Introduction to Boundary Layers
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Module - 03
Lecture – 15
Derivation of Prandtl's Laminar BL equations-III

Hi, welcome. So, we started out trying to drive the Boundary Layer equations and we started out from Navier-Stokes equations and where we reached up to was this.

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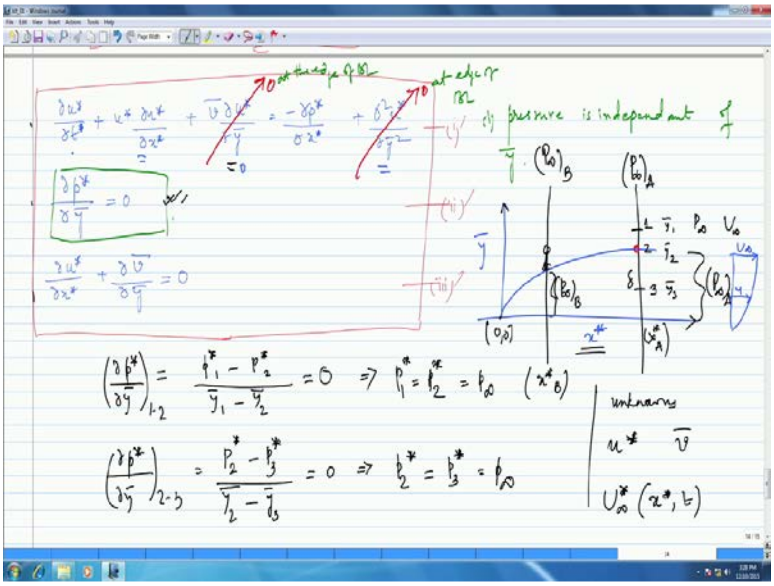

So, we got these 3 equations, and we basically began to non-dimensionalised the Navier-Stokes equations in kind of just giving an overview.

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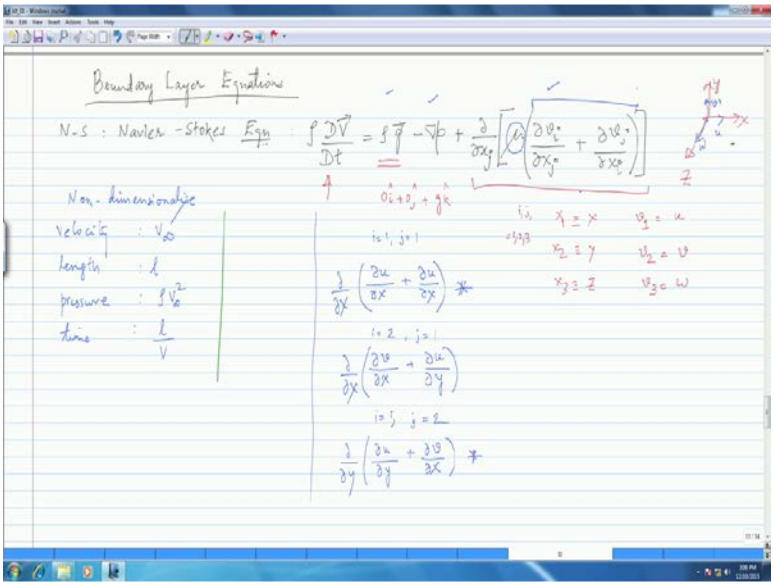

So, we non-dimensionalised it here and then we went ahead. So, this is a basic Navier-Stokes equation we wrote are the momentum equations in the x and y directions for a 2 d flow. Meaning, we have 2 components of the velocity and that is the operator, del operator; so we do that.

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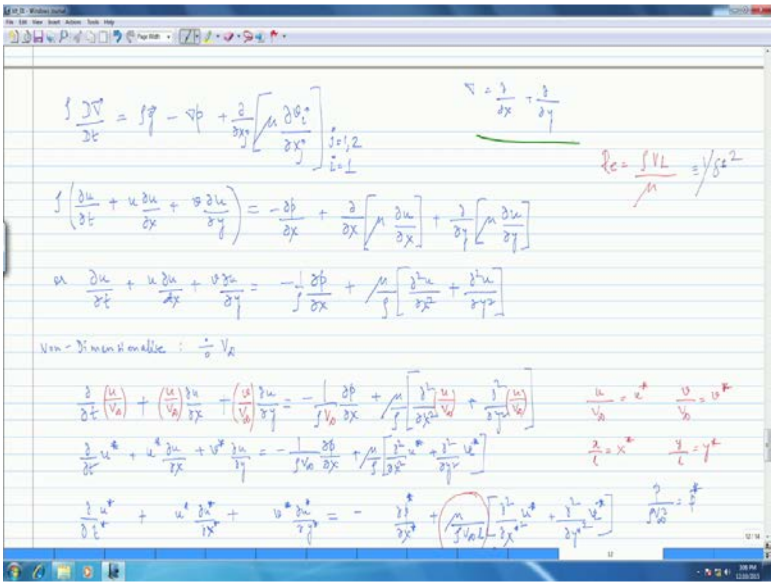

We wrote out the equation in the x direction, one direction basically. We normalize the velocities and the lengths and the pressure, but the by twice a dynamic pressure which is this and then, so finally, we came up with momentum equation in the x direction and momentum equation in the y direction. What we saw is that, since this has the viscus terms in terms of 1 by Re. So at high Reynolds number basically, the viscus terms disappears. So, if you write the equation in this form then this is basically going to reduce to the visit case and it will not account for any viscus phenomenon and therefore, we will not be able to study any boundary layer parameters, boundary layer behavior with these equations. So, we have to figure out what we would do then. So, then what we essentially did, right?

So, basically these are the 3 equations it reduces to after the first non-dimensionalisation. Now, because of this dependence of these equation on Reynolds number, and Reynolds number at basically high in a Reynolds number, the viscus term are disappearing and also at high Reynolds number the y star here, so that go in a tends to 0. So, therefore, this the Quaternary equation also becomes degenerate. So, basically we have landed a cells into little bit for troubles, how do we get out of that? So, what we essentially did was that we stretched, we basically stretched out this v star and the y star by a factor, and that factor was under root of R e, root of R e so that we came up with new non-dimensionalised term which is y bar. And y bar is nothing but mouth plying just y star by under root of R e and v bar is nothing but just multiplying v star by R e. So, we did that and hence we came up with these 3 equations.

So, now, what is interesting is this equation, right here, the second one. So, this one is very interesting and it reduces to pretty much you know, this. Now, when this happens what you can see here basically. We could you know basically say that these are they found a boundary layer equations. Now, if you look at this; the pressure, so del p star del y bar. What you see is that, if you took the derivative of the pressure with respect to y bar it is 0, which means what? Which means that the pressure here does not depend on y bar which means the pressure is independent of y bar that is interesting, that is interesting. So, therefore, you know it is a basically consent across a cross section of the boundary layer.

Now for example, if you were to say go I mean let us draw a little bit of a say, this my boundary layer and this is my y bar and this is my x star, now at a certain location and at a certain location say here, if I were to locate a quaint say you know this point here. So, if I were to locate this point here. So here, now what I will do is I will try to sort of calculate del p star del y bar here.

So, how would I do that for example, if I were to calculate it say here. I will then you know this is just simple calculation of maths. So, say I take a point here; so this is a point one and this is point 2. And this point 2 is just about at the boundary layer, just about at the boundary layer. So, then here, basically what I can say is that and then this value of Y bar here is y bar 1 and this is y bar 2, and we calculate the y bar from this axis here. So, then del p star del y bar, I can write that as p star 1 minus p star 2 by basically y bar 1 minus y bar 2 that is all. If I do that and this is supposed be 0, which means what? Which essentially means that, p star 1 is equal to p star of 2.

So, what if I want to put this in words, what I am saying is that there is a change in the pressure as we change the y bar, as we move along in the y direction. So that one star is p 2 star, but then this p 1 star is basically outside of the boundary layer and here is nothing but free stream pressure, which means that this is basically equal to free stream pressure. So, if I were to sort of use this equation right at the boundary, what I see is that the edge of the boundary just about at the boundary layer here where the boundary layer starts, it is free stream pressure.

So now, the similar way let us take another point which is set inside, let us call that 3 and this is y 3. If then let us just do this equation, apply the equation between 2 and 3. We could applied between 1 and 3 as well, so kind of you know teaches of a little bit. If I would do that then, so this one I went from 1 to 2. So now, what I am going to do is del p star go from that, go from 2 to 3 and what I get is p 2 minus p 3 by y 2 minus y 3 which is 0 which also implies that p 2 star is equal to p 3 star, which again is nothing but p infinity.

So, essentially that means, so now you can take, you could have done this by going between 1 and 3 also. So, essentially therefore, what we are saying is that the pressure does not vary with y, we not saying the pressure does not vary with x we always saying pressure is varying with y; which means that at a particular location of this location is x star a, so at this location pressure is going to be free stream pressure at this location. So, let us say this is free stream pressure at a, depends on how ever we want to write it, right. So, in case there is a pressure gradient in the free steam in the sense that I take another section here when do I call this as x star of b and this pressure here is p infinity b. When the pressure here, the pressure here will be nothing but p infinity b, like in this particular case here it is free stream pressure at a.

So, this is what it is this equation is this on, the second one. The momentum equation in the y direction goes down 2, which means that I do not have a gradient of the pressure in the y direction.

Next sense, another way of interesting you are looking at this, is that the pressure is almost you know it is like you impose the free stream pressure on to the boundary layer, you know you think this you know this is a guy who is just sitting here and is applying pressure on the boundary layer and that the pressure that this guy is applying is p infinity b, that is all. Also it is like you impose that pressure on to the boundary layer. So, this is what I understand from the second equation. So, pressure is basically a function of the x and time of course. So, now, having said that, right now pressure therefore, I think the important thing here to look at.

So, few words you look at these the 3 equations, here the unknowns were u star, v bar and p star. And now, since p star is basically now reduced to the free stream pressure we have got rate of one more unknown. So, the 2 unknowns that we have right now or the unknowns that we have are u star and v bar. We have 2 unknowns, we have. Now again, if you can make some more inferences out of this, let us look at something else as well.

Now, let us also look at, let us also look at this terms.

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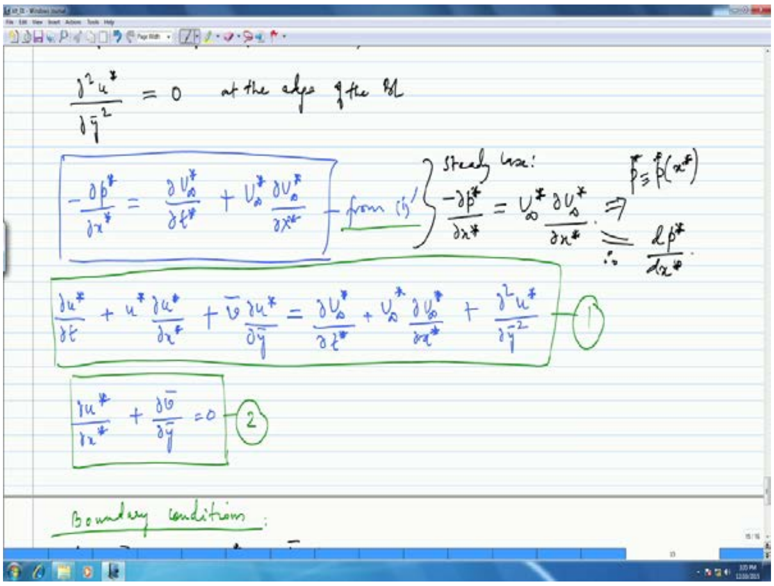

So, we have got del u star del y; del u star del y which is something, which is here. If I would have done this, so this is essentially the velocity. Now, just think about this term or this derivative; so this derivative again at the boundary. If I look at the boundary layer let us take this, let us take 1 and 2 again, so if I take that again and let us take it that. So, then this becomes, so if I do this between 1 and 2. So, that is basically what u star at 1 minus u star at 2 by y bar at 1 minus y bar at 2. Now what is this? Now, what is u one star? What is u one star?

It is nothing but free steam; this is outside of the boundary layer. So, this is nothing but free stream. So, this is you can say that, you can put that and this is free stream; so this free stream.

Now, what should be u 2? U 2 is up the boundary layer, just about at the boundary layer. And how do we define the boundary layer? How do we define this height here which is delta? That the velocity you know just about here is just about equal to the free stream, you know. So, right inside of this say line here, the velocity is 99 percentage so just of side is you know it recovers the entire velocity. So, therefore, u 2 is also equal to the free stream. And this remains and therefore, this is equal to 0 here. Now this is possible only at the boundary or edge, this is only at the edge of the boundary layer because this does not hold if you take 3 you know. Because 3, it is a different velocity.

So, in the sense that if I was to draw the velocity profile. So, this velocity profile is basically at x star a; I am just drawing it here. So, then this is essentially free stream, so this is u. So, if I do take, what I am basically saying is the derivative of u star with respect 2 y bar at the boundary, at the boundary is 0 obviously, right? Because the velocity at the boundary is equal to the free stream, there is no derivative. So, we come up with that. So, therefore this term is going to disappear. We also have this other term which is del 2 u star by del y bar. Now, that again will be 0. So, we will have del 2 u star del y bar 2. So, that again is 0 at the edge of the boundary layer, at the edge of the boundary layer. So of course, you know from all of this I think just to mention that the free stream of course, you know the free stream is a function of space x and t, it is function of x and t.

So, therefore, if we look at this equation here; let us look at this equation here. At the boundary, you know we will apply the first equation at the boundary. So, therefore, I can replace u star by the free stream, I can replace that. So, this term is going to stay, this term is going to go, this term is going to go and this is going to again stay. So, therefore, what I get is essentially an expression for del p del x. So, what I get is del p star del x star is equal to; let us go here. So, this is the term whether it is the first term which is del u star del t star; del u star del t star. Why did I miss that? So that, basically what we will say is del u infinity star del t star plus again if you go here we are going to look at this term because that is of this one actually is 0, let us cancel that out, let us do that so this basically vanishes; this also vanishes. So, we are left out with 3 terms. So, on the left hand side therefore.

So now, we are going to take the second point which is u infinity star and del u infinity star del x star, that. So, basically we get this. This we get from y, right? So, if we get this now what we will do is, we will use this expression of the del p star del x bar star, and but this pack into same equation one. That is that I can write one again as del u star del t plus u star del u star del x star plus v bar del u star del y bar is equal to, now minus del p star del x star that is what I am going to replace by this. So, which means that is del u infinity star del t star plus u infinity star del u infinity star del x star, that. And plus del 2 u star del y bar 2, and we going to call this as "1" let us call this is as 1. Now, it is very important let me just say this one more time that what basis did we get this.

Now basically, we said that the pressure you know the p star or whatever. So, that is basically does not change it a given location x, x star. So, it because; and where it we get that from, we get that from this second equation. So, therefore, p star is nothing but you know p stream pressure. So, if that is so, it does not matter. So, we do not need to write p star at all here we can just write that as free stream pressure, I can just write this as p infinity. Therefore, what I did is that I took the first equation and I impose that into the; I use that at the boundary and when I use that at the boundary again as we said, as we showed that these derivative del u star del y bar and del 2 u star these disappear. This is only at the boundary, hopefully you remember, at the edge. This is at edge of boundary layer, of boundary layer; this is valid. And also at the edge of the boundary layer u star is basically going to the free stream.

So from there I was able to write an expression for the delta, for the del p star del x star in terms of the free stream velocity. So, I can write the change in del p star del x star in terms of the free stream velocity. So this piece, it does not matter because wherever you take along, so the only different that I mean pressure does not change in the y direction; it can only change in the x direction. Therefore, at a given x this will hold and the pressure at the given x; pressure is nothing but the free stream pressure. So therefore, I can relate this free stream pressure to the free stream velocity which can change in the x direction, because we said that u infinity is dependent on x star and t.

Therefore, since I can write this pressure term this way. So, I put it back in to the first equation and we get what we get is 1. So therefore, we kind of incorporate the second equation into the 1, so we do not need that equation any more. So, what we left that with is the third equation which is nothing but the continuity equation and we will keep it that way. So, basically what we have is.

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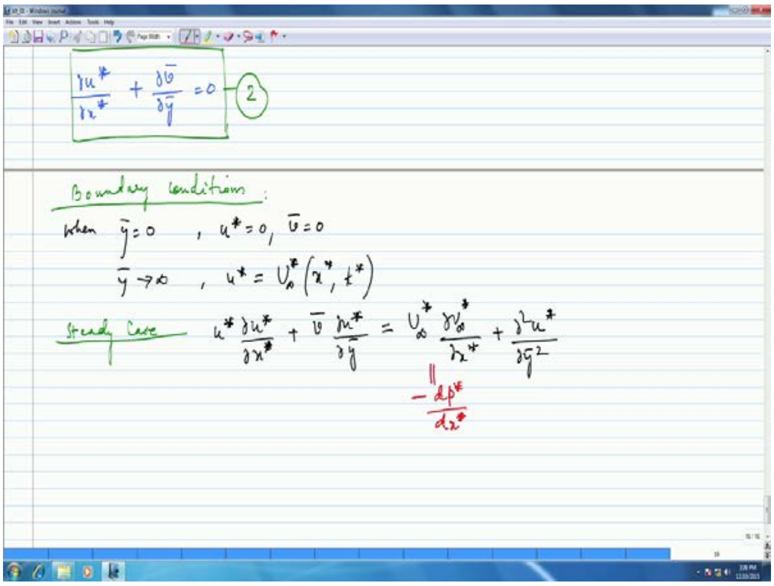

So, basically what we have is. If I have that, I am going to call it that. So, therefore, I reduce my number of equations again, 2 by 2. So essentially, my job now here is to solve; solve these 1 and 2. Now, to do that; I will do that, but every time you have to solve set of equations you also need a corresponding boundary condition and I mean that comes in numerical analysis. So, I also need a set of boundary conditions. Now, the boundary condition in this case we should be able figure that out. The boundary conditions; let us write that down what are the boundary conditions.

So, the boundary conditions, now when y bar is 0; y bar is 0 which is basically at the surface, u is 0 and v is 0. So, basically we have two equations and two unknowns. So, u star and v bar are my unknowns and we have two equations and we have got to set off solve these. My boundary conditions means that at the surface you know were of whatever lifting surface we are looking at. So, what? It may not be lifting surface or so, fine whatever surface on wards the boundary layer has to follow up. So, when y bar is 0 post these 0, and then how we do we get this? Because, this from is the basic definition of a boundary layer or friction of viscosity. So, that is what we get it there from.

And when y bar is far away, y bar is when we go really far away from the surface then this is basically nothing but free stream. Free stream which depends on this, it depends on x and t. So therefore, what we will basically do is that. So, right now the thing is that we will all have to do is solve 1 and 2 using these boundary conditions. Now, let us reduce this even further, let us reduce this even further. For example, we will say it is for steady case, let us say it is for steady state case.

So, if it is for steady case, then this the first term will you know will disappear. So, what we will then have is basically u star this, plus v bar del u star del y bar is equal to u infinity star del u infinity star x star del 2 y bar this. So, this is basically that. So, this is for the steady state case so essentially that. If you look at this, so this was my description of the pressure, the gradient of the pressure in the x direction. So, again if for steady state case, with this same expression for steady case, then del p star del x star will become u infinity star del u infinity this by del x star. So, what you can now see is that basically in my pressure is only function of x star not even t because I think when we came up here, did I write that down. So, pressure is the function of x and t because, now I can, rather in terms of the free stream.

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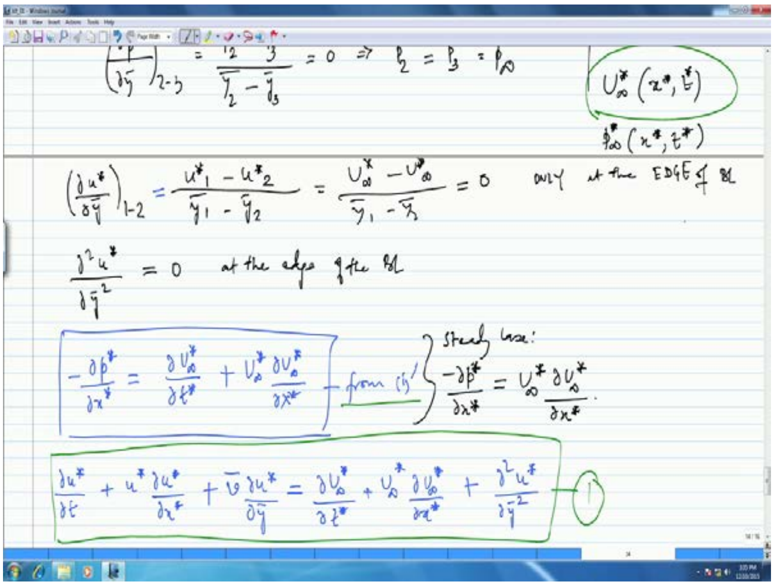

So here, pressure is also a function x star and well you know, you can write t star here also; no problem; t star. Now of course, we say that the pressure this means that pressure is a function of only x star. So, therefore, what I can write at here is a full derivative which is that this therefore equal to, therefore this equal to dp star dx star. Therefore, this term that I have written here u infinity or whatever this can also be written as. So, this thing is essentially equal to minus dp star dx star. So, that is what we reduce it to, so what I will do is set of stop here for this module and come back and look at this little more and look at the skin friction coefficient and also talk about you know what we did.

In fact, we will spend some time, we have come up with kind of simple equation looks like at least from the Navier-Stokes equation. So, let us discuss a little bit in the next module. So, we will stop here for this one.

Thanks.