide Time: 54:26)

So, the zeroth order term will be nothing but there is no viscous flux is equal to 0 and heat flux is equal to 0; that means there is no diffusion. This is nothing but the Euler equation. Whereas, if you incorporate both the zeroth order and the first order terms together you get your navies stoke equation. So, any other higher order terms that you incorporate will be giving you better solutions for higher Knudsen number.

Therefore, what people did, they just also incorporated the second order terms for larger Knudsen numbers.

(Refer Slide Time: 55:08)

So, with that in including the zeroth order, first order, and second order terms together you get what is called as the Burnett equation. I am not writing down the equation these are very very large equations because of so many terms, but never the less when you look at Knudsen numbers greater than 0.3, 0.4 people have to inevitably solve these Burnett equations. And these Burnett equations can give a better solution compare to the continuum equations.

(Refer Slide Time: 55:32)
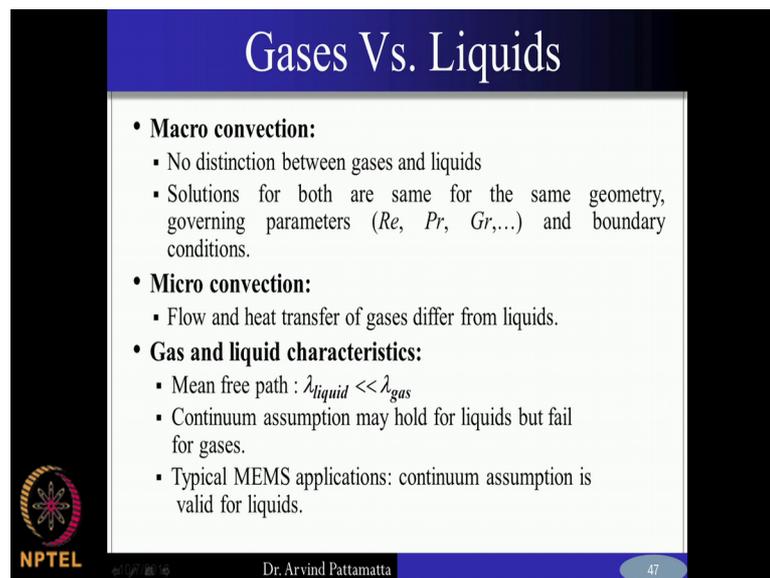
So, with that we will kind of stop this portion on gas flows in micro channels, most important aspects have covered. The next topic that we will be looking at is liquid flows and micro channels. Before we go on I think I will any way stop here, but you should understand that liquids in the sense of Knudsen number so have much smaller Knudsen numbers for the same diameter ducts that you are talking about compare to gases.

So therefore, hardly you go in to the slip regimes itself. Most of the times talking about Knudsen number values less than we know 0.01 even for that diameters of the order of few microns. So, usually in liquids flows you do not have any problem using the continuum equations also with the no slip boundary conditions.