Design and Optimization of Energy Systems

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Lecture No. # 22

Properties of Objective Function and Cardinal Ideas in Optimization

Good afternoon, so we will continue our discussion on optimization. Towards the end of the last class, I did present some ideas about optimization the general principles involved in optimization.

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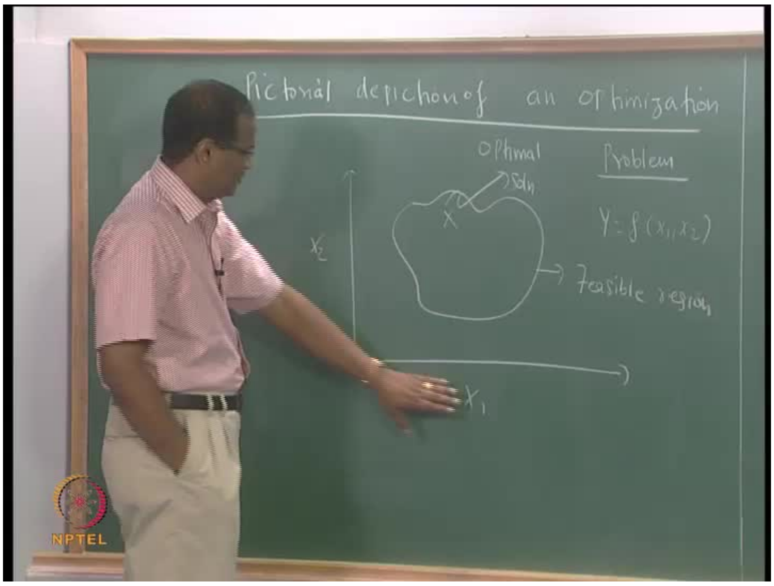

So, we looked at this pictorial depiction of an optimization problem where Y is a function of x 1 and x 2. So, if you plot all the constraints, for example if x 1 and x 2 are the two independent variables. So, there is an envelope, what is contained in this envelope are the various combinations of x 1 and x 2 which will satisfy all the constraints but each of this combination will give a different value of Y. All of which are not equally desirable because the Y keeps changing; depending on whether it is a maximization problem or minimization problem, you want to choose the best value of Y which is indicated as an optimal solution.

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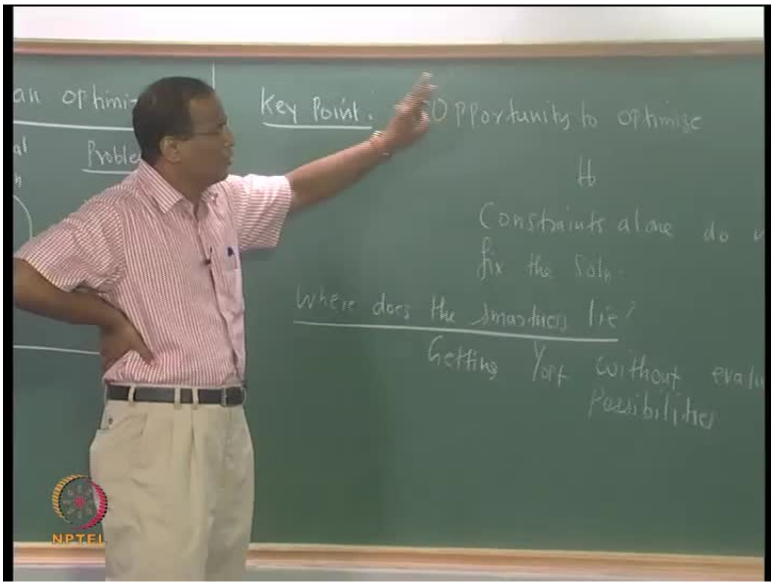

So, here are two cardinal or key point involved in optimization, that is what I have written on the board. The opportunity to optimize stems from the fact that constraints alone do not fix a solution. If the constraints alone are so tight that finally there is an envelope only around that optimum point or there is only one point, you have to live with it. That is the optimum point that is the only solution. The feasible solution becomes the optimum point automatically. The meaning of optimum then it no longer exists because there is no opportunity for you to find out whether you can improve the value of Y if it is a maximization problem or reduce the cost if it is a minimization problem. Therefore, the key point is the opportunity to optimize is afforded by the fact that the constraints alone do not fix the solution. The constraints give you enough breathing space; the constraints give you enough freedom to explore the domain.

What domain; the feasible domain or the feasible region. Now, where does the smartness lie? Now there are umpteen number of possibilities of x 1 and x 2 which will satisfy all the constraints. But it is not possible for us to explicitly find out the value of Y for all values of x 1 and x 2 because in real life x 1 and x 2 are only notional. You may have x 1 to x n and each time you have to run a CFD solution or you have to run answer is to get the value of Y; Y may be stress at a point, maximum stress in the beam or Y may be the Nusselt number or the maximum temperature in electronic chassis, the maximum temperature in data center, a super computer, whatever. Therefore, the smartness lies in not evaluating all the possible combinations of the variables which satisfy all the constraints, without going through all the possible combinations but confidently asserting that this is the optimum that is where this smartness lies. Is that clear? So, these are the two important principles.

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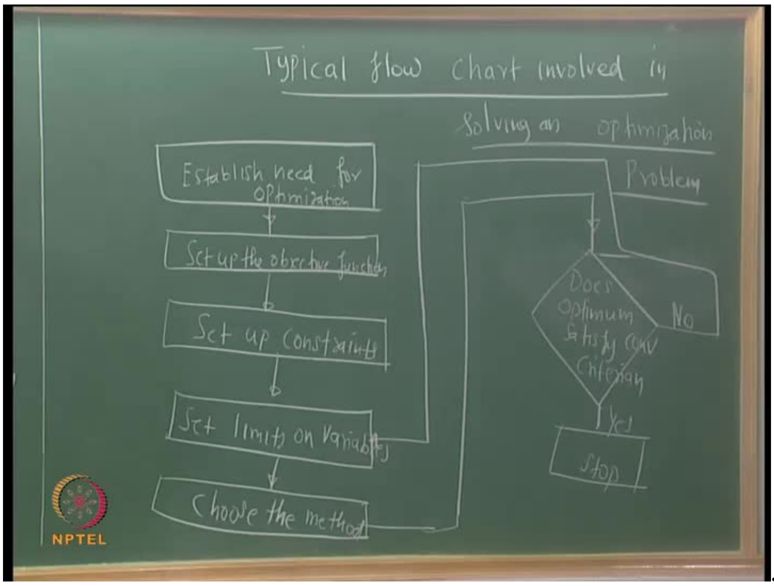

What is the typical flow chart involved in solving an optimization problem? What will be the first step?

Student: Finding out the objective.

Yeah even before that? Yeah, that is good that is an important thing. Before that I am talking about philosophically; no, before that?

Student: Why do not we do optimally?

Okay, is there a need? First, we have to establish the need for optimization whether this problem is worth optimizing. So, first we have to establish the need for optimization. Fine, so we have now decided that the optimization is required. What is the next step?

Student: Can it be obtained?

Yes. No, that is all fine. Then, once we want to optimize, then?

Student: Objective function.

Set up the objective function. Immediately following this?

Student: Set up constraints.

Set up constraints. Good, set up constraints. Then?

Student: Find the principle A.

Yes, that will come automatically. Then?

Student: Solving.

Before solving? No.

Student: Whether there exists an optimization?

That we would not know.

Student: But when constraints are fit you can find out whether

No, that is okay that is where m is less than equal to n where m number of constraints number of variables all those are there. You will set limits on the bounds for the variables non-negative constraints.

No, but bounds need not be constraints. Bounds are not there, we do not have to state everything as a constraint. Okay, if you want you can take it as a subset but, let us explicitly say set limits on variable. Then?

Student: I can determine the variable.

No, that is LP, I mean choose the method. Once you choose the method, then there is a decision box. Does the solution satisfy the convergence criteria? Right? That is the change in Y with respect to two consecutive iterations is less than this or Y new minus Y old divided by Y new modulus of 100 is less than 0.01, 0.001 whatever. Is does, what did he say?

Student: If x are convergence?

The solution, okay does optimum satisfy? If it is yes you stop; if it is no then you go back. Okay. There could be several types, several variants to that, there could be several types of flow charts but this is indicative. This gives a broad idea of how we go about solving an optimization problem, alright.

Student: Can we go in and set limits of variables.

What?

Student: If it is their condition is no value under set, we read through another method.

Do not set limits. Sometimes there is a physical limit, we do not want to pressure. Suppose you want to optimize a power plant pressure we want to set limits. Pressure is less than 160 bar, temperature is less than 150 degree centigrade which mathematical the objective function will not know. These are coming from outside the metallurgical restrictions, okay?

Student: Why are we coming back to the original and not choose the method?

Oh, that is your question?

Student: Yes.

No no, what happens is sometimes we are I mean my response to your question was with regard to issues like maximum temperature in a power plant or the admission pressure for stream and so on. In this case what happens is sometimes we do not want the solution to go out of these things. Sometimes extraneously the analyst may put some tight condition on the variables, so you relax this or you tighten that. So, I am not saying you should cheat in such a way that you say if X is 5 you say X is not less than equal to 4, X is not greater than equal to 6 and get 5. I am not saying like that but sometimes to make it more convincing we do not want Helter Skelter search in the domain, right?

Sometimes, intelligently you know suppose you are searching for you want to determine thermal conductivity of some substance and metal, you do not want an optimization to go from k is equal to 0 watts per meter per Kelvin to infinity, 0 to 1. I know it is a metal, so let me say that it will vary from 20 to 400 or something like that; it is for me to do that. As an engineer, as a mathematician, I may not want to do that; any variable can go between 0 to infinity. But as an engineer I know the bounds for viscosity, specific heat, density and so on. So, I will apply realistic bound. Is it okay, fine?

Now if you do not like it you can observe this in the constraints and then it has to come here. This is the general flow chart for solving an optimization problem. Now what could be a very desi or I am not denigrating the word desi but what, so we are all on live. What could be a naive or what could be a crude optimization method?

Student: Substitute x1 on x2.

Substitution, what man, there will be hundreds of values. So, now you have got an idea of pictorial description of an optimization problem Y is the function of x 1, x 2. Where does the opportunity to optimize arise? Where does the smartness lie? How do you go about solving? Then this recipe you can apply by so if you know various techniques for various kinds of problem, you will decide which is the most appropriate technique and solve.

Now even before starting all this, just like you did the pump and piping problem I mean we did not go through all this. You adopted some crude method maybe you plotted, it was just one variable you plotted or you wrote a program and make small increments in the variable and try to find out where it hits some maximum or minimum.

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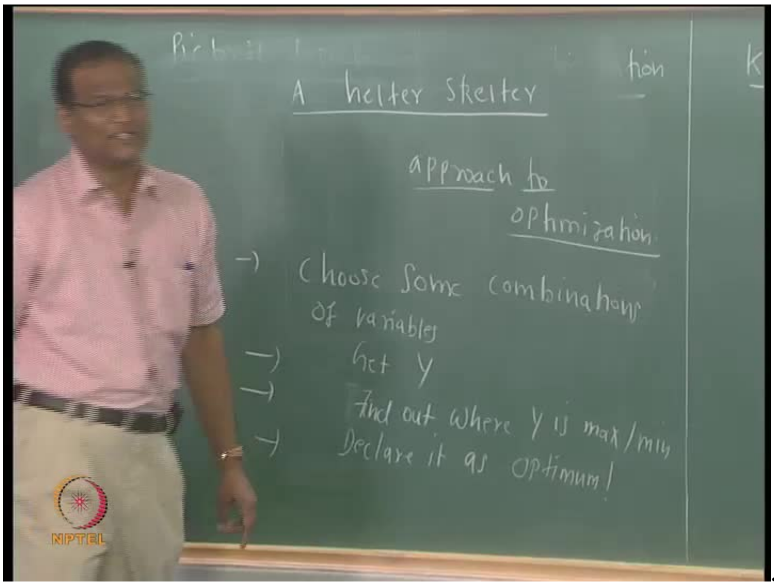

So, what could be Helter Skelter approach to optimization?

Student: We choose one point.

Okay.

Student: And then find out value of Y.

Good.

Student: And then find the value of Y at four points.

Okay.

Student: And then whichever direction it is increasing.

Oh, very good. That is what he says is a two dimensional problem. Choose one value of x 1, x 2. Find the value of Y; choose east, west, north, south. Find in which way it is moving then move in that direction, then again choose. This is actually called one kind of search which I will teach. It is called the lattice search; I will come to it a little later. A Helter Skelter approach will be choose, even before that the crudest of them all will be choose some combinations of variables, get Y. Find the highest or lowest and declare that that is the optimum. Get Y, it is very dangerous but it is better than not doing anything at all. You are not finally getting the global optimum is not guaranteed. So, this is basically a non-series approach to optimization but there may be some limitations.

So, the practicing engineer will say that it is not possible to get all combinations of variables; there is only handful of combinations. These are the combinations which I can manufacture because from machinability point or manufacturability point or when I am outsourcing some components, these are the values of x 1, x 2, x 3. If I work out the value of Y for these combinations it is enough. So, we are not talking about optimization where these techniques will work. We are talking about real series of optimization where there are many variables and then the constraints give you the freedom to search the optimum, to search the feasible domain and so on. There could be some crude approach where you will get some ideas.

So, you will get which is a better solution compared to the other; that is all, but global optimum is not guaranteed. Therefore, the next half of the course we are going to do all this only after we are convinced that there is a need for optimization and methods like this will not work. If either because of cost, money if you buy that because of money and effort you do not want to do optimization you do not need optimization or your problem is such that some engineer or some managers or some person working in the industry will tell you no no, if these 10 or 20 combinations are enough then we have to just close shop and go home.

But we are talking about the serious optimization in thermal science and this things where there are plenty of variables like you want to minimize the weight of the aircraft or spacecraft, you want to minimize the weight of the Boeing 787, you want to increase the BHP per ton of a racing car or you want to increase the fuel efficiency and all these where you cannot simply put 4 or 5 designs and you want to have a CAD design, then you do a finite element analysis, then you go through thermal dynamic analysis; you want to do all this. When you are doing all this where these techniques will not work, then it is time to understand serious optimization; that is what we are going to do in the next 7 or 8 weeks, alright?

Now let us start what are the various optimization techniques which are available? Sometimes, when you are in the business of doing research when you write some papers, when you read a serious papers on optimization, I have seen comments like this; why do the authors struggle so much? Let them take some ten possible combinations of variables and find out the value. So, I have to write two pages; whatever I told now I have to write and then say that it is not possible and this thing and sometimes this common sense will fail. So, how do you arbitrarily choose these 10 combinations and all this? So, it will not work for all otherwise the whole field would not have developed.

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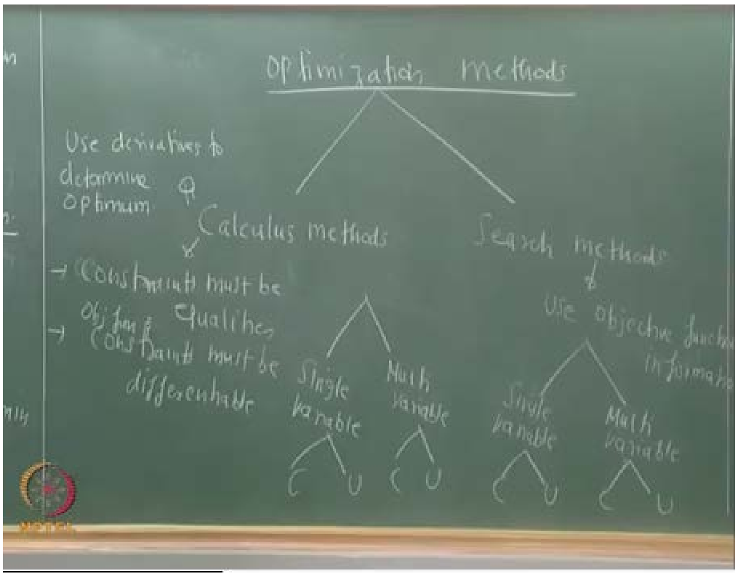

Now optimization methods, there are several ways of classifying this. Some people classify the methods, some people classify the problems. So, we will combine both optimization methods are broadly classified into calculus and non-calculus, calculus methods and search methods. What do we do in the calculus method? Use derivatives to determine optimum, correct? Here you use objective function information. Like what Abhishek said you start with a point take east, west, north, south, get the new value. But in a calculus technique we completely forget about that; we just take dou Y by dou X and find out what are the values of x 1, x 2, x 3 at which Y becomes an extremum. We do not even care about the value of the optimum. The value of the function Y at the optimum is a post process quantity in a calculus method; it is incidental. You will find out what are the values of the design variables at which the solution becomes stationary or dy by dx becomes stationary

So, the calculation of the objective function is push to the end in the calculus technique but in the search method it is the objective function always you are comparing y1 with y 2, y 3, y 4, find out which is the maximum and then PERT of that. So, that is the basic difference between these 2 methods. Having said that calculus, but what is the important requirement? First objective constraints must be differentiable; constraints must be equalities, right? Otherwise, if it is in inequality we are going to have trouble, slack variable other things but it is more difficult. Constraints must be equalities. Of course this is debatable but generally constraints must be equality and constraints must be differentiable. Why only constraints objective function, right. There is no requirement in so far as search methods are concerned. In each of this, you can have single variable multiple variables. In each of this you can have, but what are C and U? Constrained and unconstrained; so, you can have C constrained.

Please note that this division basically represents two types of methods, from here on I am classifying the problems. Now I have indicated both the type of optimization problems we will normally encounter and the types of methods used for solving this. So, you can have a multivariable unconstrained optimization problem which can be solved by a search method or a calculus method or you can have a single variable constrained optimization problem which we can solve by this or this. So, you can put actually x 1 and x 2, for example, where x 1 is the type of optimization problem and x 2 is the type of optimization method, then you go and choose whatever is most appropriate. Alright, so without much ado we will start off with the first technique the Lagrange multipliers. Where is the duster?

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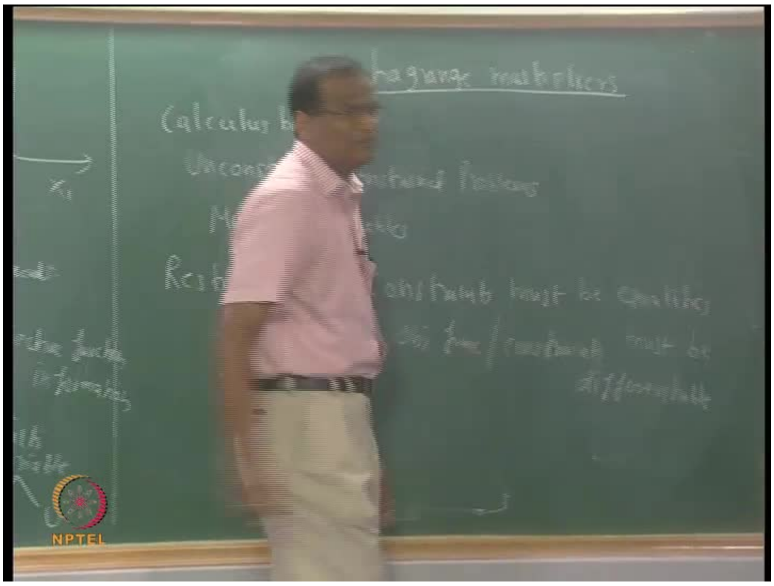

The most powerful optimization technique that is calculus based. Let us write some stories about this calculus based, unconstrained and constrained problems; I wanted to say something else unconstrained multiple variables. Is it like this swine flu vaccine? So, it is a cure all. Everything is there unconstrained, constrained, multiple variable, so, that is it, you just learn Lagrange multipliers and go home. What is the catch?

Student: Have the problem.

The constraints must be equalities, okay. Number of constraints must be less than equal to the number of variables. Constraints must be equalities; objective function constraint must be differentiable. These restrictions is not withstanding, there is a wide class of problems which fall under the category which can be imminently solved using the Lagrange multiplier method. So, what is this method?

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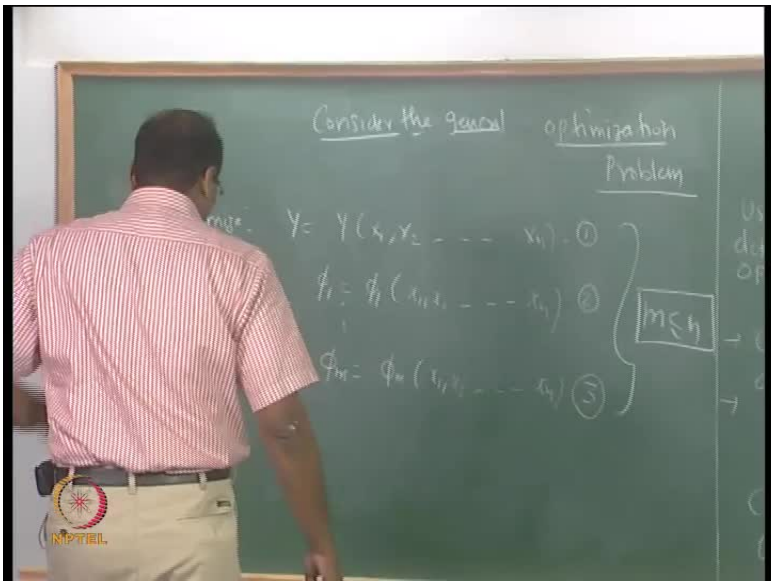

So, consider the general optimization problem. Optimize Y is Y of subject to, I am putting this as 3 it is not correct. This is some m plus 1 or something like that; it will confuse us, so I am just putting it as 3. So, the funda is m is less than equal to n, correct? If m is equal to n, the constraints themselves will fix the solution that is the optimum solution. If m is less than n, there will be a feasible domain where you can possibly explore. If m is greater than n, it is an over constraint optimization problem. So, what does the Lagrange multiplier now say? The Lagrange multiplier method says that the solution to this optimization problem of m constrained n variable optimization problem with m equalities constraints and n variables is akin to solving a set of equations which are generated like this.

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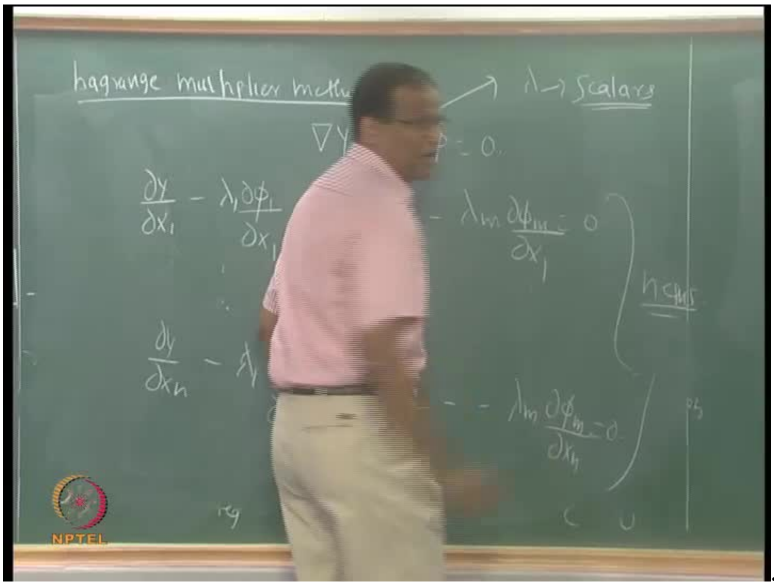

So Lagrange multiplier, suppose you have, so lambda is a scalar, it is lambdas. What does is it mean? Delta Y minus lambda delta phi equal to 0, what does is it actually mean? It actually means lambda 1, can you write for the first one? Dou phi 1, Tell me, dou phi 1 minus, not dou Y everything is dou X 1, right correct, is equal to 0. How many equations are there? How many such equations are there?

Student: n equations.

Good, n equations apart from this. So, these are n scalar equations. Apart from this what are the other equations we have? We have m constraints equations.

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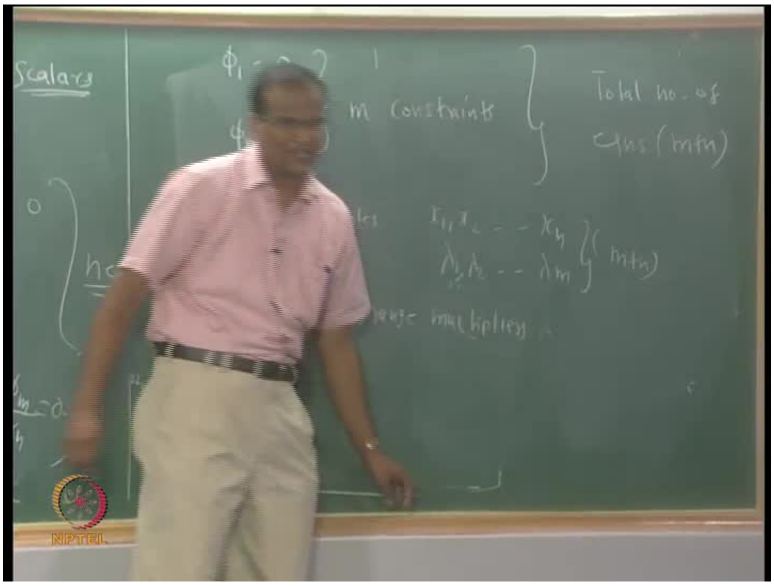

Phi 1 equal to 0, number of variables, the total number of equations is m plus n total number of variables is m plus n. So, this m plus n equations can be simultaneously solved, you will get the values of x 1, x 2 up to x n at which y become stationary. Apart from that you get an added bonus lambda 1 to lambda m; these lambdas are called the Lagrange multipliers. It is too premature at this stage to discuss what lambdas are, I will give you a physical feel for lambda after you work we a problem you get an intuitive field for what lambda is and then I will turn back and say what is the interpretation we have for lambda.

This is the procedure for solving the Lagrange multiplier method. Now it may look nebulous, cloudy, vague and all that; once you solve problems it becomes easy. So, we can extend it for the unconstraint problem also; let us extend it to the unconstraint problem which makes life a lot easier. This comes from vector calculus; I will also prove this in one of the later classes, what is the proof of the Lagrange multiplier method, what is the graphical interpretation of the Lagrange multiplier method, what is the physical significance or the economic significance of lambda and so on. Vinay, you have a problem? Okay.

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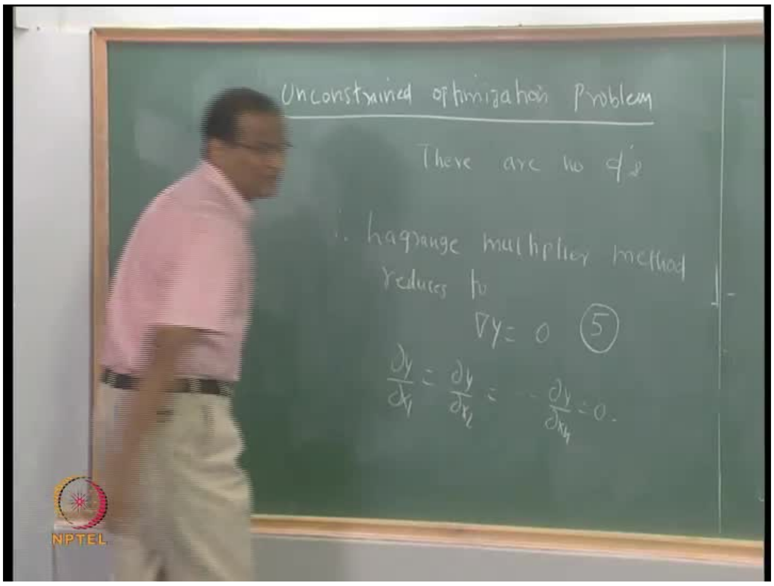

I expected a question here. I expected of question at least from one of you.

Student: What does that equation means by if are solving that optimization.

No; all that I am going to explain. Why not substitute the m constraints into the objective function and reduce the order of the objective function? That is the question that I expected; are you getting the point? You can substitute the constraints and reduce the order of the equation. You can substitute this m for these m constraints into the objective function and reduce but this is not always possible because sometimes the constraints are too difficult to mess around with; I mean they are very complicated and it is very difficult to do so. Otherwise conceptually and mathematically it is perfectly legal for you to substitute. For example, if you have finally n minus m is equal to 1 if you substitute m constraints at the n finally you will be left with only one variable you can solve. It is just like our substitution solution of simultaneous equations and so on. Unconstrained optimization problem there are no phi‘s. Therefore, Lagrange multiplier reduces to, it is very simple.

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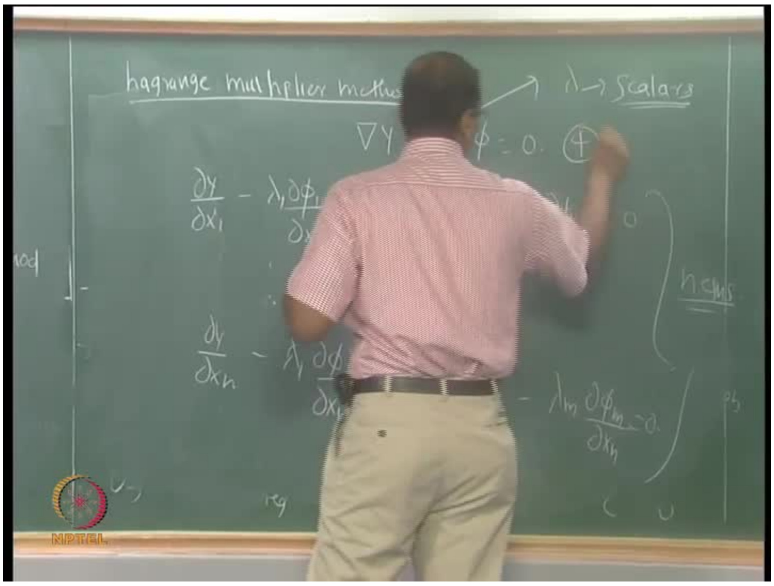

So, I think you want to give some numbers for this. So, I will just call this 5. So, it is very simple. So, this reduces to.

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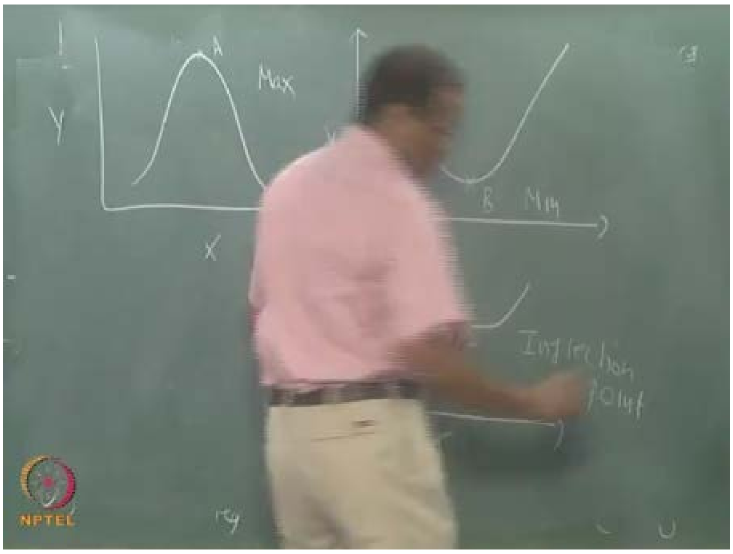

So, it is just a one variable problem. It could be like this, it could be like this or it could be like this A, B and C Y versus X. So, A is the maximum, B is the minimum and C is the inflection point where unfortunately the second derivative is also zero. So, the Lagrange multiplier method will give you only the values of the independent variables at which the function becomes stationary. It helps you to locate the extremum; however, second order necessary and sufficient conditions are required to determine whether it is in the optimum is a maxima or minima or optimum or an inflection point; I will come again. We are working only with first order conditions, we are making the first derivative stationary; therefore, it will not tell you whether it is maximum minimum or an inflection point.

So, higher order necessary and sufficient conditions are required for doing this; however, in many engineering problems it is possible for you. Once have made something stationary it is possible for you to figure out. You are working so much, you have done a complete analysis IC engine, you did simulation this thing, you struggled in mat lab and finally you have optimized. You will not get such solution where the specific we will consumption is the maximum, right. When you hit an extremum it should be S of C is minimum, right. It is very unlikely that after all this hard work you will end up with solution where S of C is maximum, right.

Student: In order differential equation when there are number of variables is difficult no, sir.

Second order differential equation.

Student: Yes sir if we differentiate, when you find out the second order equation, when there are number of variables.

Yeah.

Student: How can we come to a judgment whether it is reaching?

No problem, we have to evaluate what is called the Hessian matrix; I will teach you. So your Hessian matrix must be positive definite. Is it going above your head? Do not worry. So, there is something called the Hessian matrix. So, you can look at it; do not worry, it is not for two variables problem it is pretty straight forward.

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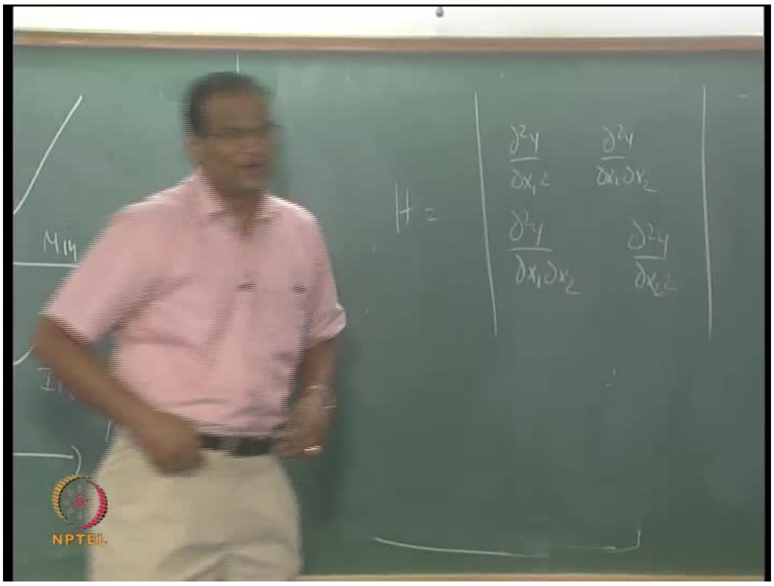

You will get the determinant; this is called the Hessian matrix. You look at the determinant of this and see whether it is positive definite, whether the leading diagonal this is positive and the whole thing is determinant is positive negative. We will do all this later on.

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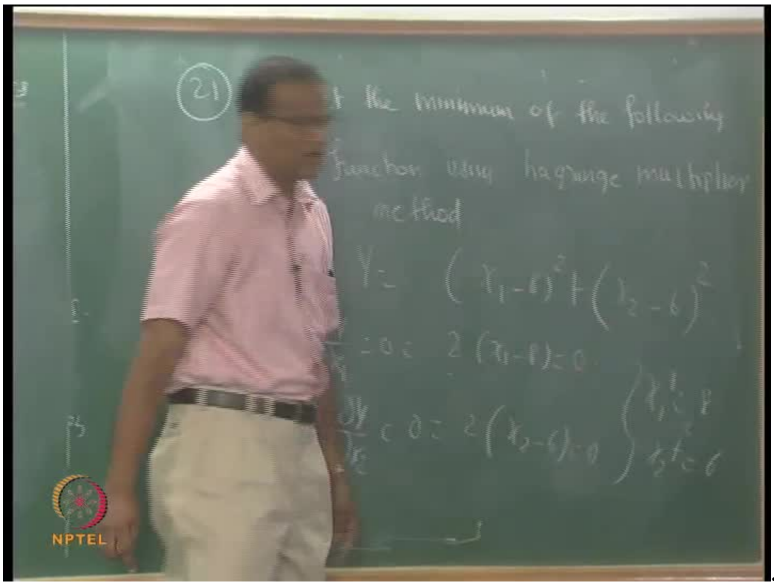

So, problem number 21: Get the minima of the following function using Lagrange multipliers. First decide whether it is single variable, multiple variable, constrained, unconstrained; go ahead shoot, 5 minutes. What is it equation of a circle. It is equation of a circle. Get the minima of the following using Lagrange multiplier x 1 minus 8 whole square plus X 2 minus 6 whole square, zero; when does it become stationary?

Student: 8 and 6.

8 and 6, so it is pretty straight forward man and you thought Lagrange multiplier is very difficult? Yeah, finish it finish it. Do the dou Y by dou x 1 equal to 0 dou Y by dou x 2 is equal to 0, get the value of x1 x 2 optimum, substitute, get the value of Y. If you want, go ahead and take the second derivatives. No, I have not taught you Hessian and all that, forget about it. So, I denote the optimum by plus, you can use star or whatever opt, subscript of whatever, to distinguish it from general x 1 and x 2.

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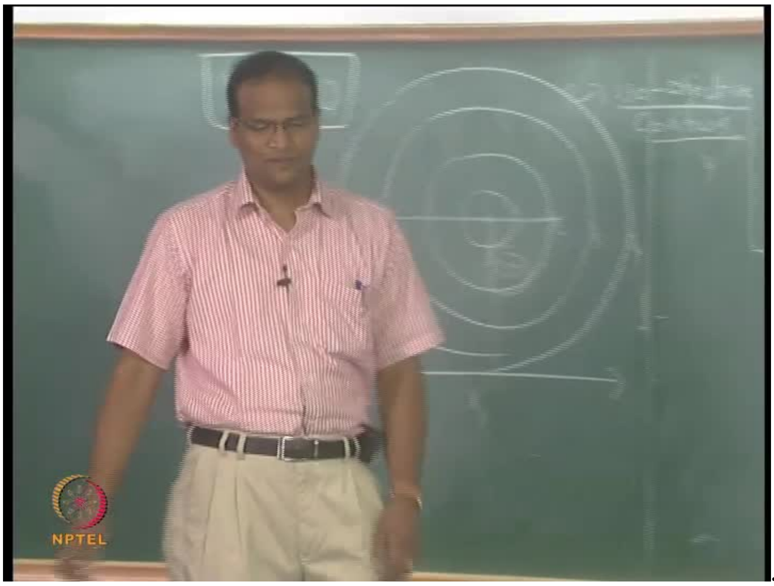

What is the solution? So Y 1, Y 2, Y 3, Y 4 and all that, right? This is Y equal to 0. What are the circles?

What?

Student: Y equal to constant.

Yeah, give a name iso-objective contours. Any point on that has the same objective function iso-objective contours, lines joining points having equal rainfall is called?

Student: Isohyets.

Yeah, people who have taken atmospheric science should know. They are called isohyets.

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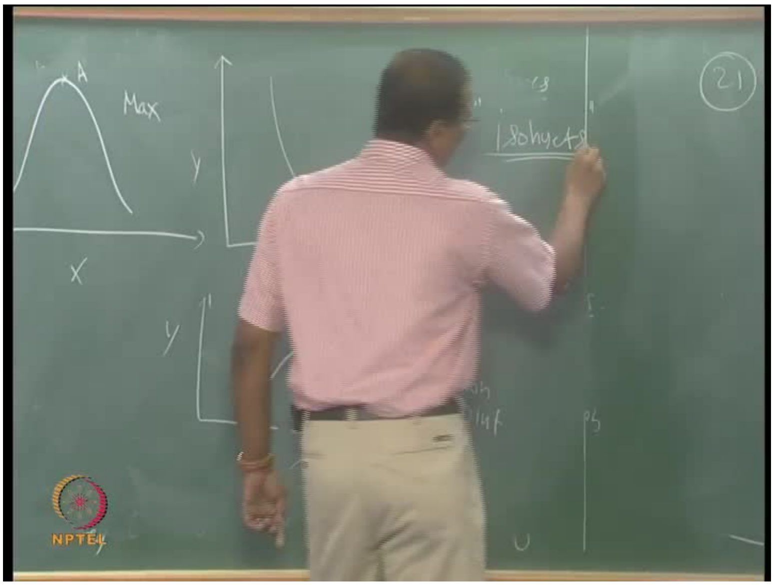

They are called isohyets, isobars, isochor, isotherm, isoentropy, isoefficiency, isocost isocgpa, whatever, isocgpa contours. So, this is a simple unconstrained optimization problem but I will give a little more I will give a little twist to it and give a slightly more involved unconstrained optimization problem before we break and then we come and take more serious problems. We will start looking at some heat transfer problems and use Lagrange multiplier method. This Lagrange multiplier method is important; we will go on for 4, 5 classes. So, surely there is one question in second quiz on Lagrange multiplier.

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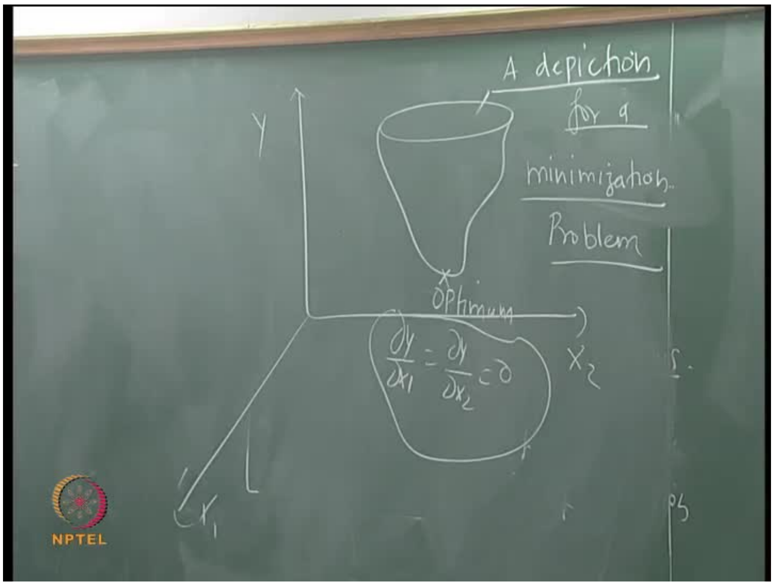

Second quiz ends up in now the most important topic in optimization. So, 2 variables problem, so this x 1, this is x 2, it is Y, optimum a depiction; I will just say a depiction. Generally you can indicate like this.

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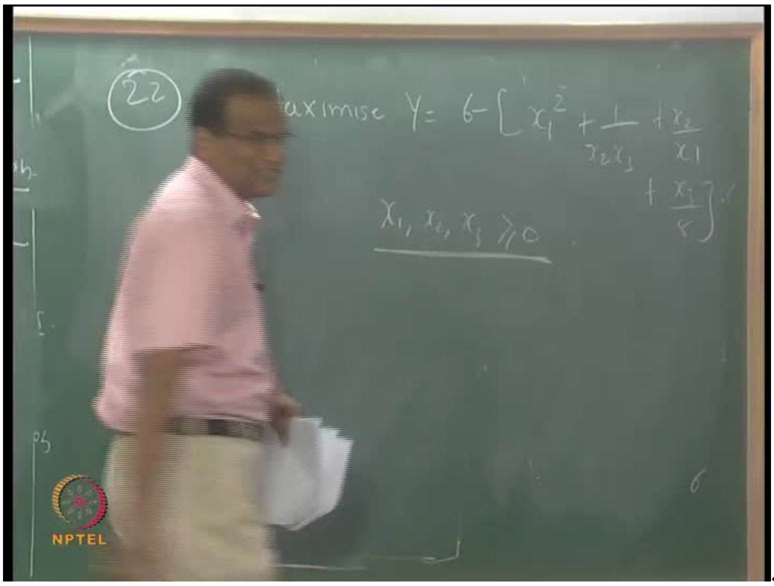

Maximize, of course it is needless to say using the Lagrange multiplier method; is it constrained or unconstrained? Do not worry about x 1, x 2, x 3, is it constrained or unconstrained? It is an unconstrained optimization problem. I am giving x 1, x 2, x 3 greater than zero because this fellow can give you can put minus values and say that some. X 1, x 2, x 3 are physical variables; you cannot produce minus 4 tables and minus 6 chairs. Maximization of a plus y is equal to a plus maximization of y, correct? Therefore, minimization of a minus y is equal to a minus minimum of, I mean, you can use.