APPLIED ELASTICITY

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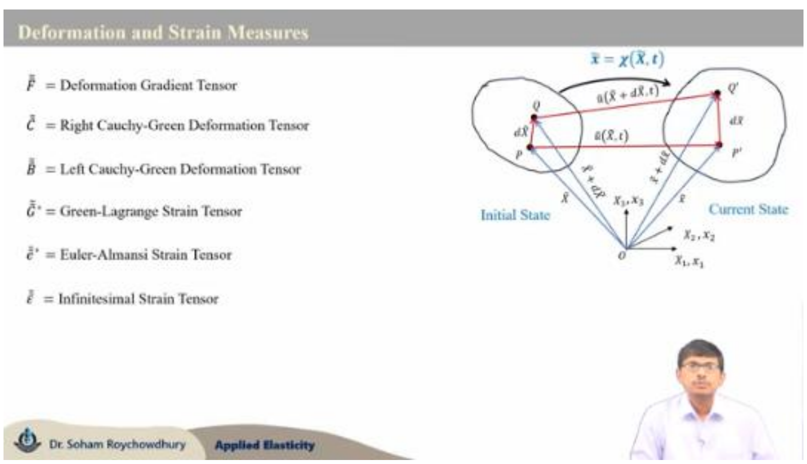
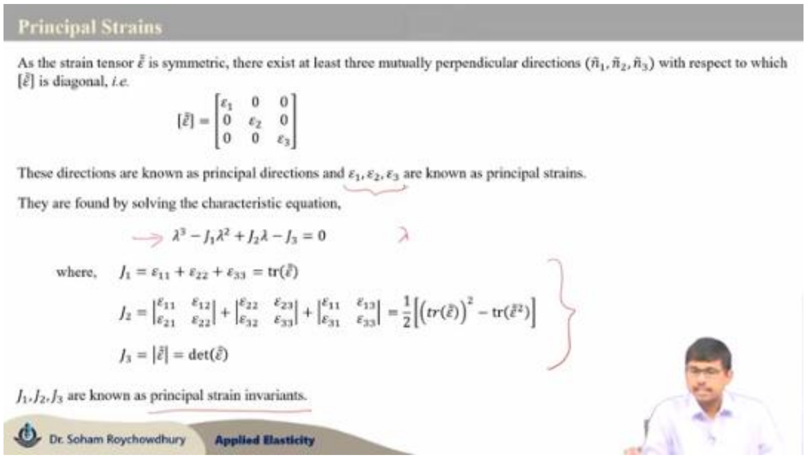
Week 2
Lecture 09: Strain Measures II

Welcome back to the course on Applied Elasticity. Today, we will continue with our lecture on strain measures.



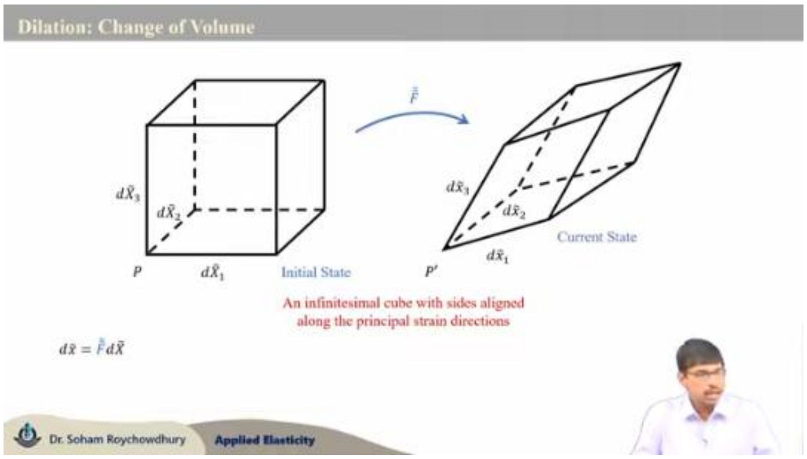
So, in the last lecture, we started our discussion on strain measures and defined various types of strains, such as: the Green-Lagrange strain tensor, which is valid for the undeformed state of stress, then the Euler-Almansi strain tensor, which is valid for the deformed or current state of stress. So, $\tilde{\tilde{\varepsilon}}$ was defined as another small linear infinitesimal strain tensor, which is the same in both initial and current states.



Now, we will first look into the concept of principal strains. So, if you consider any small strain tensor $\tilde{\tilde{\varepsilon}}$, which is always symmetric, as we saw in the last lecture. For such symmetric $\tilde{\tilde{\varepsilon}}$, there always exist at least three mutually perpendicular directions—let us say $\tilde{n}_1$, $\tilde{n}_2$, and $\tilde{n}_3$, with respect to which the strain tensor $\tilde{\tilde{\varepsilon}}$ can be written as a diagonal tensor of this particular form: all the non-diagonal terms are zero; only along the diagonal locations do we have non-zero terms of ε_1, ε_2, and ε_3. So, if it is possible to find one such direction—one such triad of $\tilde{n}_1$, $\tilde{n}_2$, $\tilde{n}_3$ directions—in which $\tilde{\tilde{\varepsilon}}$ looks like this, for that particular case, we call $\varepsilon_1, \varepsilon_2, \varepsilon_3$ the principal strains and the corresponding $\tilde{n}_1, \tilde{n}_2, \tilde{n}_3$ directions, the directions of the principal strains.

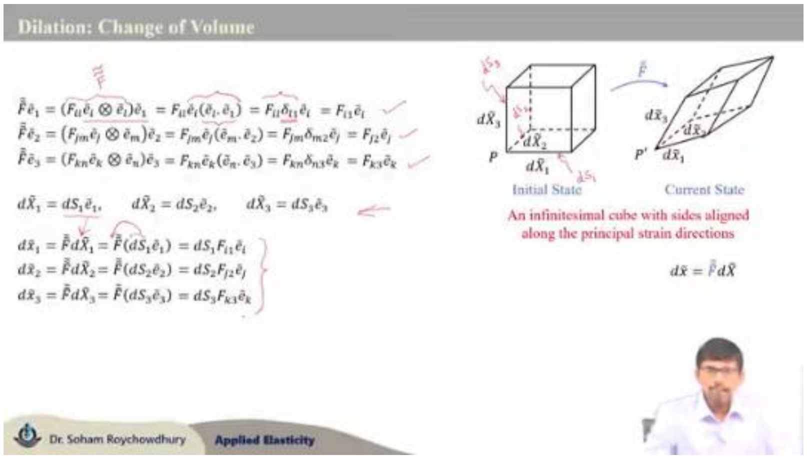
Now, this can be obtained by solving the following characteristic equation: $\lambda^3 - J_1\lambda^2 + J_2\lambda - J_3 = 0$, where J_1, J_2, and J_3 are known as principal strain invariants, and they are explicitly given in this particular form: J_1 is the summation of the diagonal terms of $\tilde{\tilde{\varepsilon}}$, which is $\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$. It can also be written in terms of $\mathrm{tr}(\tilde{\tilde{\varepsilon}})$. The second strain invariant, J_2, can be expressed as $\frac{1}{2}[tr(\tilde{\tilde{\varepsilon}})^2 - \mathrm{tr}(\tilde{\tilde{\varepsilon}}^2)]$. And the third one, J_3, is nothing but the determinant of $\tilde{\tilde{\varepsilon}}$.

So, these three quantities—J_1, J_2, and J_3—are known as principal strain invariants, and these are independent of the choice of the reference frame. So, after finding J_1, J_2, and J_3, then, substituting those in this characteristic equation, you can find out three roots of λ, and those three are nothing but the three principal strains: ε_1, ε_2, and ε_3. They are associated with the principal strain directions $\tilde{n}_1$, $\tilde{n}_2$, and $\tilde{n}_3$.



Now, we are proceeding toward the change or deformation of a small volume element. So, let us consider an infinitesimal (meaning small) cube whose sides are aligned along the principal strain directions. So, dX_1, dX_2, and dX_3 are in the undeformed or initial state, and these are aligned along the principal strain directions ε_1, ε_2, ε_3 or $\tilde{e}_1$, $\tilde{e}_2$, $\tilde{e}_3$.

Now, upon deformation, which is defined by the deformation gradient tensor $\tilde{\tilde{F}}$, this particular small volume element is looking like this in the current state. So, in the previous or the initial configuration, all three sides, dX_1, dX_2, dX_3 were orthogonal, whereas in the current state, dx_1, dx_2, and dx_3, they no longer remain orthogonal. Now, our objective is to get transformation formula for transforming this initial small volume element from its initial state to the current state.



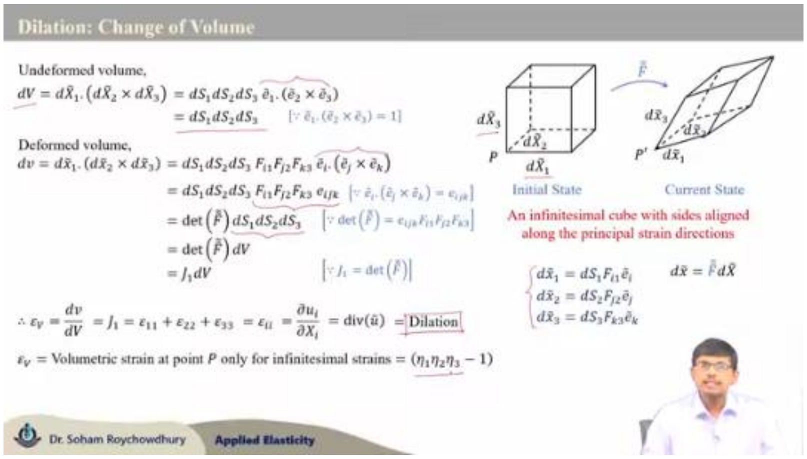
Now, we are applying this $\tilde{\tilde{F}}$, deformation gradient tensor, over three unit vectors $\tilde{e}_1$, $\tilde{e}_2$, and $\tilde{e}_3$, which are aligned along dX_1, dX_2, dX_3. So, if I consider the first one, $\tilde{\tilde{F}}$ acting over $\tilde{e}_1$ vector, this we can write with the definition of tensor components of $\tilde{\tilde{F}}$ as $F_{il}\tilde{e}_i \otimes \tilde{e}_l$ with the help of dyadic product of two unit vectors $\tilde{e}_i$ and $\tilde{e}_l$. So, this is nothing but $\tilde{\tilde{F}}$ which is acting over $\tilde{e}_1$. Now, by using the definition of dyadic product $\tilde{e}_i \otimes \tilde{e}_l$ acting over $\tilde{e}_1$, this can be written as $\tilde{e}_i\tilde{e}_l.\tilde{e}_1$. This is by definition of the dyadic product.

Now, this $\tilde{e}_l.\tilde{e}_1$, this particular dot product of two unit vectors is nothing but Kronecker delta, δ_{l1}. So, this term is equal to $\tilde{e}_l.\tilde{e}_1$. So, thus $\tilde{\tilde{F}}\tilde{e}_1$ becomes $F_{il}\delta_{l1}e_i$ and multiplying F_{il} with δ_{l1}, this first term would become $F_{i1}e_i$. Thus, $\tilde{\tilde{F}}\tilde{e}_1 = F_{i1}\tilde{e}_i$. Similarly, $\tilde{\tilde{F}}\tilde{e}_2$ and $\tilde{\tilde{F}}\tilde{e}_3$ can also be obtained like this, $F_{j2}\tilde{e}_j$ and $F_{k3}\tilde{e}_k$.

Now, writing $d\tilde{X}_1$ as $dS_1\tilde{e}_1$. So, the length of this dX_1 is taken to be dS_1, the length of this (dX_2) to be dS_2, and the length of this (dX_3) to be dS_3. So, all these three vectors—$d\tilde{X}_1$, $d\tilde{X}_2$, $d\tilde{X}_3$ in the initial state—can be written in this particular form. And then, the deformed small vectors $d\tilde{x}_1$ is $\tilde{\tilde{F}}d\tilde{X}_1$, writing $d\tilde{X}_1$ as $dS_1\tilde{e}_1$. So, substituting this here, it would become $\tilde{\tilde{F}}dS_1\tilde{e}_1$.

Now, dS_1 being a constant, we will take it out, and $\tilde{\tilde{F}}$ would act over just $\tilde{e}_1$, which is equal to $F_{il}\tilde{e}_i$ as already derived. So, $d\tilde{x}_1$ would thus become $dS_1F_{i1}\tilde{e}_i$. Similarly, $d\tilde{x}_2$ and $d\tilde{x}_3$ vectors can also be obtained like this. So, now, we have expressed all $d\tilde{X}_i$

vectors and $d\tilde{x}_i$ vectors in terms of the deformation gradient tensor $\tilde{\tilde{F}}$ and unit vectors $\tilde{e}_1$, $\tilde{e}_2$, $\tilde{e}_3$.



Now, moving forward, we are going to calculate the undeformed volume of the small cube in the initial state. We are defining the undeformed volume to be dV, which is nothing but the scalar triple product of these three vectors defining the cube. So, $d\tilde{X}_1.\left(d\tilde{X}_2 \times d\tilde{X}_3\right)$. So, the scalar triple product of these three vectors defines the total volume of the cube or parallelepiped formed by these three vectors. Now, substituting the expressions of $d\tilde{X}_1$, $d\tilde{X}_2$, and $d\tilde{X}_3$ derived in the last slide, if we substitute all of them, it would look like this. Now, this part $\tilde{e}_1.(\tilde{e}_2 \times \tilde{e}_3)$, where $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ are three orthogonal base vectors in the initial state. This scalar triple product is equal to unity, and thus the undeformed volume element becomes $dS_1 dS_2 dS_3$. So, this is very obvious because the volume of the cube dV is simply the product of the three side lengths dS_1, dS_2, dS_3 in the undeformed state.

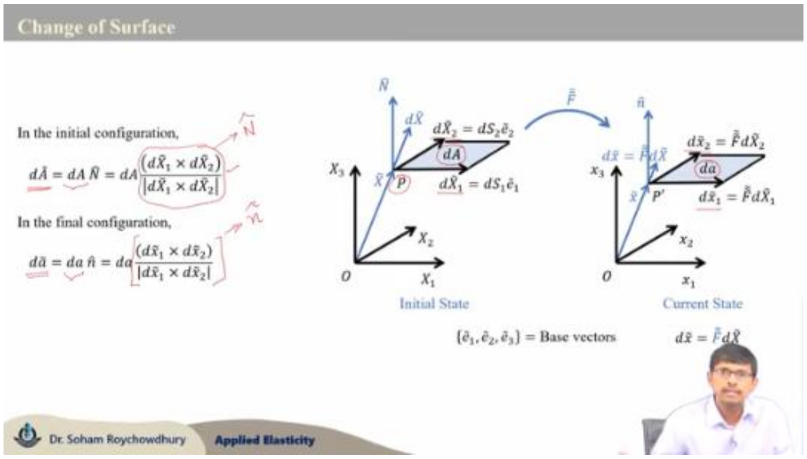
Now, coming to the deformed state, following a similar approach, the deformed volume dv can be written as the scalar triple product of $d\tilde{x}_1, d\tilde{x}_2, d\tilde{x}_3$ as $d\tilde{x}_1.(d\tilde{x}_2 \times d\tilde{x}_3)$, and substituting the expressions of $d\tilde{x}_1, d\tilde{x}_2, d\tilde{x}_3$ in the deformed state as obtained in the previous slide, this equation of dv can be expressed like this, where $\tilde{e}_i.\left(\tilde{e}_j \times \tilde{e}_k\right)$ is the scalar triple product of the three vectors $\tilde{e}_i, \tilde{e}_j, \tilde{e}_k$, where i, j, k can take any values of 1, 2, and 3. This can be defined with the help of a permutation symbol. So, the scalar triple product of any three vectors $\tilde{e}_i, \tilde{e}_j, \tilde{e}_k$, with i, j, k varying from 1 to 3, this can be written or substituted as the permutation symbol e_{ijk}.

Thus, the expression becomes $e_{ijk} dS_1 dS_2 dS_3 F_{i1} F_{j2} F_{k3}$. Now, if you recall the expression derived in one of the initial slides, we had expressed the determinant of any tensor as this. The determinant of the $\tilde{\tilde{F}}$ tensor can be written as e_{ijk} (permutation symbol) $F_{i1} F_{j2} F_{k3}$. Thus, this part is nothing but $\det\left(\tilde{\tilde{F}}\right)$. So, hence $dv = \det\left(\tilde{\tilde{F}}\right) dS_1 dS_2 dS_3$, and this $dS_1 dS_2 dS_3$ is nothing but dV, the undeformed volume.

So, the deformed volume dv can be written as $\det\left(\tilde{\tilde{F}}\right) dV$. $\det\left(\tilde{\tilde{F}}\right)$ can also be written as J_1. Now, $\varepsilon_V = J_1$ is a kind of Jacobian. So, ε_V is equal to the volumetric strain, which is defined as the deformed volume divided by the undeformed volume, $\frac{dv}{dV}$, which is the Jacobian or $\det\left(\tilde{\tilde{F}}\right)$, where $\tilde{\tilde{F}}$ is the deformation gradient tensor.

This is defined as $\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$, the same as the first strain invariant of the small strain tensor $\tilde{\tilde{\varepsilon}}$. We can also write this as ε_{ii} using the summation convention, and by the definition of $\tilde{\tilde{\varepsilon}}$, we can write this as $\frac{\partial u_i}{\partial X_i}$. So, this quantity ε_V defines the volumetric strain, and characterizes the change in the small volume element for the small cuboid from the initial configuration to the current configuration.

This volumetric strain is also named dilation. So, dilation is nothing but the ratio of the deformed volume to the undeformed volume. Now, it can also be shown that by using the definition of stretch and its relation with the normal strain, we can show that the volumetric strain at any point P is equal to the product of the three principal stretch ratios η_1, η_2, η_3 minus 1. This is left as an exercise; you can try to prove this.



Now, coming to the change of surface. So, just before this, we had discussed how one small volume element is getting transformed from the undeformed to the deformed configuration. Now, we are going to consider a surface element in the initial state and then its transformation to the deformed state. In the initial state, let us consider a point P denoted by $\tilde{X}$, and at point P, a small vector $d\tilde{X}$ is also assumed. Now, a surface is defined at point P with the help of these two vectors: $d\tilde{X}_1$ and $d\tilde{X}_2$.

These two vectors are aligned along the $\tilde{X}_1$ and $\tilde{X}_2$ directions, meaning base vectors $\tilde{e}_1$ and $\tilde{e}_2$ with side lengths as dS_1 and dS_2. So, the $d\tilde{X}_1$ vector is $dS_1\tilde{e}_1$, and the $d\tilde{X}_2$ vector is $dS_2\tilde{e}_2$. With the help of these two elements, the total surface area $d\tilde{A}$ is defined around point P, where $d\tilde{X}_1$ and $d\tilde{X}_2$ are two small vectors. Now, we are going to consider the unit normal vector $\hat{N}$, which is normal to the surface area dA at point P.

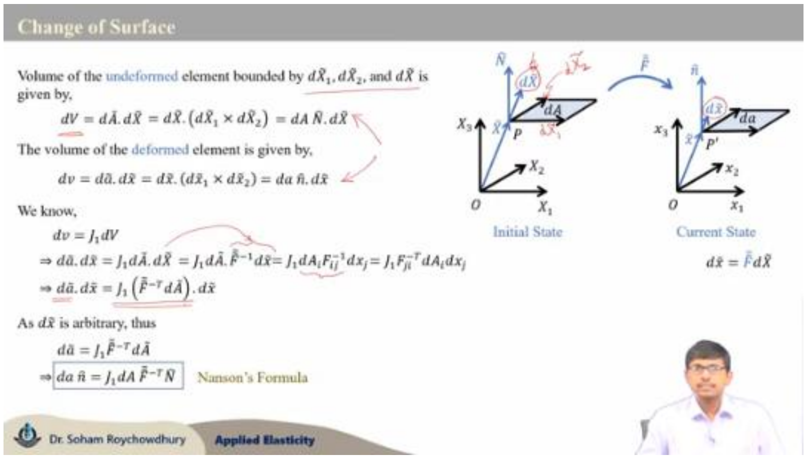
Upon transformation, which is characterized by the deformation gradient tensor $\tilde{\tilde{F}}$, this particular surface element deforms like this. Point P moves to a new point P' defined by the position vector $\tilde{x}$, and then $d\tilde{x}$ is the deformed small line element, which is defined as $\tilde{\tilde{F}}d\tilde{X}$. The deformed area is now named as $d\tilde{a}$, which is defined by $d\tilde{x}_1$ and $d\tilde{x}_2$, and those are defined as $\tilde{\tilde{F}}d\tilde{X}_1 = d\tilde{x}_1$, and $\tilde{\tilde{F}}d\tilde{X}_2 = d\tilde{x}_2$.

And $\hat{n}$ is the unit vector in the current state on the deformed area da. Now, we can write the explicit expressions for $\hat{N}$ and $\hat{n}$ as this. So, $\hat{N}$ being the unit vector on the undeformed area, that can be written as $\frac{d\tilde{X}_1 \times d\tilde{X}_2}{|d\tilde{X}_1 \times d\tilde{X}_2|}$. So, this total expression, $d\tilde{X}_1$ and $d\tilde{X}_2$

being two vectors in a plane, taking their cross product and then, dividing it by the magnitude of the cross product will give us the unit normal $\widehat{N}$, which is orthogonal to both $d\tilde{X}_1$ and $d\tilde{X}_2$. So, any area vector, including the undeformed area vector $d\tilde{A}$, note that here we are defining area as vector quantities. $d\tilde{A}$ vector is the undeformed area vector, which is defined as the undeformed area (scalar value of the area) times the unit normal vector $\widehat{N}$, which is a unit normal on that particular undeformed area, and that is given by this particular expression.

Similarly, in the deformed configuration, $d\tilde{a}$ vector, the deformed area vector, is defined as the scalar deformed area $da\,\hat{n}$, where $\hat{n}$ is the unit normal defined as $\frac{d\tilde{x}_1 \times d\tilde{x}_2}{|d\tilde{x}_1 \times d\tilde{x}_2|}$, similar to the initial configuration. We can, here, also define the unit normal on the deformed area da. Now, the volume of the undeformed element bounded by these three vectors, vector $d\tilde{X}_1$, vector $d\tilde{X}_2$, and this particular vector $d\tilde{X}$ at point P, using these three vectors, we can find out a small undeformed volume element, which is given by $d\tilde{A}.d\tilde{X}$. So, $d\tilde{A}$ is basically this vector of the surface area in the initial or undeformed state.

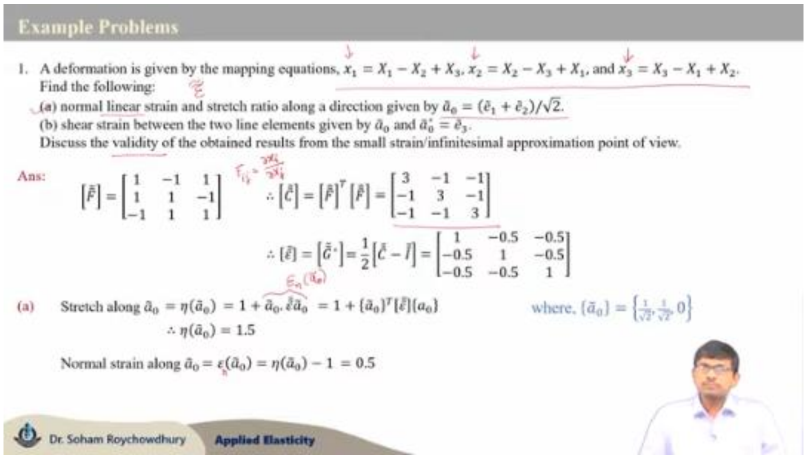
So, the surface area vector $d\tilde{A}$ dot $d\tilde{X}$ will give us the volume of the small element bounded by these three vectors. Basically, it is bounded by the surface area $d\tilde{A}$ and this vector $d\tilde{X}$. Now, expanding $d\tilde{A}$ as $d\tilde{X}_1 \times d\tilde{X}_2$, and using the definition of $\widehat{N}$, dV can be written as $dA\widehat{N}$, the unit normal in the undeformed state, dot $d\tilde{X}$. Similarly, the volume of the deformed element, which is obtained by transforming this dV, can be written as dv as the deformed area vector $d\tilde{a}$ dot this deformed small vector $d\tilde{x}$, and $d\tilde{a}$ is once again written in terms of $d\tilde{x}_1 \times d\tilde{x}_2$, and this is dotted with the $d\tilde{x}$ vector. In terms of the unit normal to the deformed area $d\tilde{a}$, we can write $dv = da\hat{n}.d\tilde{x}$.



Now, as we had already related the volume elements with the help of Jacobian J_1 or dilation. (By definition of dilation, $dv = J_1 dV$). These two expressions, dV and dv, are related through the Jacobian. We are substituting those here, and $d\tilde{a}.d\tilde{x} = J_1 d\tilde{A}.d\tilde{X}$. Writing $d\tilde{X}$ as $\tilde{\tilde{F}}^{-1}d\tilde{x}$, with the help of the definition of the deformation gradient tensor, and then, writing these in the indicial notation as $J_1 dA_i F_{ij}^{-1} dx_j$. Now, we will rewrite this as F_{ji}^{-T}.

So, F_{ij}^{-1} is written as F_{ji}^{-T} and taken before dA_i. And thus, in the vector form, this equation's right-hand side can be written as $J_1 \tilde{\tilde{F}}^{-T} d\tilde{A}.d\tilde{x}$. The left-hand side is the same: $d\tilde{a}.d\tilde{x}$. Now, if you compare both sides. Since $d\tilde{x}$ is arbitrary, $d\tilde{a}$ must be equal to this total quantity.

So, $d\tilde{a}$ vector (deformed area vector) should be $J_1 \tilde{\tilde{F}}^{-T} d\tilde{A}$. And in terms of unit normals in the undeformed and deformed state, we can write $da\,\hat{n} = J_1 dA \tilde{\tilde{F}}^{-T}\hat{N}$. So, with the help of this formula, which is also known as Nanson's formula, we can transform any surface element from the initial state to the final state, and all the areas are defined like area vector here, which is equal to the magnitude of the area multiplied by the unit normal of that particular area in both initial and current configurations.



Now, with this, we will move toward the solution of a few example problems on the strains. So, in the first problem, this deformation mapping is given: x_1, x_2, x_3 are given as functions of X_1, X_2, X_3. This is basically a description in the material or Lagrangian form. Now, you are asked to find out the normal linear strain and stretch ratio along the direction $\tilde{a}_0 = \frac{\tilde{e}_1+\tilde{e}_2}{\sqrt{2}}$. Note this term—linear strain; linear means we need to find out $\tilde{\tilde{\varepsilon}}$.

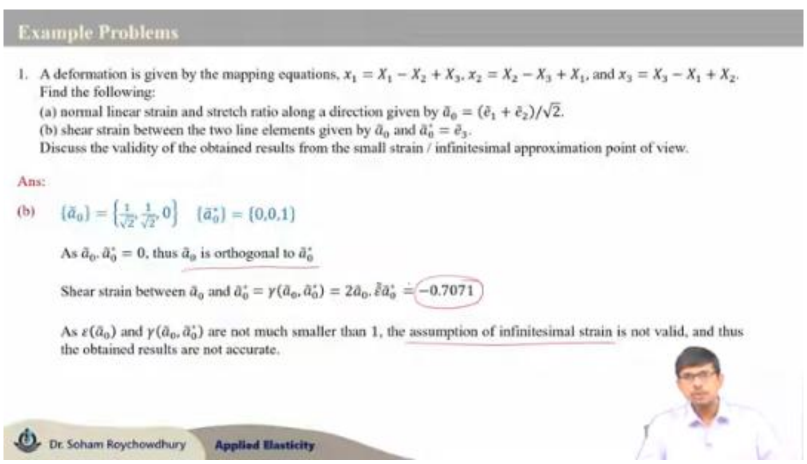
$\tilde{\tilde{\varepsilon}}$ is small linear strain and second part is shear strain between two given line elements $\tilde{a}_0$ and $\tilde{a}_0^*$. So, we are asked to take linear small infinitesimal strain, obtain that, and at the end, you are also asked to discuss the validity of the obtained result, i.e., assumption of this small strain. Now, from the given deformation mapping, $\tilde{\tilde{F}}$, the deformation gradient tensor, which can be easily obtained as this: $F_{ij} = \frac{\partial x_i}{\partial X_j}$.

So, from the given expressions of x_1, x_2, x_3, you can easily obtain $\tilde{\tilde{F}}$ as $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$, and then from that, $\tilde{\tilde{C}}$, the right Cauchy-Green deformation tensor, which is defined as $\tilde{\tilde{F}}^T\tilde{\tilde{F}}$, that can be obtained like this. Now, from $\tilde{\tilde{C}}$ we will go for the definition of $\tilde{\tilde{\varepsilon}}$, which is the linear strain tensor, and that is same as $\tilde{\tilde{G}}^*$, if we drop the non-linear terms. So, this is defined as $\frac{1}{2}\left(\tilde{\tilde{C}} - \tilde{\tilde{I}}\right)$, where $\tilde{\tilde{C}}$ is this one, which is right Cauchy-Green deformation tensor.

So, obtaining $\tilde{\tilde{\varepsilon}} = \frac{1}{2}\left(\tilde{\tilde{C}} - \tilde{\tilde{I}}\right)$, we got the linear strain tensor $\tilde{\tilde{\varepsilon}} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 1 & -0.5 \\ -0.5 & -0.5 & 1 \end{bmatrix}$.

These are the elements of $\tilde{\tilde{\varepsilon}}$. Now, coming to the first part, part (a) of the solution. Stretch along $\tilde{a}_0$ direction, where $\tilde{a}_0 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ (no component along $\tilde{e}_3$). That $\eta(\tilde{a}_0)$ can be obtained as $1 + \tilde{a}_0 . \tilde{\tilde{\varepsilon}} \tilde{a}_0$.

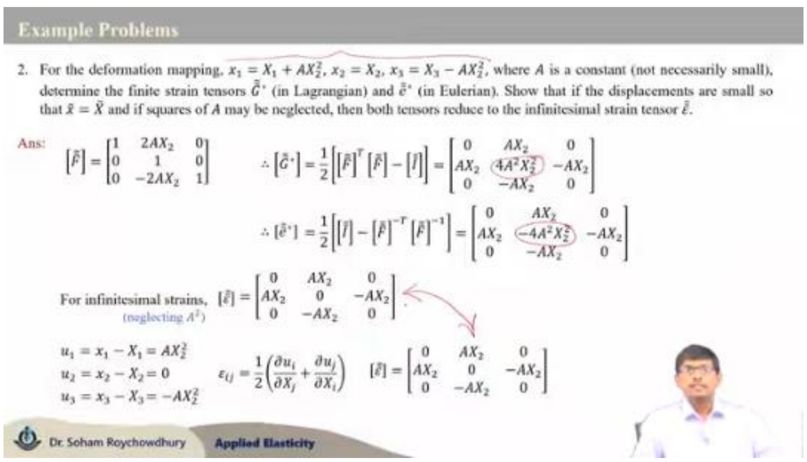
This we had discussed while deriving or discussing the physical interpretation of the components: normal components and shear components, diagonal and non-diagonal components of $\tilde{\tilde{\varepsilon}}$. So, this part is basically ε_n along $\tilde{a}_0$—normal strain along $\tilde{a}_0$ vector, and stretch along $\tilde{a}_0$ vector is $1 + \varepsilon_n$. Now, $\tilde{a}_0 . \tilde{\tilde{\varepsilon}} \tilde{a}_0$, in the matrix or vector form, can be written as $(\tilde{a}_0)^T \tilde{\tilde{\varepsilon}} \tilde{a}_0$. And putting $\tilde{a}_0$ and $\tilde{\tilde{\varepsilon}}$, all these vectors and tensors, and simplifying this, the stretch will come out to be, $\eta(\tilde{a}_0) = 1.5$. Now, using the expression or relation between η and ε_n, you can obtain ε_n, the normal strain, along $\tilde{a}_0$ direction to be $\eta(\tilde{a}_0) - 1$, which would be equal to 0.5. So, 0.5 is the normal strain along the $\tilde{a}_0$ direction. That completes the first part of the problem.



Now, coming to the second part, where we are asked to find out the shear strain for these two given directions: $\tilde{a}_0$ and $\tilde{a}_0^*$. If you recall, we had derived the formula for engineering shear strain with the help of an assumption that the two initial line elements in the undeformed configuration are orthogonal to each other.

So, that must be valid here. So, that has to be checked. If you check this in between given $\tilde{a}_0$ and $\tilde{a}_0^*$, you can verify that $\tilde{a}_0 . \tilde{a}_0^* = 0$. So, these two vectors are orthogonal to each other, and thus, we can write the shear strain between these two orthogonal vectors as $\gamma(\tilde{a}_0, \tilde{a}_0^*)$, which is defined as $2\tilde{a}_0 . \tilde{\tilde{\varepsilon}}\tilde{a}_0^*$. So, substituting the matrix of epsilon and these two vectors: $\tilde{a}_0$ and $\tilde{a}_0^*$, this shear strain can be obtained as -0.7071. Now, if you look at the values of both ε_n and this γ, both of them are not much smaller to 1. Thus, this assumption of small or infinitesimal strain is not perfectly valid, or obtained results are not accurate as this value is not much smaller than 1.

So, we should not go with the assumption of small strain while solving this problem.

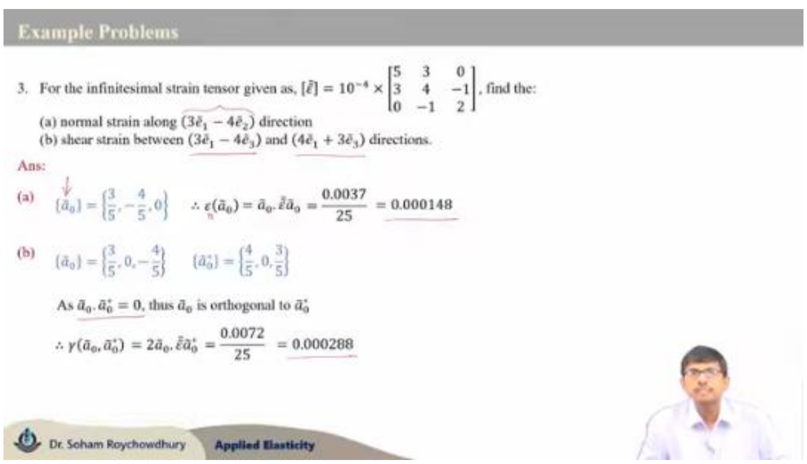


Now, moving to the next problem, where the deformation gradient tensor or deformation mapping is given through these expressions, where A is a constant which may and may not be necessarily small. So, first you are asked to find out the finite strain tensors $\tilde{\tilde{G}}^*$ and $\tilde{\tilde{e}}^*$ in Lagrangian and Eulerian formulation. Then you are asked to show that if the displacements $(\tilde{u})$ are small, then neglecting the square order terms of A, both $\tilde{\tilde{G}}^*$ and $\tilde{\tilde{e}}^*$ reduce to an infinitesimally small strain tensor, $\tilde{\tilde{\varepsilon}}$.

So, for the given mapping, the deformation gradient tensor, $\tilde{\tilde{F}}$, can be obtained as $\begin{bmatrix} 1 & 2AX_2 & 0 \\ 0 & 1 & 0 \\ 0 & -2AX_2 & 1 \end{bmatrix}$. From this, using the formula of $\tilde{\tilde{G}}^* = \frac{1}{2}\left(\tilde{\tilde{F}}^T\tilde{\tilde{F}} - \tilde{\tilde{I}}\right)$, we can obtain the Green-Lagrange strain tensor $\tilde{\tilde{G}}^*$ as this. And using the definition of the Euler-Almansi strain tensor $\tilde{\tilde{e}}^*$ as $\frac{1}{2}\left(\tilde{\tilde{I}} - \tilde{\tilde{F}}^{-T}\tilde{\tilde{F}}^{-1}\right)$, we can obtain $\tilde{\tilde{e}}^*$ as this. Now, this completes the first part of the problem.

Now, for small strains (linear strains), we need to neglect the A^2 terms. So, A^2 terms are present here and here. Neglecting those A^2 terms, $\tilde{\tilde{\varepsilon}}$ becomes this, which is the same from both $\tilde{\tilde{G}}^*$ as well as $\tilde{\tilde{e}}^*$. If you neglect this A^2 term from both $\tilde{\tilde{G}}^*$ and $\tilde{\tilde{e}}^*$, you will get the same linear strain tensor, $\tilde{\tilde{\varepsilon}}$. This was one of the points we had discussed during the derivation of $\tilde{\tilde{\varepsilon}}$.

This is independent of the initial state or the current state; $\tilde{\tilde{\varepsilon}}$, small strain, is the same in both states. You can also derive $\tilde{\tilde{\varepsilon}}$ with the help of its definition. u_1, u_2, u_3 are the three displacement components. These can be obtained for the given deformation mapping as $u_1 = AX_2^2$, $u_2 = 0$, $u_3 = -AX_2^2$. Now, using the definition of the components of $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$, you can obtain this $\tilde{\tilde{\varepsilon}}$, and if you compare them, the $\tilde{\tilde{\varepsilon}}$ obtained using its definition, and the $\tilde{\tilde{\varepsilon}}$ obtained by neglecting the non-linear terms of $\tilde{\tilde{G}}^*$ and $\tilde{\tilde{e}}^*$, they are identical. So, it can be shown that by neglecting the non-linear terms of the finite strain tensor, we can arrive at the small strain tensor.



Coming to the next problem, here, the small strain tensor, $\tilde{\tilde{\varepsilon}}$, is given to be this, where you are asked to find out the normal strain along the $3\tilde{e}_1 - 3\tilde{e}_2$. So, in this problem, $\tilde{a}_0$, the unit vector along the given direction $3\tilde{e}_1 - 4\tilde{e}_2$, is this. Note that $\tilde{a}_0$ is always defined to be the unit vector. So, you should not take this given vector to be $\tilde{a}_0$. You need to find the unit vector along the direction of this given vector and name that as $\tilde{a}_0$.

So, for finding the normal strain along this direction $3\tilde{e}_1 - 4\tilde{e}_2$, that is ε_n, the normal strain along the $\tilde{a}_0$ direction. This is defined as $\tilde{a}_0 . \tilde{\tilde{\varepsilon}}\tilde{a}_0$, and just by putting the values of $\tilde{\tilde{\varepsilon}}$ and A, both of them, this normal strain will come out to be 0.000148. Coming to the second part, once again, two vectors are given: $3\tilde{e}_1 - 3\tilde{e}_3$, the unit vector along that is $\tilde{a}_0$. The second vector is $4\tilde{e}_1 + 3\tilde{e}_3$; the unit vector along that is $\tilde{a}_0^*$. Here also, we need to verify or validate that these two vectors are orthogonal to each other to apply the definition of engineering shear strain.

And that is true for these given vectors $\tilde{a}_0$ and $\tilde{a}_0^*$. So, thus, γ, the shear strain between $\tilde{a}_0$ and $\tilde{a}_0^*$, can be obtained as $2\tilde{a}_0 . \tilde{\tilde{\varepsilon}}\tilde{a}_0^*$. And putting the values of the components of $\tilde{a}_0$, $\tilde{\tilde{\varepsilon}}$, and $\tilde{a}_0^*$, this shear strain can be obtained as 0.000288. So, this completes the present problem.

Here, you note both these values of ε_n and γ are much smaller than unity, and thus, the assumption of small strain or linear strain is valid for this particular problem, unlike the previous problem.

So, in this lecture, we discussed the concept of principal strain. We discussed the change of volume and change of surface, and finally, solved a few example problems on the strain tensors.

Thank you.