

APPLIED ELASTICITY

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Week 2

Lecture 07: Kinematics of Deformation II

COURSE ON:
APPLIED ELASTICITY

Lecture 7
KINEMATICS OF DEFORMATION II

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Welcome back to the course on Applied Elasticity. In the last lecture, we started our discussion on the kinematics of deformation. In this lecture as well, we will continue with the discussion on kinematics of deformation.

Deformation Gradient Tensor

The deformation of any material point P is completely characterized by the deformation gradient tensor \tilde{F} as $d\tilde{x} = \tilde{F}d\tilde{X}$.

$$d\tilde{x} = OQ' - OP' = (OQ + QQ') - (OP + PP')$$

$$= \tilde{X} + d\tilde{X} + \tilde{u}(\tilde{X} + d\tilde{X}, t) - \tilde{X} - \tilde{u}(\tilde{X}, t)$$

$$\Rightarrow d\tilde{x} = d\tilde{X} + \tilde{u}(\tilde{X} + d\tilde{X}, t) - \tilde{u}(\tilde{X}, t) \Rightarrow d\tilde{x} = d\tilde{X} + (\tilde{\nabla}\tilde{u})d\tilde{X}$$

Second order displacement gradient tensor

$$\Rightarrow d\tilde{x} = (\tilde{I} + \tilde{\nabla}\tilde{u})d\tilde{X} = \tilde{F}d\tilde{X}$$

$$\Rightarrow d\tilde{x} = \tilde{F}d\tilde{X} \text{ or } d\tilde{X} = \tilde{F}^{-1}d\tilde{x}$$

$$\therefore \tilde{F} = \tilde{I} + \tilde{\nabla}\tilde{u} \text{ where } \tilde{\nabla} \text{ is defined in initial description with respect to } \tilde{X}$$

Initial State: \tilde{X} (coordinates X_1, X_2, X_3)
Current State: \tilde{x} (coordinates x_1, x_2, x_3)

Displacement vectors: $\tilde{u}(\tilde{X}, t)$ and $\tilde{u}(\tilde{X} + d\tilde{X}, t)$

Position vectors: $\tilde{x} = \chi(\tilde{X}, t)$

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Now, first, we are going to describe the deformation gradient tensor. So, we are considering a body in the initial state, where P to Q is a small elementary length dX . So, as discussed in the previous lecture, \tilde{X} denotes the position vector of the material point P in the undeformed or initial state, which upon transformation moves to the deformed or the current state, where \tilde{x} is the position vector of the corresponding point P' . Now, P to Q —this particular small line element deforms to $P'Q'$, which is defined as dx , i.e., the deformed line element. Let us choose the length of the undeformed line element $d\tilde{X}$ to be dS and the length of the deformed line element $d\tilde{x}$ to be ds . The displacement is defined by this quantity, $\tilde{u} = \tilde{x} - \tilde{X}$.

Now, the deformation is completely characterized with the help of a tensor known as the deformation gradient tensor, which is described by \tilde{F} , and through this particular transformation law, $d\tilde{x} = \tilde{F}d\tilde{X}$ for any small elementary vector $d\tilde{X}$ in the initial state. If we act \tilde{F} , the deformation gradient tensor, on that $d\tilde{S}$, this will result in the small line element in the current state or the deformed state, that is $d\tilde{x}$. From geometry, $d\tilde{x}$ can be written as $OQ' - OP'$. You can see P' to Q' is $d\tilde{x}$.

So, $OQ' - OP'$ defines $d\tilde{x}$. Now, OQ' can be written as $OQ + QQ'$, where OP' can be written as $OP + PP'$. Now, OQ , this is nothing but $\tilde{X} + d\tilde{X}$, $OP = \tilde{X}$, $OQ' = \tilde{x} + d\tilde{x}$, $OP' = \tilde{x}$. Substituting all these in this equation and cancelling this \tilde{X} , this would become $d\tilde{x} = d\tilde{X} + \tilde{u}(\tilde{X} + d\tilde{X}, t) - \tilde{u}(\tilde{X}, t)$.

Now, if you try to recall the definition of the second order displacement gradient tensor, the gradient of any vector \tilde{u} can be defined as: $\tilde{\nabla}u$, which is a second order tensor, acting over this $d\tilde{X}$, and $\tilde{\nabla}u = \tilde{u}(\tilde{X} + d\tilde{X}, t) - \tilde{u}(\tilde{X}, t)$. So, by definition of the gradient of a vector, which is a second order tensor, which we are naming here as displacement gradient tensor, with the help of that, this expression can be written as $d\tilde{x} = d\tilde{X} + \tilde{\nabla}u d\tilde{X}$. Now with the help of this, we are going to define the deformation gradient tensor.

Now, this first term, $d\tilde{X}$, this can be written as identity tensor $\tilde{I}d\tilde{X}$ by the property of the identity tensor, and then, combining both the terms, $d\tilde{x} = (\tilde{I} + \tilde{\nabla}u)d\tilde{X}$. And this is the

deformation gradient tensor $\tilde{\tilde{F}}$ which is $\tilde{\tilde{I}} + \tilde{\nabla} \tilde{u}$. So, $d\tilde{x} = \tilde{\tilde{F}} d\tilde{X}$ or $d\tilde{X} = \tilde{\tilde{F}}^{-1} d\tilde{x}$, where $\tilde{\tilde{F}} = \tilde{\tilde{I}} + \tilde{\nabla} \tilde{u}$ and this gradient operator is defined in the material or initial description.

So, this $(\tilde{\nabla})$ refers to $\frac{\partial(\quad)}{\partial X_k}$; the derivative is with respect to X components in the definition of the deformation gradient tensor in the initial or Lagrangian description. Now, coming to the components of this $\tilde{\tilde{F}}$ tensor. So, $\tilde{\tilde{F}} = \tilde{\tilde{I}} + \tilde{\nabla} \tilde{u}$. So, we can write F_{ij} as a component of $\tilde{\tilde{F}}$, as $F_{ij} = \delta_{ij} + \frac{\partial u_i}{\partial X_j}$, where δ_{ij} is Kronecker delta. As I had already mentioned, this gradient operator refers to the partial derivative with respect to \tilde{X} .

Deformation of a Continuum

Deformation Gradient Tensor:

$$\tilde{F}(\tilde{X}, t) = \frac{\partial \tilde{x}(\tilde{X}, t)}{\partial \tilde{X}} = \tilde{I} + \tilde{\nabla} \tilde{u}$$

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = \delta_{ij} + \frac{\partial u_i}{\partial X_j}$$

The determinant of \tilde{F} is called the Jacobian of the motion, and it is denoted by $J = \det(\tilde{F})$. If $J \neq 0$, then \tilde{F} is invertible.

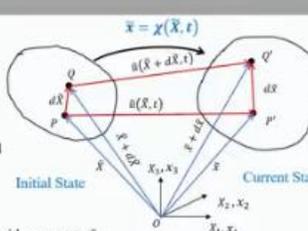
$$d\tilde{X} = (\tilde{I} - \tilde{\nabla}_x \tilde{u}) d\tilde{x} = \tilde{F}^{-1} d\tilde{x}$$

where $\tilde{\nabla}_x$ is defined in spatial description with respect to \tilde{x}

$$F_{ij}^{-1} = \frac{\partial X_i}{\partial x_j} = \delta_{ij} - \frac{\partial u_i}{\partial x_j}$$

Isochoric Deformation:

If $J = \det(\tilde{F}) = 1$, then the deformation is occurring without any change in volume of the continuum. Thus, this refers to isochoric deformation.







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So, F_{ij} , the components of the deformation gradient tensor, is equal to δ_{ij} (Kronecker delta), plus $\frac{\partial u_i}{\partial X_j}$. And this J is called the determinant of $\tilde{\tilde{F}}$, which is called the Jacobian of the motion. The determinant of $\tilde{\tilde{F}}$ is called the Jacobian of the motion, and if $J \neq 0$, then only $\tilde{\tilde{F}}$ is invertible. So, it is possible to define $d\tilde{X} = \tilde{\tilde{F}}^{-1} d\tilde{x}$ only if the $\det(\tilde{\tilde{F}})$ (or J , the Jacobian of the system) is not equal to 0.

Now, in this particular spatial description, when $\tilde{\tilde{F}}$ is written as a function of \tilde{x} , for that case, using a similar approach, it can be shown that $\tilde{\tilde{F}} = \tilde{\tilde{I}} - \tilde{\nabla}_x \tilde{u}$, but here, this gradient is defined in the spatial coordinate or in the spatial description. So, that is why I have added an x in the subscript of the grad operator: this refers to the partial derivative with respect to x components, $\frac{\partial(\quad)}{\partial x_k}$. So, this is the definition of F_{ij}^{-1} , which would be $\delta_{ij} - \frac{\partial u_i}{\partial x_j}$;

this partial derivative in this gradient is with respect to the spatial or deformed coordinate vector.

Now, coming to the definition of isochoric deformation. If we have a deformation for which $\det(\tilde{\tilde{F}})$, or the Jacobian of the system, is equal to 1, then during that deformation, no change in the total volume of the continuum is allowed. So, for such deformations, we call it isochoric deformation because the term isochoric refers to the case of constant volume. No change in the total volume of the continuum is allowed for isochoric deformation, and that can be achieved if $\det(\tilde{\tilde{F}})$, or the Jacobian of the system, is equal to 1.

Now, coming to homogeneous deformation. So, if the deformation gradient tensor is the same at every material point of a continuum, then we call that a homogeneous deformation. For such cases, $\tilde{\tilde{F}}$ is only a function of time; it is not a function of \tilde{X} , the material point coordinate. For all the material coordinates, we have the same $\tilde{\tilde{F}}$, which only changes with time, independent of \tilde{X} ; for such cases, this is called homogeneous deformation.

Deformation of a Continuum

Homogeneous Deformation:
 If \tilde{F} is same at every material point, i.e., independent of \tilde{X} , then it is known as homogeneous deformation.
 $\tilde{F} = \tilde{F}(t) \rightarrow$ Independent of \tilde{X}

Non-Homogeneous Deformation:
 It is a deformation in which all the material points do not deform in an identical manner.
 $\tilde{F} = \tilde{F}(\tilde{X}, t)$

Polar Decomposition Theorem:
 Any real tensor \tilde{F} with non-zero determinant can be decomposed into the product of a proper orthogonal tensor \tilde{Q} and a symmetric positive definite tensor \tilde{U} or \tilde{V} so that $\tilde{F} = \tilde{Q}\tilde{U} = \tilde{V}\tilde{Q}$
 where \tilde{U} and \tilde{V} are referred as right and left stretch tensors respectively.
 Thus, $\tilde{U} = \tilde{Q}^T\tilde{V}\tilde{Q}$ or, $\tilde{V} = \tilde{Q}\tilde{U}\tilde{Q}^T$



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So, the deformation is the same for all the material points, whereas, for the case of non-homogeneous deformation, $\tilde{\tilde{F}}$ is a function of both \tilde{X} , the material point coordinate, as well as time. And in this particular case of non-homogeneous deformation, all the material points do not deform in an identical manner.

Now, coming to a very important theorem called the polar decomposition theorem. This particular theorem states that any real tensor \tilde{F} with a non-zero determinant ($J \neq 0$) can be decomposed, or can be written, as a product of a proper orthogonal tensor \tilde{Q} and a symmetric positive definite tensor, either \tilde{U} or \tilde{V} . Now, what is an orthogonal tensor? That was already defined in the first week. Whereas, what is a symmetric positive definite tensor? A symmetric tensor we had defined as $\tilde{T} = \tilde{T}^T$.

If this identity is satisfied, then \tilde{T} is a symmetric tensor. Now, a tensor is called positive definite if all the eigenvalues of the tensor are positive. So, \tilde{U} or \tilde{V} being two such positive definite tensors and \tilde{Q} being an orthogonal tensor, $\tilde{F} = \tilde{Q}\tilde{U}$ or $\tilde{F} = \tilde{V}\tilde{Q}$, where \tilde{U} and \tilde{V} are known as the right and left stretch tensors, respectively. So, in any deformation, using the polar decomposition theorem, we are decomposing the deformation into two parts:

\tilde{Q} is responsible for the rotation of the body, whereas \tilde{U} or \tilde{V} is responsible for the stretching or change of shape of the body. So, the total deformation can have first stretching followed by rotation, or it can also have first rotation followed by stretching or change in shape. So, that is the physical significance of the polar decomposition theorem. And pre-multiplying this equation, $\tilde{F} = \tilde{Q}\tilde{U} = \tilde{V}\tilde{Q}$, with \tilde{Q}^T on both sides, we can show that $\tilde{U} = \tilde{Q}^T\tilde{V}\tilde{Q}$, or we can also show that $\tilde{V} = \tilde{Q}\tilde{U}\tilde{Q}^T$.

Now, after deformation, we come to the definition of different strain measures. The first strain measure we will discuss is the Cauchy-Green strain measure, known as \tilde{C} , the Cauchy-Green deformation tensor \tilde{C} . So, we consider the same body where the undeformed line element $d\tilde{X}$ is deformed to $d\tilde{x}$ with the help of the deformation gradient tensor \tilde{F} , which defines the mapping from the initial state to the current state.

Strain Measures

Cauchy-Green Deformation Tensors (\tilde{C}):
(Defined in initial configuration)

An infinitesimal line element in the initial and current configurations can be expressed as:

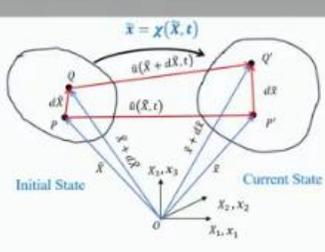
$$dS = PQ$$

$$ds = P'Q'$$

$$\therefore (dS)^2 = d\tilde{X} \cdot d\tilde{X} = dX_i dX_i$$

$$(ds)^2 = d\tilde{x} \cdot d\tilde{x} = \tilde{F}^T d\tilde{X} \cdot \tilde{F} d\tilde{X} = F_{ij} dX_j F_{ik} dX_k = dX_j F_{ji}^T F_{ik} dX_k$$

$$= dX_j (\tilde{F}^T \tilde{F})_{jk} dX_k = d\tilde{X} \cdot (\tilde{F}^T \tilde{F}) d\tilde{X}$$

$$\Rightarrow (ds)^2 = d\tilde{X} \cdot \tilde{C} d\tilde{X} \quad \Rightarrow \tilde{C} = \tilde{F}^T \tilde{F}$$


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So, dS is PQ , the undeformed line element length, whereas ds is the deformed line element length $P'Q'$. Now, as dS is the length of $d\tilde{X}$, we can write $(dS)^2 = d\tilde{X} \cdot d\tilde{X}$. With the help of indicial notation for the dot product of two vectors, this can be written as $dX_i dX_i$. Now, if you also write $(ds)^2$, the square of the length of the line element in the current state, as $d\tilde{x} \cdot d\tilde{x}$. Now, $d\tilde{x}$ is nothing but \tilde{F} acting over $d\tilde{X}$.

So, this quantity is $\tilde{F}^T d\tilde{X} \cdot \tilde{F} d\tilde{X}$. And if you try to write this in indicial notation, the first \tilde{F} acting over $d\tilde{X}$ is written as $F_{ij} dX_j$, and the second one is written as $F_{ik} dX_k$. So, dX_j and dX_k are two vectors, and F_{ij} and F_{ik} are two tensors. Now, we rewrite this $F_{ij} F_{ik}$ as $F_{ji}^T F_{ik}$. Why? So that we can take the product of these two.

The second index of the first tensor is i , and the first index of the second tensor is i . In this fashion, we can write this as the product of $\tilde{F}^T \tilde{F}$. So, $(ds)^2 = dX_j (\tilde{F}^T \tilde{F})_{jk} dX_k$.

This can also be written as $d\tilde{X} \cdot \tilde{F}^T \tilde{F} d\tilde{X}$. And this quantity— $\tilde{F}^T \tilde{F}$ —is defined as \tilde{C} , the Cauchy-Green deformation tensor.

Now, this particular Cauchy-Green deformation tensor is known as the Right Cauchy-Green deformation tensor, which is defined as $\tilde{F}^T \tilde{F}$, and component-wise, $C_{ij} = F_{ik}^T F_{kj}$.

And by definition of \tilde{F} as $\frac{\partial x_k}{\partial X_i} \frac{\partial x_k}{\partial X_j}$, this would be the components of C_{ij} . Now, if you recall, \tilde{F} was also defined as $\tilde{I} + \tilde{\nabla} u$. Now, replacing that expression of \tilde{F} in the definition of \tilde{C} , \tilde{C} is $\tilde{F}^T \tilde{F}$.

Strain Measures

Right Cauchy-Green Deformation Tensor :

$$\tilde{C} = \tilde{F}^T \tilde{F} \quad \Delta C_{ij} = F_{ik}^T F_{kj} = \frac{\partial x_k}{\partial X_i} \frac{\partial x_k}{\partial X_j}$$

\tilde{C} is a second order symmetric tensor

$$\tilde{C} = \tilde{F}^T \tilde{F} = (\tilde{I} + \tilde{\nabla} \tilde{u})^T (\tilde{I} + \tilde{\nabla} \tilde{u}) \quad [\text{as } \tilde{F} = \tilde{I} + \tilde{\nabla} \tilde{u}]$$

$$= \tilde{I} + \tilde{\nabla} \tilde{u} + (\tilde{\nabla} \tilde{u})^T + (\tilde{\nabla} \tilde{u})^T (\tilde{\nabla} \tilde{u})$$

Left Cauchy-Green Deformation Tensor :

Transpose of \tilde{C} is called Left Cauchy-Green deformation tensor and denoted as, $\tilde{B} = \tilde{C}^T = \tilde{F} \tilde{F}^T$

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So, the transpose of $(\tilde{I} + \tilde{\nabla} \tilde{u})$ multiplied by $(\tilde{I} + \tilde{\nabla} \tilde{u})$. Expanding this... So, $(\tilde{I} + \tilde{\nabla} \tilde{u})^T = \tilde{I}^T + (\tilde{\nabla} \tilde{u})^T$, and $\tilde{I}^T = \tilde{I}$. So, if you expand this, we will be getting four terms: $\tilde{I} + \tilde{\nabla} \tilde{u} + (\tilde{\nabla} \tilde{u})^T + (\tilde{\nabla} \tilde{u})^T (\tilde{\nabla} \tilde{u})$.

Now, this is the definition of the right Cauchy-Green deformation tensor. In the same fashion, we can define another Cauchy-Green deformation tensor, which is named the left Cauchy-Green deformation tensor, and denoted by \tilde{B} , which is nothing but \tilde{C}^T and it is $\tilde{F} \tilde{F}^T$. So, the right Cauchy-Green deformation tensor $\tilde{C} = \tilde{F}^T \tilde{F}$, whereas the left Cauchy-Green deformation tensor $\tilde{B} = \tilde{F} \tilde{F}^T$.

Example Problems

(1) For the deformation mapping $x_1 = X_1 + 2X_3, x_2 = X_2 - 2X_3, x_3 = -2X_1 + 2X_2 + X_3$, determine the following

- Displacement components in both material and spatial descriptions.
- Deformation gradient tensor (\tilde{F}), left and right Cauchy Green deformation tensors (\tilde{B} and \tilde{C}).

Answer: $x_1 = X_1 + 2X_3, x_2 = X_2 - 2X_3, x_3 = -2X_1 + 2X_2 + X_3$ $u_i = x_i - X_i$

(a) $u_1 = x_1 - X_1 = 2X_3, u_2 = x_2 - X_2 = -2X_3, u_3 = x_3 - X_3 = -2X_1 + 2X_2$ [u in material description]

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ -2 & 2 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

Inverting the deformation mapping, we get

$$\Rightarrow \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 5 & 4 & -2 \\ 4 & 5 & 2 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow$$

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Now, with this definition of the deformation gradient tensor and the right and left Cauchy-Green deformation tensors, we will try to solve a few example problems. So, let us consider this deformation mapping: $x_1 = X_1 + 2X_3, x_2 = X_2 - 2X_3, x_3 = -2X_1 +$

$2X_2 + X_3$. You need to determine the displacement components in both Eulerian and Lagrangian formulations, and you also need to determine \tilde{F} , \tilde{C} , and \tilde{B} . So, first starting with part 'a': the description of the deformation is given in the material coordinates because x is given as a function of X . The displacement components of u are defined as $u_i = x_i - X_i$.

So, with the help of this definition of displacement components in the given material description, $u_1 = x_1 - X_1 = 2X_3$, and so on. In the same fashion, you can obtain all three deformation components in the material description, where they are functions of X_i . For finding u in the spatial description, we need to invert the equations of the given mapping.

So, $\tilde{x} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ -2 & 2 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}$. This matrix refers to the deformation gradient tensor.

Now, if you invert this matrix, then $\{X_1, X_2, X_3\}$ vector can be written as function of $\{x_1, x_2, x_3\}$ vector. So, this particular, $\frac{1}{9}$ times this matrix, is nothing, but inverse of this particular matrix. So, with the help of this \tilde{F}^{-1} , we can write $\{X_1, X_2, X_3\}$ - the material position vector components - as function of $\{x_1, x_2, x_3\}$. Now, if we try to write u , the displacement components, we had already described u in the material description as function of X_i . Now, we know what these X_i are as function of x_i .

Example Problems

$$\left. \begin{aligned} u_1 &= x_1 - X_1 = 2X_3 = \frac{4}{9}x_1 - \frac{4}{9}x_2 + \frac{2}{9}x_3 \\ u_2 &= x_2 - X_2 = -2X_3 = -\frac{4}{9}x_1 + \frac{4}{9}x_2 - \frac{2}{9}x_3 \\ u_3 &= x_3 - X_3 = -2X_1 + 2X_2 = -\frac{2}{9}x_1 + \frac{2}{9}x_2 + \frac{8}{9}x_3 \end{aligned} \right\} [u \text{ in spatial description}]$$

$$\begin{aligned} X_1 &= \frac{5}{9}x_1 + \frac{4}{9}x_2 - \frac{2}{9}x_3 \\ X_2 &= \frac{4}{9}x_1 + \frac{5}{9}x_2 + \frac{2}{9}x_3 \\ X_3 &= \frac{2}{9}x_1 - \frac{2}{9}x_2 + \frac{1}{9}x_3 \end{aligned}$$

(b) $\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ -2 & 2 & 1 \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \Rightarrow \tilde{F} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ -2 & 2 & 1 \end{bmatrix}$ [Deformation Gradient Tensor]

$dx = \tilde{F} dX$

$[\tilde{C}] = [\tilde{F}]^T [\tilde{F}] = \begin{bmatrix} 5 & -4 & 0 \\ -4 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ [Right Cauchy Green Deformation Tensor]

$[\tilde{B}] = [\tilde{F}] [\tilde{F}]^T = \begin{bmatrix} 5 & -4 & 0 \\ -4 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ [Left Cauchy Green Deformation Tensor]

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So, if you replace $\{X_1, X_2, X_3\}$ as function of x components in the expression of this u and simplify them, you will be getting all three displacement components $\{u_1, u_2, u_3\}$ as function of $\{x_1, x_2, x_3\}$, i.e., in the spatial description.

Now, coming to the second part. As I had already told, $\tilde{\mathbf{x}}$ is equal to this matrix times $\tilde{\mathbf{X}}$ and that matrix is nothing, but the matrix of the deformation gradient tensor, because by definition $d\tilde{\mathbf{x}} = \tilde{\mathbf{F}}d\tilde{\mathbf{X}}$.

So, this particular matrix is nothing, but the matrix of deformation gradient tensor $\tilde{\mathbf{F}}$. Now, you can easily find out $\tilde{\mathbf{C}}$, the right Cauchy-Green deformation tensor, by $\tilde{\mathbf{F}}^T\tilde{\mathbf{F}}$, that will come out to be this, and $\tilde{\mathbf{B}}$, the left Cauchy-Green deformation tensor, by $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^T$ which will come out to be this. And if you compare them for this particular problem, both right Cauchy-Green deformation tensor and left Cauchy-Green deformation tensor comes out to be same because this is a symmetric tensor.

Example Problems

(2) Consider uniform deformation of a square block of sides 2 unit and initially centered at $\tilde{\mathbf{X}} = (0, 0, 0)$. Determine the deformation gradient tensor $\tilde{\mathbf{F}}$, and sketch the deformed square if the deformation mapping is given as $x_1 = 3.5 + X_1 + 0.5X_2, x_2 = 4 + X_2, x_3 = X_3$.

Answer: $x_1 = 3.5 + X_1 + 0.5X_2, x_2 = 4 + X_2, x_3 = X_3$

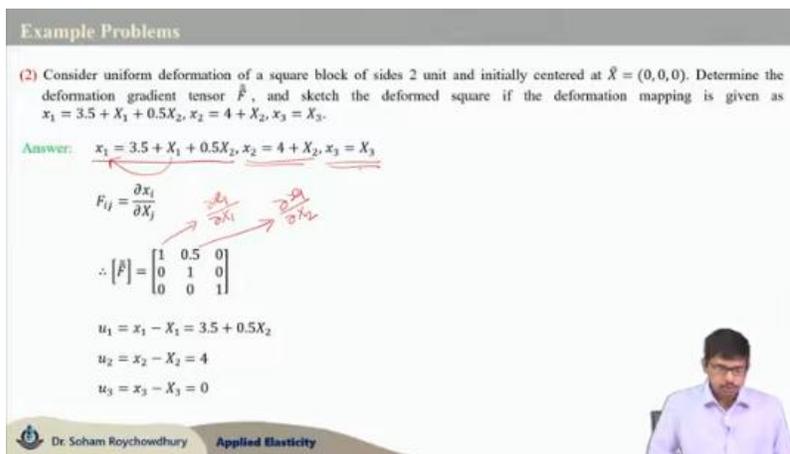
$$F_{ij} = \frac{\partial x_i}{\partial X_j}$$

$$\therefore [\tilde{\mathbf{F}}] = \begin{bmatrix} 1 & 0.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$u_1 = x_1 - X_1 = 3.5 + 0.5X_2$

$u_2 = x_2 - X_2 = 4$

$u_3 = x_3 - X_3 = 0$



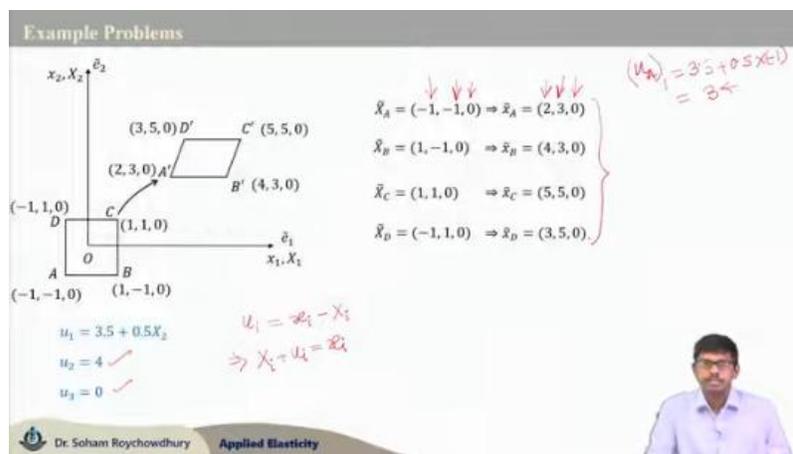
Now, coming to the next example problem, where we are given a square block with sides of 2 units, initially centered at the origin $(0, 0, 0)$. You are asked to determine the deformation gradient tensor $\tilde{\mathbf{F}}$ and sketch the deformed square, if the deformation mapping is given as $x_1 = 3.5 + X_1 + 0.5X_2, x_2 = 4 + X_2$, and $x_3 = X_3$. So, this is the given deformation mapping, which is defined in the material time frame. By definition, the components of the deformation gradient tensor, $F_{ij} = \frac{\partial x_i}{\partial X_j}$.

So, from that, you can easily find out the deformation gradient tensor $\tilde{\mathbf{F}}$ as this. So, the first component of $\tilde{\mathbf{F}}$ is $\frac{\partial x_1}{\partial X_1}$, the second component is $\frac{\partial x_2}{\partial X_2}$, and so on. Like that, you can obtain all the components of this deformation gradient tensor, and that would come out to

be $\begin{bmatrix} 1 & 0.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Now, coming to the displacement fields: $u_1 = x_1 - X_1 = 3.5 + 0.5X_2 -$

this can be obtained from here, $x_1 = 3.5 + X_1 + 0.5X_2$.

If you take X_1 on the left-hand side, that would give us u_1 , the first displacement component, which would be $3.5 + 0.5X_2$. In the same fashion, we can obtain the second displacement component, u_2 , to be a constant 4 from this. And the third displacement component, u_3 , is $x_3 - X_3 = 0$. So, here, only one displacement component is a function of the position vector X_2 ; the other two - u_2 and u_3 - are constants, 4 and 0.



Now, we are first sketching the given square in the undeformed configuration. So, the center was given at $(0, 0, 0)$, i.e., at the origin. A, B, C, D - these form the given square with a side length of 2. So, the coordinates of the four corner points A, B, C, D are $(-1, -1, 0), (1, -1, 0), (1, 1, 0), (-1, 1, 0)$, respectively. Now, we had already defined the displacement components u_1, u_2, u_3 in the material description, and with the help of this, we can obtain the deformed coordinates.

So, u of any point is defined as $u_i = x_i - X_i$. Thus, $X_i + u_i$ will give us the deformed location. We know the undeformed material coordinates for all these four points A, B, C, D , and we also know the displacement vector. So, by adding the displacement vector to the undeformed coordinate X_i , we can get the deformed coordinate x_i . So, for the first point A , the undeformed coordinate is given as $(-1, -1, 0)$, and the deformed coordinate would come out to be $(2, 3, 0)$.

How is it coming? So, if you are considering $\tilde{u}(A)$, the displacement of point A, the first component of u , that is $u_1 = 3.5 + 0.5X_2$. The second position vector for point A is equal to -1 . So, this would be 3. So, -1 was the undeformed coordinate, and the corresponding displacement is 3; if you add both of them, you will get 2.

The second displacement component is 4 for all. So, this second position vector component is $-1 + 4$, which results in 3, and the third displacement component is 0. So, here that remains the same. So, like that, the undeformed coordinate $(-1, -1, 0)$ for point A changes to the deformed location $(2, 3, 0)$. In the same fashion, for point B, the deformed coordinate can be obtained as $(4, 3, 0)$.

For point C, the deformed coordinate can be obtained as $(5, 5, 0)$, and for point D, the deformed coordinate can be obtained as $(3, 5, 0)$. Now, we know the complete deformation mapping and have obtained the location of all four deformed points, let us say A', B', C', D' — they are new names. If we sketch it, that will look something like this. So, A' is $(2, 3, 0)$, B' is $(4, 3, 0)$, C' is $(5, 5, 0)$, and D' is $(3, 5, 0)$. You can see $A'B'$ and $C'D'$ are parallel to the x -axis, whereas $B'C'$ and $A'D'$ —these two sides—will no longer remain parallel to the y -axis.

So, from a square, it is changing to a parallelogram. So, the $ABCD$ square maps to this particular shape $A'B'C'D'$, where the coordinates for all the endpoints, all four corners, are given like this. So, this completes the present problem.

The image shows a presentation slide with a grey header bar containing the text "Example Problems". Below the header, there is a list of three bullet points: "• Deformation Gradient Tensor", "• Cauchy-Green Deformation Tensor", and "• Example Problems on Deformation Gradient Tensor". In the bottom right corner of the slide, there is a small video inset showing a man with glasses and a mustache, wearing a light purple shirt, speaking. At the bottom left of the slide, there is a logo and the text "Dr. Soham Roychowdhury Applied Elasticity".

So, in this particular lecture, we introduced the concept of the deformation gradient tensor, its components, and physical significance, followed by the definition of both left and right Cauchy-Green deformation tensors with the help of the polar decomposition theorem. And we have solved example problems involving this deformation gradient tensor and the Cauchy-Green deformation tensor. Thank you.