Design and Optimization of Energy Systems

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Lecture No. # 37

Dynamic programming

So, we will continue with our discussion on linear programming. So, we will solve one more problem. We will revisit a problem which we formulated earlier so that it strengthens your understanding of how the algorithm works, right.

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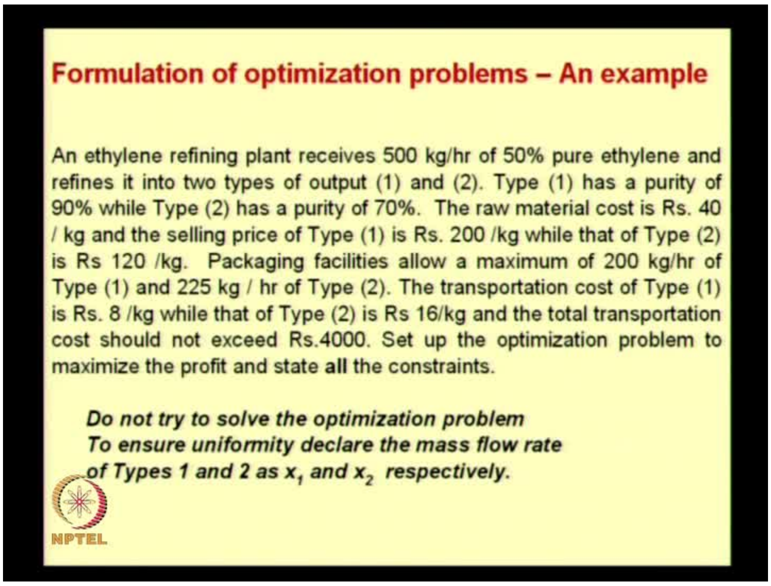

What is the problem number? 40. So, you can take it as problem number 40. So, we are familiar with this problem. This concerns an ethylene refining plant. It receives 500 kg per hour of 50 percent pure ethylene and it refines it into two types of output; type 1 has 90 percent purity while type 2 has 70 percent. Raw material cost and profit everything the selling prices is given. There are constraints in the form of packing and transportation, packaging facilities how much can be packed per hour and then the transportation cost for type 1 and 2.

So, we formulated the optimization problem in one of the earlier classes in the semester. I said at that point do not try to solve the optimization problem. To ensure uniformity declare the mass flow rate of types1 and 2 as x1 and x 2.We have plotted all the constraints; x1 and x 2 are the two types of output. We declared the two types of output as x 1 and x2. So, please plot these constraints in graph sheet and try to get the optimum. So, I will just change the problem formulation essentially the same problem. People who have not got the graph sheet, how many of you have not got? Does everyone have one, okay? It should be there in your notes, right. We have already done this, okay.

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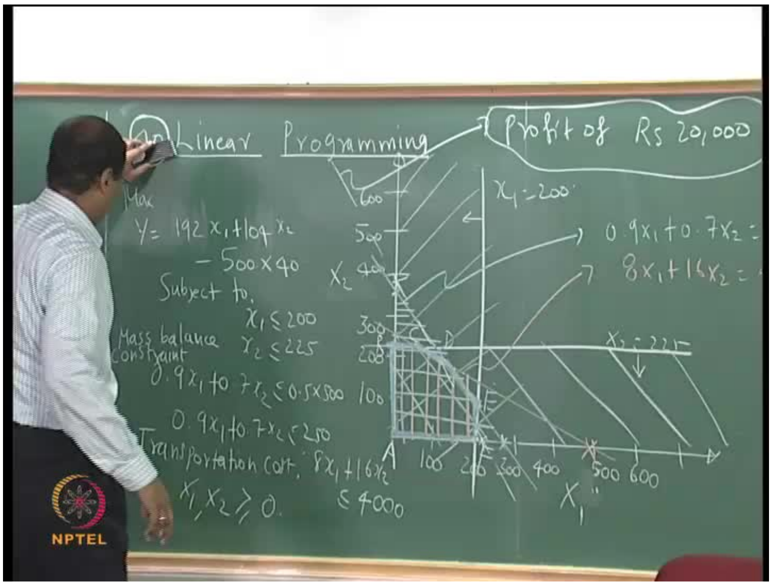

So, now that we have some scale like device we can use this. So, 100, 200, 300, 400, 500, 600, so, you have to plot all the constraints.

First identify the feasible region and then plot an iso-objective line and move the iso-objective line till it cuts the feasible region at the farthest point because you are moving away from the origin because when you are moving away from the origin each objective line iso-objective line represents a higher and higher profit. The highest profit subjected to the constraint that is what you are seeking, okay. So, there are two constraints in this problem. Max Y selling price is 200 per kg. So, 200 x 1 plus 120 x 2 minus the raw material cost is 40 per kg, okay. 500 into you are buying 500 kg for the raw material minus 500 into 40, correct; subject to first is the packaging constraint. Is there one more graphic pads doing the rounds? I gave 2 pads, what happened to the other one? Is it done or? There is still some sheets left? Yeah Vaibhav, you have to get those sheets.

Transportation cost is how much? So, we changed it 192 x 1 plus?

Student: 104.

Yeah, I hope all of you are able to get this. The selling price is 200 per kg for x1 and the selling price is 120 per kg for x2, right. But there is a transportation cost involved which is 8 rupees per kg for x1 and it is 16 per kg for x 2, right. So, you have to put 200 minus 8 and120 minus 16.This is only the transportation cost, then you have to subtract the raw material cost which is 500 into 40, right; subject to x1 is less than equal to 200, x2 is less than equal to 225. So, these two take care of the constraints involved in the packaging. Now there is a mass balance constraint, what is the mass balance constraint? 0.9 x 1, x 1 has got 90 percent purity plus 0.7 x 2 must be less than equal to 0.5 into 500, okay.

What about the transportation cost; 8 x 1 less than equal to 4000, right. Now x 1 is I have to plot. So, I can choose the region only here, correct, because it is x 1, okay. And of course non-negativity constraints x 1, x 2 must be greater than equal to 0. So, first we have to identify without plotting constraints or the other constraints itself we have got, without plotting the transportation or packaging constraints we have a restricted region. Already the feasible region is reduced. Now we have to put in the constraints. So, 0.9 x 1 plus, can you plot that? 0.9 x 1 plus 0.7 x 2 is less than equal to250. Where does it cut the X-axis?

We plotted know, we worked it out in one of the earlier classes, right. Tell me, 250, 275 is it; 250, 277 and x 2?

Student: 357.

Correct 357. Now we will use the same scale. So, what is this? Alright, now the other constraint is. We will use the brown color. Other constraint is 8 x 1 plus 16 x 2 less than equal to 400. So, x 1 will be equal to 500, right; x 2 equal to 250, correct. Now we will use the blue color chalk; we will use blue color chalk to identify our feasible region, okay. So, this feasible region has to enclose all the lines, chocolate brown, blue and the two the horizontal and the vertical white lines, right.

So we have to evaluate, how many corner points are there now. I told you there is a property for linear programming problem where you will have to search only at the vertices, right. That can be proved; if you take a LP course somebody will prove it. So A, B, C, D, E, F, so there are six vertices A, B, C, D, E, F; so, you evaluate the objective function at each of this six vertices. One of this is a trivial solution x 1 equal to 0, x 2 equal to 0; profit becomes minus because you bought 500 kg of raw material, you did not do anything with it, okay. That is the trivial solution.

Anyway for the sake of the record you just work out the solution and say that profit is minus. Then, you evaluate the Y for all the other solutions and invariably the solution will be that which is farthest from the origin because the iso-objective lines will move in this fashion. Now how will the iso-objective lines look like? Yeah take one iso-objective line. Please take x 1 equal to 500. Can somebody do that? x 1 equal to 500 and x 2equal to 2. Abinanthan, can you do this? x 1 is equal to 500 and x 2 equal to 2. Now x 2 equal to 0. What will be Y?

Sorry?

Student: x 1 should be less.

No, no, I am drawing some line; I want to know the slope of some line, okay. I want to draw a line. What is the y? Okay, that is what I am saying Y value 20000 will come man. Okay, Y value is 20000. So, this will be totally it will be 40000, right. Take Y equal to 20000; that is a good way to start with. Take Y equal to 20000; add this 20000 to the left side. So, 192 x 1 plus 104 x 2 will be 40000.Tell me the two points, so you can join these two points and have a straight line and then we can move this line up and down, correct. But even in the exam I expect you to draw all the constraints and I expect you draw at least one iso-objective line, so that I know that you understood how this thing works. Even without that you can just evaluate all the vertices and get it but that is not completely a graphical method, right. I want you to draw one iso-objective line. Please remember one iso-objective line you have to draw. Can you tell me x 1 and x 2?

Student: 0, 385.

Which is 0?

Student: x 1 equal to 0.

x 1 is 0, x 2 is?

Student: 384.

Okay.

Student: 208.3.

Oh, this is very crowded. 208 is it?

Student: Yes.

So, this is the. So, this may not be clear. Please look at the board. This is the profit of 20000, okay. So, you will get iso-objective lines which are parallel to each other. So, when you move like this you will profit less than 20000, 15000, 10000, like this but you cannot go left of this because x 1, x 2 have to be greater than 0. Now you can keep on moving it till it just escapes the feasible region. When it just escapes the feasible region since it is a convex polygon, since it is convex it will touch; it will touch the feasible region at one point or it could touch at several points, if one of the constraints it parallel to the iso-objective line in which case you will get alternate optima. But in this case it does not happen to be so.

So, I guess E must be the solution. Is that correct? E is the farthest. So, I think when you move like this, it looks like E is the solution, right. This is the graphical method. But in the exam you can just draw one iso-objective line and just say that when you move it, it will meet but the exact solution can be worked out by finding Y at A, YB, YC, YD, YE and YF. Please do that. Okay, is that clear? So, there will be a question on linear programming in the exam10 to 12 marks; I think all of you should be able to crack it. One on Golden section search or Fibonacci method will be there at one. So, straightaway out of 100 you can get 22 marks. Do not mess up these two. Do not go to the Lagrange multiplier. First take the linear programming and solve it. First take the linear programming, then go to the golden section search, then you have passed then course. Then [FL] you do whatever you want. Do not give me trouble because last 10 years nobody has failed in this course. So, this is the.

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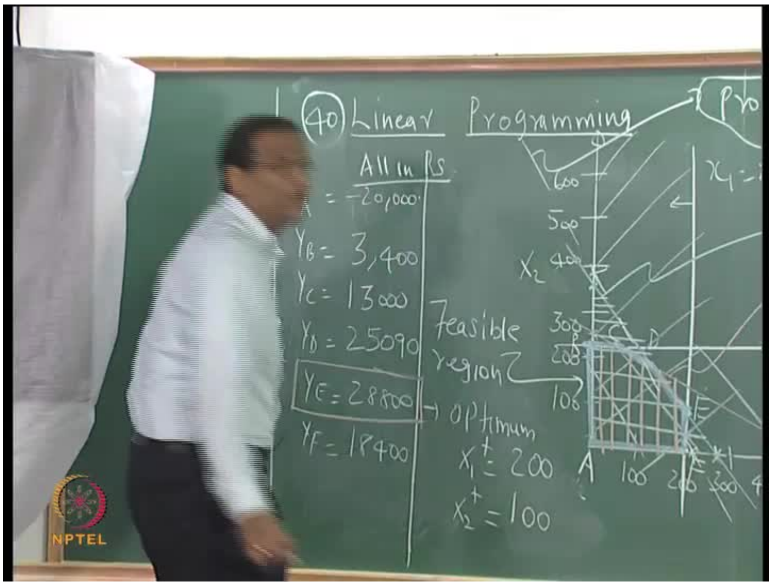

Now Y A, so A is 0, A is minus 20000; that is very bad. Chaithanya you are awake? So, what is B? B, Abishek did you calculate? Yes, why you do not you?

Student: 3400.

3400 all in Rupees, C is how much? Alok did you get it?

F is easy. Tell me what is F? C is?

Student: 13000.

13000 okay, D should be more I do not know.

Student: 25090.

25000?

Student: 90.

90?

Okay 25090, is it. Because of the decimals, okay. E, E must be?

Student: 28800.

Okay good. F?

Student: 18400.

Now you evaluated Y at the six points. Now you know where Y is maximum; that is the optimal solution to this problem. This is how you solve this problem using the graphical method. You can introduce slack variables, S 1, S 2, S 3, S 4 like what we have discussed in the previous class and solve it algebraically also. A very efficient method of introducing slack variables and solving it efficiently is called a simplex method which can be used for any number of variables and any number of constraints; so long as both the objective function and the constraints are written as linear combinations of the variables, okay.

So, at the optimum x 1plus equal to, what is the x 1?

Student: 200.

200, 100 in appropriate units kg per hour, right. Can we switch this off, but I will use it again. I want the board now. So, we have solved two problems. So, this gives you a good idea of Linear Programming, right. It is very popular. It is in fact it so happened that yesterday one of my students, some of you may be knowing Persharma; Persharma he called me from Singapore. So, he is working for Unilever. So, they have a problem. So, they are getting some raw material from 28 suppliers located in 25 countries some 20 different items and then they are trying to minimize the cost and then he says ‘Sir, I want to know some optimization technique’. I said you go to the Google and get a free download LP. So, he wanted to know whether he has to use some other technique; I said LP will work and he is working on it. If you do not get the answer, you call me back. So, people are using LP man.

So, Unilever is interested. When I tried to call him back I got 3 missed calls. Generally Professors are very poor in picking up calls, right. So, we are either at home or in the office. So, we do not travel that much. So, then I call back then it says this number goes to Unilever Singapore; it is busy. I do not know what; why I should I did not do anything in Singapore, nobody. Anyhow, some odd number it was, so I called back. Then I was in Bangalore yesterday. So, when I was going to the airport this guy called me and said that he wanted some clarification.

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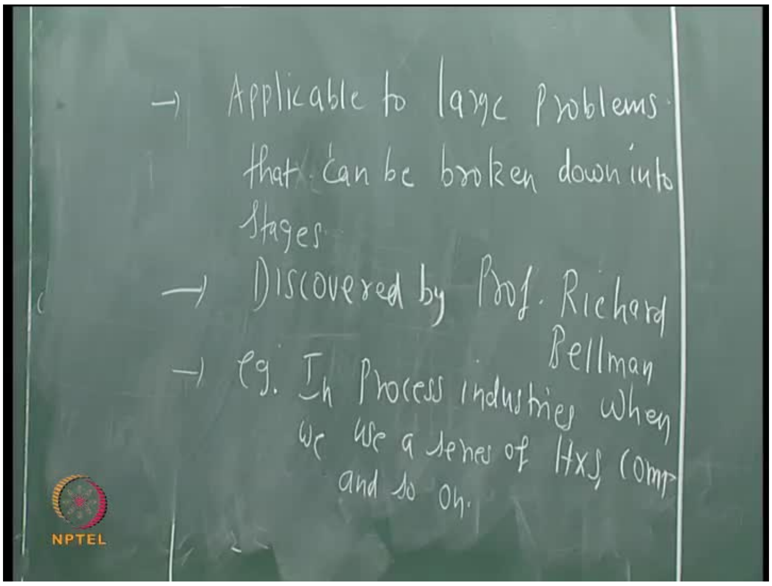

Now we will look at some other special techniques. One of them is a dynamic programming. So, it is applicable to large problems that can be broken down into stages. A complex problem can be broken down into stages and you look at the optimal solution, for each of the sub stages and keep proceeding. The belief is if you continue to subdivide a complex problem into stages, and you optimize each of these sub stages when you reach the end, that is from start you go to the end the overall optimum you will get. So, this can be proved for certain classes of problems and this was discovered by Prof. Richard Bellman Applied mathematician and Professor of Applied Mathematics in the University of Southern California, right.

So, it is called the Bellman equation. He figured out the Bellman optimality theorem, Bellman condition and so on, right. So, example in process industries when we use a series of heat exchangers, compressors and so on, what is the minimum cost of transportation? Like that. When in process industries when we use a or you want to lay a gas pipe line between two cities and if you want to lay a gas pipe line between two cities, but you want it to pass through some intermediate cities so that you want to have quality control and all that, and then there are various ways to go from one city to the another city through these intermediate cities which will be the optimum path for going from one city to another city, okay. So, this will be.

So, what you do is in each you subdivide that while going from city 1 to city 2 in between you have got three other cities A, B, C, then what is the shortest path from 1 to C; what is the shortest path from C to D, D to E, stage-wise you optimize so that suboptimal solutions at each stage are eliminated. This way there is a computational gain. What is the funda I told you? Optimization is all about trying to get the optimum without working out all the possibilities. If all the possibilities had to be worked out it is called exhaustive search; it will no longer be called dynamic programming.

So, there is a computational gain or the simulation gain. In simulation we call it simulation gain; here we call it as a computational gain. The computational gain is what is the gain you achieve with respect to performing an exhaustive search? For problems involving many such networks, it may be extremely complicated for you; the problem may be so formidable that it is impossible for you to do an exhaustive search. So, this dynamic programming is also used extensively by computer science people and example, this is one example.

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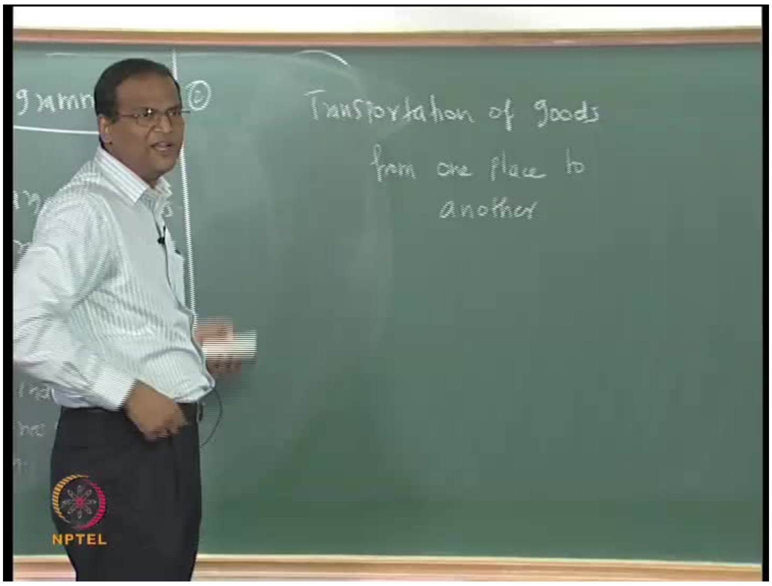

Then the other example is transportation of goods from one place to another. Of course there are some constraints that it has to pass through some intermediates cities. There are only some fixed paths where each of the fixed paths you know the costs and so on, alright. Now it can be mathematically derived and we can have an algorithmic approach but we will just take up a simple problem and then using a commonsense solution without going through rigorous numerical evaluation of the functions, objective function and so on, the technique will become very clear to you, right. Dynamic Programming is also used in the Duckworth-Lewis method. What is the Duckworth-Lewis method?

Student: Unsuccessful method.

No unsuccess, why?

Student: Some Indians tried a better method know sir.

It is published in which journal?

Student: On ESPN they mentioned.

No, maybe it is. But Duckworth and Lewis they used dynamic programming and this paper is in the Journal of Operations Research; I have this paper I will show you towards the end of the class if I have time. Basically there are some control variables; they are optimally dividing, right. In the 50 over match, there are 10 wickets, how much some team will score?

If you have only 40 overs and 10 wickets, 40 overs and 6 wickets, so he has an exponential distribution for each of this. There is a double exponential distribution when you worked out at using past data from rain interrupted matches. Duckworth and Lewis figured out what should be the constants of these curves and then they convinced the Wales and what is the cricket bowler.

Student: ICC.

Then ICC and then they tried it and now it is and then look-up table is available. So, you cannot expect umpires to solve dynamic programming problems. Every day they have to take quiz. So, it is easy to implement and it has been successful. Of course if there is some other better method which comes up and already so many people, there was an average run rate previously; ARR average run rate was used. Basically for people who do not know cricket, so this Duckworth-Lewis method is used to determine the target for the team playing second in a rain interrupted match. So, I do not know for T20 it has not come, right.

Student: They use the same data sir, minimum 5 overs.

Oh, so anyway now. So, increasing use of mathematics and operations research in sport, right, in cricket also, fine. Let us take an example.

Problem number 41: Determine the minimum cost of transportation of goods from city A to city B while passing one node each, of B, C and D as shown in this figure. Lots of you are putting up blank pages. So, I think for strategic reasons first I will draw the picture and then continue.

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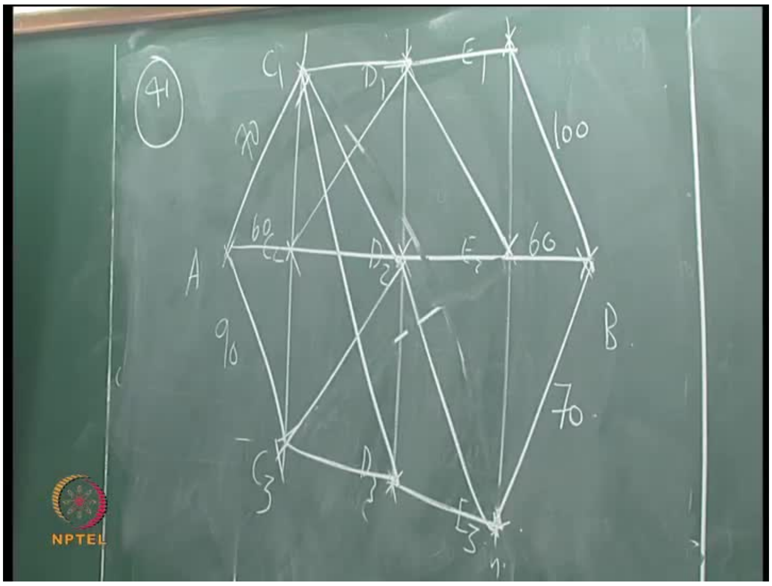

So, 41, okay. So, you start from city A. You want to go from city A, you want to go to city B. But the funda is you have to cross through there are three intermediate stages you have to cross. So, there are three possibilities here. C 1, C 2, C 3, D 1, D 2, D 3, E 1, E 2, E 3, okay. So, several paths are possible, right. This is one path, right. This is one path. There is the r g path 4 but actually I may give C 2 D 2 very high cost, so that it does not work out, right. Then, you can go from A to this thing, then you can go from here, you can go from here, you can go from here, you can go from here, you can go from here, you can go from here; did I miss out anything? But you cannot jump directly from C to E. For strategic reasons you have to pass through one of this C's, D's and that is why through one of the nodes; either it should be C 1, C 2 or C 3 at this stage; it should be D 1, D 2, D 3 and E 1, E 2, E 3.Now we will assign cost. What is that? It may be the cost of laying the pipeline or in this case the transportation of goods; it may be the cost of laying the pipeline.

Student: C 2 E 2.

C2?

Student: C1 to E3, yeah.

That is not allowed.

Student: C 1 to D 2, C 1 to D 3.

That is also not allowed.

Student: Why C1 to D2 not allowed?

Student: Go from C 1 to D 3.

C1 to?

Student: D2, D3.

Yeah, that is allowed, correct.

Student: C1 to D2, D2, C1 to D2.

Yeah, you please draw them, how many paths are there?

Student: 27.

27 paths are there. If you draw 27 paths we will go wild. So, we will just indicate. Now the funda is if you are able to calculate all the 27, there is no need for dynamic programming. But imagine that instead of three choices you had six choices or seven choices. And, instead of C, D, E, you have seven intermediate stages, then you will have to keep on doing it; it will not and if the objective function also involves some calculations, then it is worthwhile to develop an algorithm for this. Now so this is 70. You can take this to be in1000s of rupees or whatever, okay. But we will take it as70, 60; the cost is 70, 60, 90 and this is100, 60, 70, okay.

So, we will continue with the problem. From city A to city B, while passing through one node each of B, C and D as shown in the figure, the cost in appropriate units. Some costs in appropriate units are given on the figure and the rest are tabulated, I will give you the table. Some costs in appropriate units are given on the figure and the rest are tabulated. Using DP Dynamic Programming, determine the optimal path to minimize total cost, right. I will give you the other costs.

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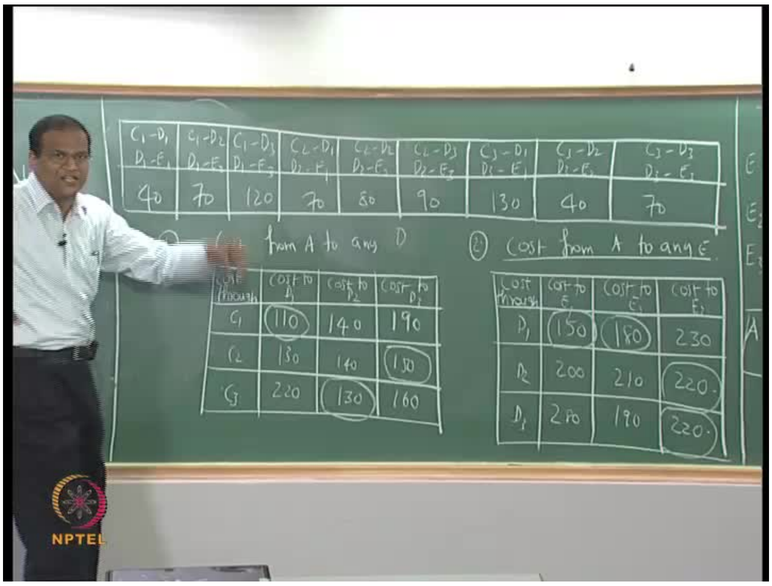

C 1 to D 1, D 1 to E 1; C 1 to D 2, D 1 to E 2; C 1 to D 3, D1 to E3; C 2 to D 1, D 2 to E 1; C 2 to D 2, D 2 to E 2; C 2 to D 3, D2 to E 3; C 3 to D 1, D 3 to E 1; C 3 to D 2. To make matters simple, we have assigned C 1 to D 1 is the same as D 1 to E 1 and so on. I will you give the cost 40, alright. So, all the costs are given. Now you have to apply the algorithm.

Now I will show you the first step and then it will become very clear to you. The problem has to be subdivided, right, into stages. So, you have to find out the cost from A to any D first because the final optimal path must pass through either D 1, D 2 or D 3. First get the optimal, first get the total cost to reach any D, right. How many paths are there?

Student: 9.

3 into 3, right, there are 9 paths. So, there will be a tabular column with 9 entries. Then we will start proceeding from there. So, the first step is dividing into sub problems. So, we will check cost from A to any D.

So, cost through cost to D1, cost to D2; are you getting the point? You have to evaluate the cost from A to any D; A to any D can be reached by starting from A and taking any C or any D. There are three possibilities of C and three possibilities of D. So, there are nine possibilities. A C 1 D 1, A C 1 D 2, A C 1 D 3, correct, A C 2 D 1, A C 2 D 2, A C 2 D 3, A C 3 D 1, A C 3 D 2, A C 3 D 3.You evaluate the nine entries, right. So, what is the cost first entry, 110? So, this is 70 plus 40. So, A C 1 D 1 look at the board. A C 1 D 1 will be A to C is 70 plus C 1 to D 1 is 40. So, this is 110, fine. A C 1 D 2 140, A C 1 D 3, 190, good; yeah, please tell me the other entries.

Student: 130, 140, 150.

Very good, 130.

Student: 220, 130, 160.

Very good, is everybody through with this?

Student: Yes sir.

Pretty straightforward, okay.

Now we have to say circle the optimal path to D1, optimal path toD2 and optimal path toD3. At this stage it is too premature to look at the minimum and then proceed only from there because we do not know what is the story from D to E. So, we should not try to get premature convergence by going to local optima but one thing is pretty sure; if you want to reach the end, it has to be through any D, okay. So, to reach each of these D's, what is the optimum path? Once you determine the optimum path from each of this D, when you go from D to E, you will use only the optimal path and the suboptimal paths will be left out; that is a dynamic program.

It will be very clear now. What do you want to circle here, the lowest cost to D1? Are you getting the point? Cost to D 2, 130; cost to D 3,150, correct? Now cost from A to any E. Till the evaluation of these nine entries no dynamic program; we did not apply any algorithm. Now when I circle this I applied the algorithm already. There are two things. I am looking at the minimum cost to D 1, D 2, D 3, but I am not looking at the minimum cost in the table and removing the other things; that may result in a suboptimal solution. These are the two cardinal points in the dynamic programming.

So, now how do you reach E 1, E 2 or E 3? It has to be through either D1, D2 or D3, alright? But when I reach from D 1to E 1, the other paths like A C 2 D 1 E 1, A C 3 D 1 E 1, I do not evaluate because I have already eliminated that. There is no point in calculating again, then it is exhaustive search because anyway up to D1 I know what is the optimal path. Why do you have to worry about that suboptimal path by the time I reach D 1; that is dynamic programming, are you getting the point? So, now D1 to E1 110 plus, good, 150, 180, 230; Rajesh, you got it? Second one: 200.

Student: 280, 190, 220.

280, here it is 180. Here we may have two. Now we want to use this.

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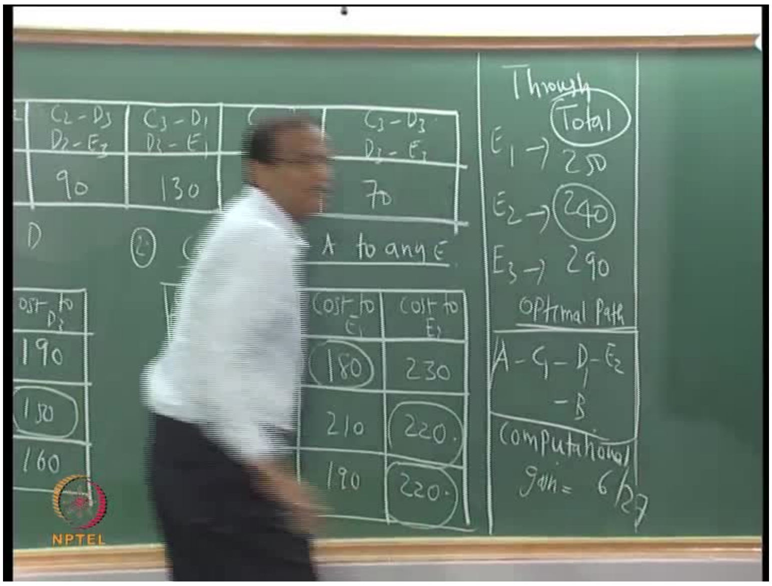

Now through E1 suppose you want to calculate the total; through E1 already it is 150, it is 150 to reach E1, 150 plus 100, 250, correct. Through E2, it is?

Student: 240.

How? 180 plus 60; up to here it is 180 plus 60, 240, E3. So, the optimal path is A C 1 D 1 E 2 B, it is not A C 1 D 1 E 1 B, right because anyway we can change the coefficients. So, the control variables were basically the cost associated with each step. But in the cricket match there are two control variables, number of wickets in hand and the number of overs. So, it will be a lot more difficult. So, they must have written a program and solved it. So, you can have any number of variables. Here this is simple one variable, okay. For the cricket match there are two variables. The resources of a team basically dependent on two factors, the number of wickets and number of overs left, okay.

So, what is the advantage, what is big thing we have done? How many evaluations did we do? We did 9 here, we did 9 here18 and we have added 3.How many evaluations? 21. If you did in the exhaustive search, how many evaluations? 27. What is the computational gain? 6 on 27, are you getting the point? So, there is a reduction of 28.52 percent effort because you applied dynamic programming. At each and every stage you identify the optimal solutions and at a subsequent stage you carry over from the optimal stage left behind in the previous stage and there is no distinction between the start and the finish. You could have started from B and then founded what is the optimal path from B to E, B to D, B to C, B to A; it will give same solution.

It does not matter whether you start from left or whether you start from right. At each and every stage whatever is the optimum you get, you proceed from there and discard the suboptimal solution at a particular stage. But do not discard because necessarily it has to be through 1 D 1 D 2 D 3. Do not remove all the things and look at the minimum; that may mislead you, are you getting the point? So, this is our computational gain 6 on 27. Can I quickly have them; I will just finish in 3 minutes.

I encourage you to read this interesting paper by Duckworth and Lewis; a fair method for resetting the target in one day matches. So, this is in the Journal of the Operations Research Society in 1997. So, they are from UK. Where is the percentage here?

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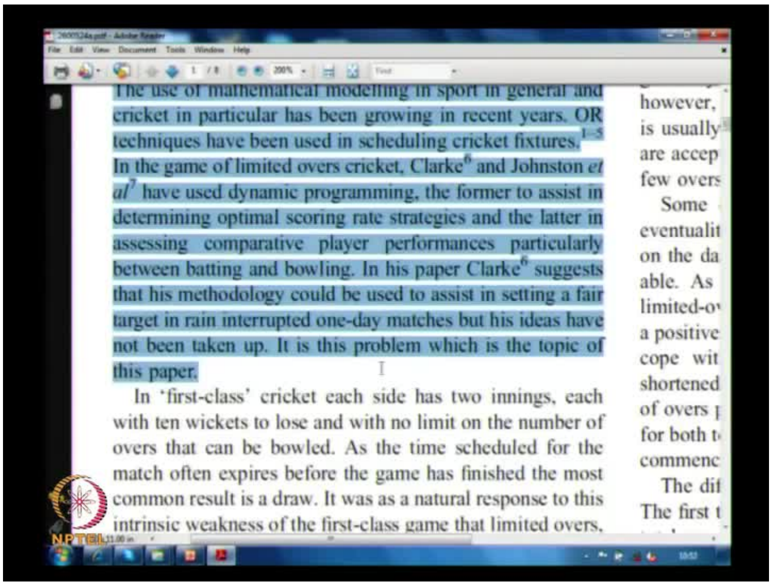

So, the game of mathematical modeling in sport, you can see Clarke and they have used dynamic programming to assist and basically they are saying that the use of mathematical modeling in sport and cricket has increased. So, that is this thing and how other people have used DP and all that.

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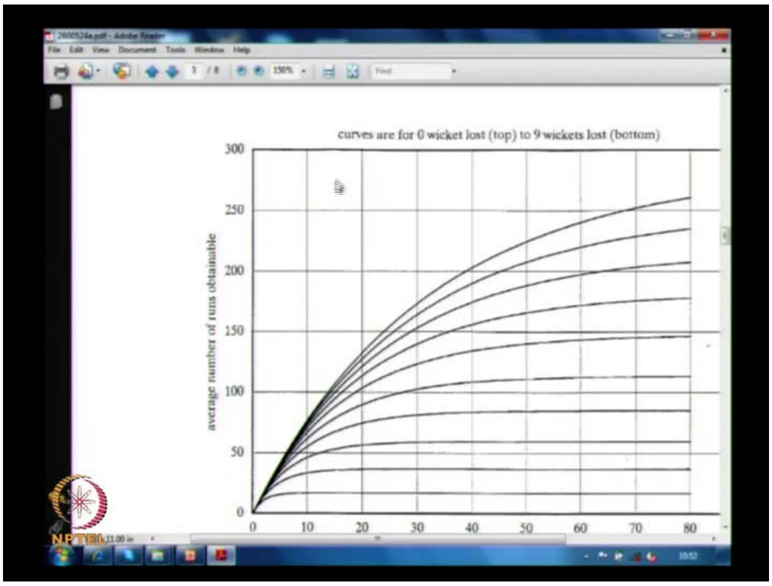

So, the key here; this is the key figure for the Duckworth-Lewis method. So, the number of runs which a team can score, average number of runs attainable, it is basically a function of two variables; the overs remaining and the wickets in hand, okay. So, in a 50 over match, if you have 30 overs left, if you have 9 wickets left; so therefore, 0 wickets lost is at top and the all the wickets lost is your trivial solution, okay. So, it goes exponentially like this. So, the resources are two resources; it is either that it is the wickets and the number of overs.

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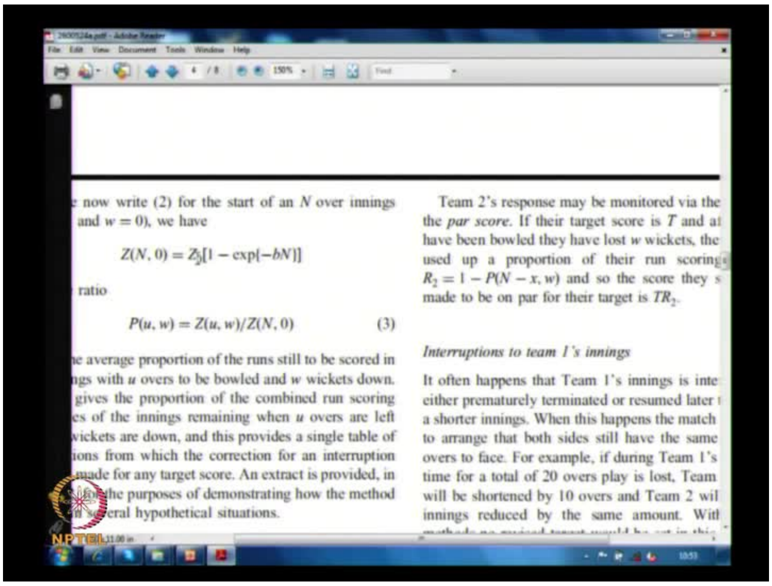

So, they have put an exponential model Z N, 0 is 1 minus E to the power of minus beta N and b. They have put an exponential for this and they say that confidentiality agreement will not tell us how the data was, how the constants were fixed and all that. But, no, no, they say that.

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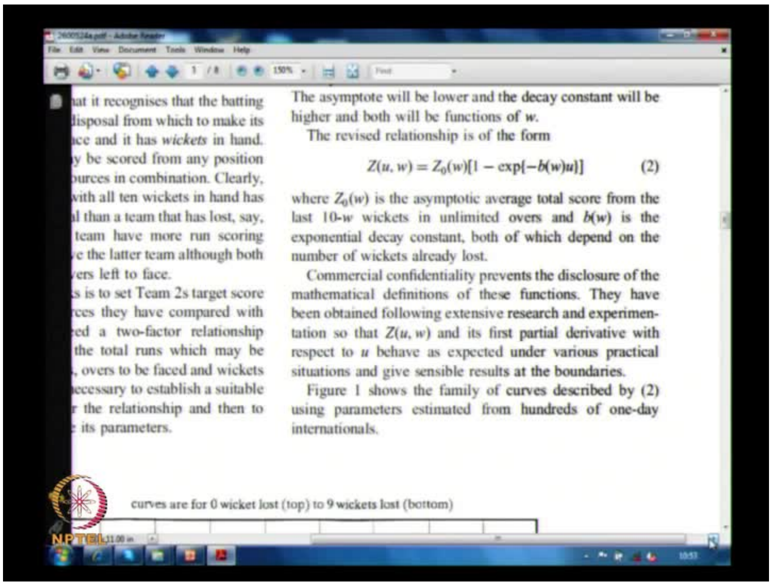

Commercial confidentiality prevents the disclosure of mathematical definition of this function; they may obtain the following extensive research, they are right so that partial derivative and all that. Good to know, right, so much of mathematics in cricket. So, this is not being challenged; some of you who are doing your M.S or PhD in OR, then go ahead and write a paper which will tear this apart, okay. And so, you can see that dynamic programming is very much used.

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So, this is our Bellman equation Richard Bellman, right.

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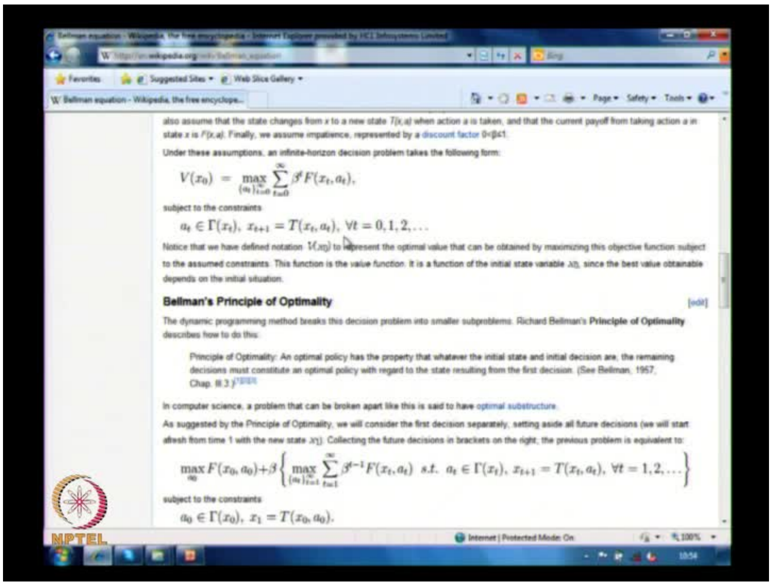

The principle of optimality says an optimal policy has the property that whatever the initial state and initial decision are, this is basically from Wiki; the remaining decisions must constitute an optimal policy with regard to the state; that is what we have applied, right. Regardless of the initial condition when you reach a particular state, at that state whatever is optimal, for the next stage you carry over from that optimal stage. Do not keep evaluating other suboptimal solutions.

So, he was in the USA Southern California, right. So, he became a fellow of the I EEE for his contribution because the computer science people recognized that dynamic Programming is a very useful tool, right. So, tomorrow we will start with Genetic algorithms.