APPLIED ELASTICITY

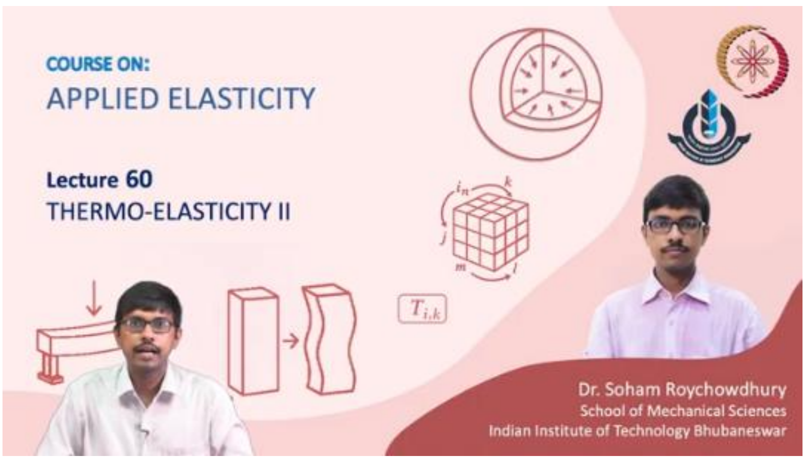
Dr. Soham Roychowdhury

School of Mechanical Sciences

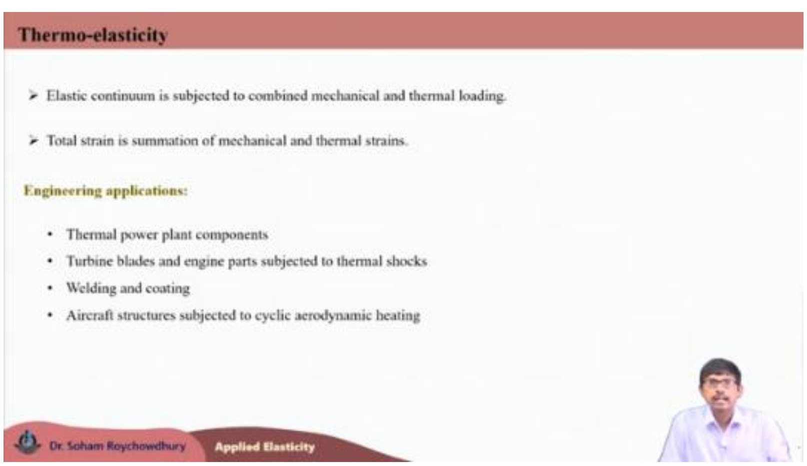
IIT Bhubaneswar

Week 12

Lecture 60: Thermo-elasticity II



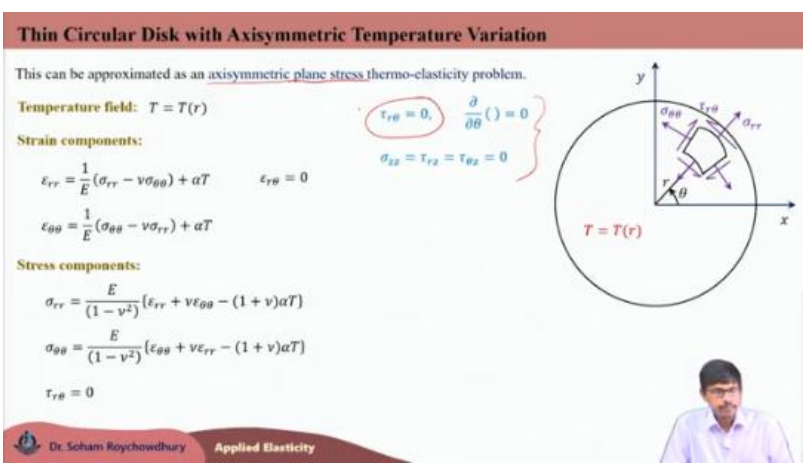
Welcome back to the course on Applied Elasticity. In today's lecture, we are going to continue our discussion on the topic of thermo-elasticity, which we had started in our last lecture.



Thermo-elasticity deals with the solution of the problems of the deformation of the elastic continuum when it is subjected to combined mechanical and thermal loading. Both mechanical and thermal loads are acting on the elastic body, and if we are interested in solving the stress distribution and deformation of such an elastic continuum subjected to combined thermo-mechanical loading, that is called the thermo-elasticity problem. The total strain for a thermo-elasticity problem is the summation of the mechanical strain and the thermal strain components.

The study of thermo-elasticity is important for various engineering applications, such as thermal power plant component analysis, turbine blades, and different I.C. engine parts which are subjected to thermal shocks, aircraft structures being subjected to cyclic aerodynamic heating, and welding and coating for these different applications. The components are subjected to thermal loading as well as mechanical loading. The solution approach of thermo-elasticity is required to be used for solving such problems.

In the last lecture, we had obtained the thermo-elastic constitutive relation, that is, the relation between the stress and strain components for any thermo-elastic problem, and also discussed the basic formulation of planar thermo-elasticity under two types of assumptions: plane stress problem and plane strain problem.



In this lecture, we are going to solve a specific problem, which is a thin circular disc subjected to axisymmetric temperature variation. The thermal loading is applied to a thin circular disc, and we are interested in finding out the stress distribution generated and the resulting displacement of this thin circular disc subjected to axisymmetric temperature variation.

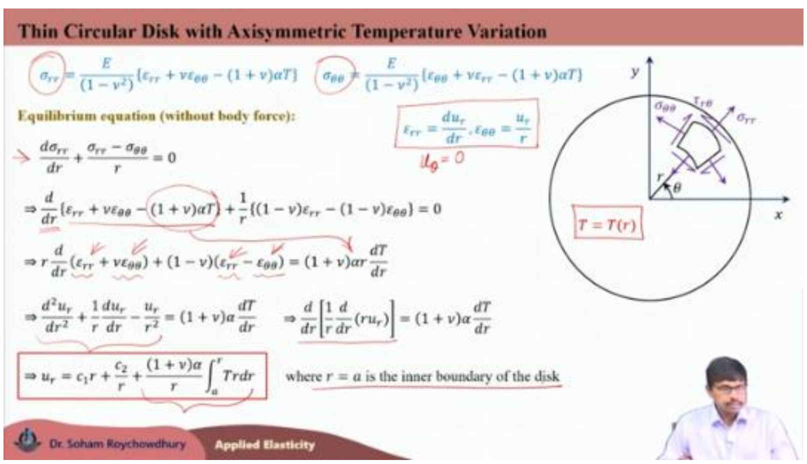
Let us consider this circular disc, which has some small thickness in the z-direction. xy is the in-plane direction. We will be using the polar coordinate, that is, r-θ coordinate, for solving this problem, and as you know, the thickness of the disc is assumed to be small. We can model this as a plane stress problem. Also, as it is subjected to axisymmetric temperature variation and the geometry of the body is also axisymmetric, the disc is axisymmetric. This overall problem can also be treated as an axisymmetric problem. So, this problem is an axisymmetric plane stress thermo-elasticity problem. It is a plane stress problem; it is also an axisymmetric problem which is subjected to the axisymmetric temperature variation.

For the plane stress problem, we have all three out-of-plane stress components, σ_{zz}, τ_{rz}, and $\tau_{\theta z}$, to be 0. These three out-of-plane stress components are 0 from the plane stress assumption. From the axisymmetric assumption, another stress component, $\tau_{r\theta}$, is 0, and all the quantities or field variables, are independent of θ. $\frac{\partial(\ )}{\partial\theta}$, the partial derivative of any quantity with respect to θ, is 0. That is the assumption coming from the axisymmetric problem. $\tau_{r\theta} = 0$, $\frac{\partial(\ )}{\partial\theta}$ of any quantity equals 0, this is the axisymmetric assumption. σ_{zz}, τ_{rz}, $\tau_{\theta z}$ equals 0, this is the plane stress assumption. As this present problem, for the thin circular disc subjected to axisymmetric temperature variation, is an axisymmetric plane stress problem, all these assumptions are valid here.

Now, we are assuming the temperature field to be a function of the radial coordinate, r, independent of the field variable θ, and only then it is an axisymmetric temperature variation. If, in case, you are having T as a function of both r and θ, then, we cannot call this an axisymmetric problem because T would be dependent on θ. As it is axisymmetric, temperature variation T can only be a function of r. So, along the radius, from the center to the outer radius, we are having a variation in the temperature of the thin disc, and we are interested in finding out the stress and deformation of this particular disc.

Strain components for this axisymmetric plane stress problem can be written like this. As it is a plane stress problem, the out-of-plane strain components are not there. We are only going to consider the in-plane strain components. We are interested in the in-plane strain components: the normal strains ε_{rr} and $\varepsilon_{\theta\theta}$, which are written as the summation of mechanical strain and thermal strain as: $\varepsilon_{rr} = \frac{1}{E}(\sigma_{rr} - \nu\sigma_{\theta\theta}) + \alpha T$, $\varepsilon_{\theta\theta} = \frac{1}{E}(\sigma_{\theta\theta} - \nu\sigma_{rr}) + \alpha T$, and $\varepsilon_{r\theta}$ is 0. For axisymmetric problems, $\varepsilon_{r\theta}$ must be 0, as we have $\tau_{r\theta}$, the corresponding in-plane shear stress, to be 0.

Moving to the stress components: in terms of the strain components, we can write the stress component σ_{rr} as $\frac{E}{(1-\nu^2)}[\varepsilon_{rr} + \nu\varepsilon_{\theta\theta} - (1+\nu)\alpha T]$. The other normal stress, $\sigma_{\theta\theta}$, can be written as $\frac{E}{(1-\nu^2)}[\varepsilon_{\theta\theta} + \nu\varepsilon_{rr} - (1+\nu)\alpha T]$, and $\tau_{r\theta}$, the shear stress, is 0.



Moving forward, these stress components are now substituted in the equilibrium equation. As it is a polar coordinate problem, which is axisymmetric in nature, the equilibrium equation in the theta direction is automatically satisfied.

Also, as this is a body with small thickness in the z-direction, thus with the plane stress assumption, the equilibrium equation for the z-direction would also be satisfied automatically. In the absence of any body force, the only equilibrium equation which we need to satisfy under this assumption of an axisymmetric plane stress problem is the r-direction equation of equilibrium, which is $\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} = 0$. Note that here, the stress components, σ_{rr} and $\sigma_{\theta\theta}$, are functions of the single variable r; they are independent of the second variable θ.

Replacing σ_{rr} and $\sigma_{\theta\theta}$, these stress components, in terms of the strain components ε_{rr} and $\varepsilon_{\theta\theta}$, in this equilibrium equation, in the absence of any body force, it would be like this. I am replacing both the normal stress components in terms of normal strain components in the equation of equilibrium. And then, if I simplify this, the first term has a derivative with respect to r, the radial variable. If I take the derivative of all these three terms with respect to r, then these equations can be rewritten like $r\frac{d}{dr}(\varepsilon_{rr} + \nu\varepsilon_{\theta\theta}) + (1-\nu)(\varepsilon_{rr} - \varepsilon_{\theta\theta}) = (1+\nu)\alpha r\frac{dT}{dr}$.

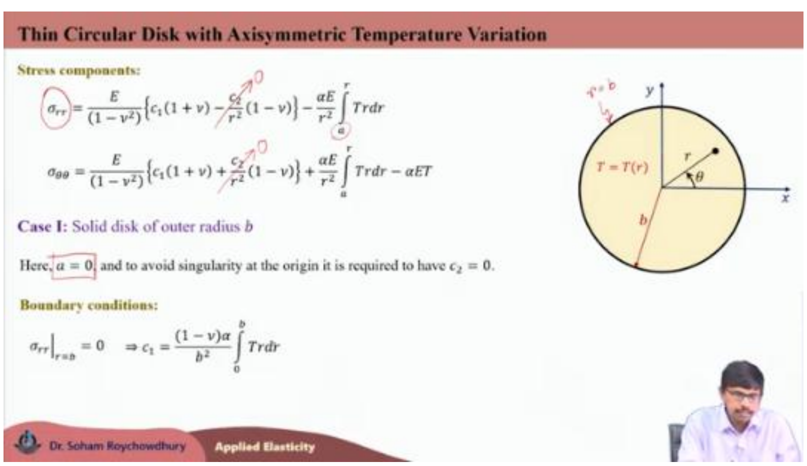
Note that this T is a function of the radial coordinate r; this is not a constant. Once you are taking the derivative of this term with respect to r, the radial variable, we would be treating $1+\nu$ and α as constants, but T is dependent on r; it is a function of r. That is why we have a non-zero $\frac{dT}{dr}$ term coming from this, which we are writing on the right-hand side. This is a temperature field with a variation along the radial coordinate.

Moving forward, these strain components ε_{rr} and $\varepsilon_{\theta\theta}$ in these equations are now written in terms of displacement components. If you recall the axisymmetric formulation discussed earlier, for axisymmetric problems, we have u_θ, the displacement along the angular or tangential direction, must be 0 to maintain the axis symmetry of the deformation. Hence, the only non-zero displacement component is u_r, which is the radial displacement, with respect to which, the strain components can be written as $\varepsilon_{rr} = \frac{du_r}{dr}$ and $\varepsilon_{\theta\theta} = \frac{u_r}{r}$. These are the two normal strain components written in terms of the radial displacement component u_r for the axisymmetric problem.

Replacing ε_{rr} and $\varepsilon_{\theta\theta}$ here as $\frac{du_r}{dr}$ and $\frac{u_r}{r}$, respectively, this equation can be simplified in this form as $\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} = (1+\nu)\alpha\frac{dT}{dr}$. Writing the left-hand side as this particular function of u_r, we rewrite it in this form: $\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru_r)\right]$. That is the left-hand side, and this similar form of governing equation for radial displacement was earlier used for other axisymmetric mechanical problems, such as thick cylinder deformation. Here also, the left-hand side is written in a similar fashion.

The right-hand side is non-zero here. For purely mechanical problems, the $\frac{dT}{dr}$ term is 0, so the right-hand side was 0. As this is a thermo-mechanical problem, we have the right-hand side as $(1+\nu)\alpha\frac{dT}{dr}$. Solving this equation, the radial displacement of the disc can be obtained as $c_1 r + \frac{c_2}{r}$. These are the two terms coming from the homogeneous solution. These are the complementary functions. If the right-hand side were 0, then these two terms, $c_1 r + \frac{c_2}{r}$, would be the solution, as in the purely mechanical plane stress axisymmetric problem.

Along with that, since the right-hand side is non-zero, due to the temperature variation, we have an extra term, the last term, which is the particular integral term in u_r: $\frac{(1+\nu)\alpha}{r}\int_a^r Trdr$. What is this a? a defines the inner boundary of the disc. The integral starts from the inner boundary of the disc and goes up to any radial coordinate value r, giving us u_r, the radial displacement at any value of r. If it is a solid disk, the a value is 0 (the inner limit of $r = 0$). For an annular disk, r starts from some inner radius r_i, which equals a. Thus, this equation gives the displacement field for a thin circular disk subjected to axisymmetric temperature variation.



Substituting this result for u_r into the strain components, we can obtain ε_{rr} and $\varepsilon_{\theta\theta}$ in terms of c_1 and c_2. Then, substituting those back into the stress components, we can obtain the stress components σ_{rr} and $\sigma_{\theta\theta}$ of this problem in terms of these two constants c_1 and c_2. σ_{rr} is obtained as $\frac{E}{(1-\nu^2)}\left[c_1(1+\nu) - \frac{c_2}{r^2}(1-\nu)\right]$. This part was there earlier for mechanical problems. Now, an additional term in the radial normal stress is $-\frac{\alpha E}{r^2}\int_a^r Trdr$. This term is the contribution from the temperature field variation.

Similarly, $\sigma_{\theta\theta}$ would also have two terms: one was the existing term for the mechanical problem, which is $\frac{E}{(1-\nu^2)}\left[c_1(1+\nu) + \frac{c_2}{r^2}(1-\nu)\right]$. And apart from the first term, these two extra terms, second and third, are coming in $\sigma_{\theta\theta}$ due to temperature variation, which are $\frac{\alpha E}{r^2}\int_a^r Trdr - \alpha ET$, where a defines the inner boundary of the disk.

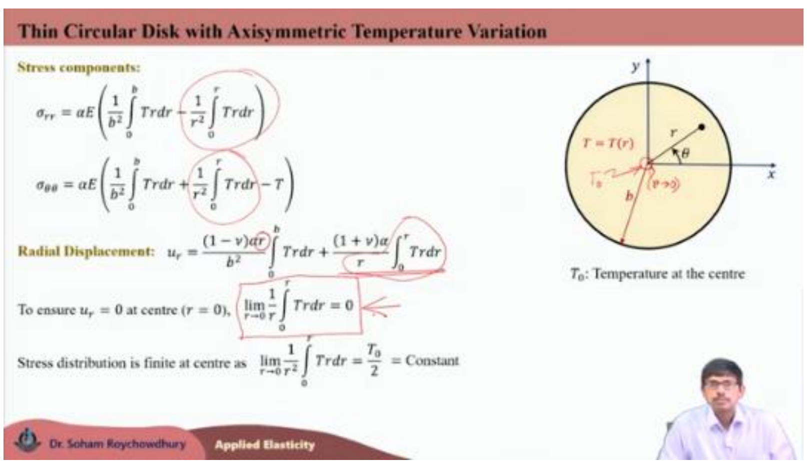
Note that these first two terms, c_1, c_2 being constants, and E and ν being material constants, these first two terms for σ_{rr} can be written as some $A_1 - \frac{B_1}{r^2}$, and for $\sigma_{\theta\theta}$, this is $A_1 + \frac{B_1}{r^2}$, which are the typical forms of the two in-plane normal stresses for any axisymmetric 2D problem. Along with that, we are getting these additional extra terms because of the temperature variation. For σ_{rr}, only one extra T-dependent term is there, whereas for $\sigma_{\theta\theta}$, two such T-dependent terms are there. This is the general formulation for any axisymmetric plane stress problem subjected to temperature variation.

Now, let us consider two cases: one case is a solid disk subjected to temperature variation; another case would be an annular disk subjected to temperature variation. The first case we are considering is a solid disk of outer radius b, which is subjected to the radial temperature field T as a function of r.

For this particular case, the radial variable r is varying between 0 to b. Hence, the lower limit of r being 0, the value of a would be 0. For such cases, if the origin of the domain is included within the domain (the origin of the coordinate system is within the domain). So, $r = 0$ is a point within the domain. To ensure the finite value of stresses or to avoid the singularity at the origin, $r = 0$, point, we must force this constant c_2 to be 0, and with that the infinite stress values at the origin can be avoided. So, one of the constant c_2 is forced to 0 in this stress equation σ_{rr} and $\sigma_{\theta\theta}$. Thus, we would be having only one constant left, that is equal to c_1, which is required to be evaluated. So, c_2 is 0 for the solid disk problem.

For finding c_1, we need to use the surface traction boundary conditions. Here, the problem which we are considering is only subjected to temperature variation; no other mechanical loads, no other pressures are acting. So, this outer boundary at $r = b$ is stress free. This is free of any kind of normal or shear stresses. Using that boundary condition, that is σ_{rr}, the normal radial stress, on outer boundary $r = b$ is 0, we can evaluate this value of c_1.

Substituting σ_{rr} at $r = b$ to be 0, we can evaluate this constant c_1 as $\frac{(1-\nu)\alpha}{b^2}\int_0^b Trdr$. Note that in this integral limit, a is nothing but 0 for the solid disk. We got this expression of c_1 using the mechanical stress traction free boundary condition.



Substituting this c_1 back into the expression of σ_{rr}, $\sigma_{\theta\theta}$, and u_r, we can obtain the stress and displacement components for this particular problem of the solid disk subjected to temperature variation. The stress components would be obtained as follows: $\sigma_{rr} = \alpha E\left(\frac{1}{b^2}\int_0^b Trdr - \frac{1}{r^2}\int_0^r Trdr\right)$, and $\sigma_{\theta\theta} = \alpha E\left(\frac{1}{b^2}\int_0^b Trdr + \frac{1}{r^2}\int_0^r Trdr - T\right)$, where T is the radial temperature variation field. These are the two stress components along the two normal directions, r and θ, for the present problem.

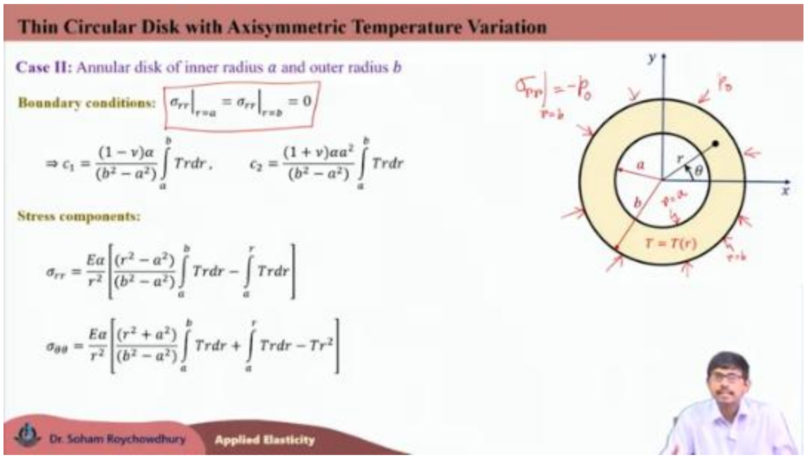
Substituting the obtained expression of c_1 in the radial displacement u_r, with c_2 set to 0, yields u_r, the radial displacement for this solid circular disc, as: $\frac{(1-\nu)\alpha r}{b^2}\int_0^b Trdr + \frac{(1+\nu)\alpha}{r}\int_0^r Trdr$. This is the radial displacement field for this particular solid disk.

Considering the centre at point O, which refers to r tending to 0 or $r = 0$, if we examine the second term of the displacement, we observe that r is in the denominator, and with r tending to 0, this particular term will approach ∞ unless we enforce that the total r-dependent part equals 0 as r tends to 0. Hence, to ensure zero displacement at the centre, this problem being axisymmetric, the centre point must have zero radial displacement.

In u_r, the first term, due to the presence of r in the numerator, will directly go to 0 if $r = 0$, with no issues. However, the second term can only be 0 if we ensure that the limit as r tends to 0 of $\frac{1}{r}\int_0^r Trdr$ equals 0. Thus, the given temperature field $T(r)$ must satisfy this condition to avoid the ∞ or large value of radial displacement at the centre. $\lim_{r\to 0}\frac{1}{r}\int_0^r Trdr$ must be equal to 0 for the given temperature distribution.

However, this problem will not be there in the stress field, even though in the stress field you can see, this second term is having this distribution $\frac{1}{r^2}\int_0^r Trdr$. At $r \to 0$, this $\frac{1}{r^2}$ will also try to go to a higher value. However, for these variations, this $\lim_{r\to 0}\frac{1}{r^2}\int_0^r Trdr$, is a constant value, which is normally equal to $\frac{T_0}{2}$, with T_0 being the temperature at the centre.

So, for all possible radial temperature variations, with T_0 being the temperature at the centre, and then it is varying either linearly, parabolically, exponentially, whatever, that r function can be anything. For all such cases, we can verify that this $\lim_{r\to 0}\frac{1}{r^2}\int_0^r Trdr$ would come out to be $\frac{T_0}{2}$, which is a constant. So, stress distribution is finite at the centre, and to ensure the 0 radial deformation at the centre, we must enforce this condition: $\lim_{r\to 0}\frac{1}{r}\int_0^r Trdr = 0$. This is the discussion about the solid disk.



Moving to the second case of axisymmetric annular disk of inner radius a and outer radius b subjected to axisymmetric temperature variation $T(r)$. This annular disc is considered with inner radius a, outer radius b. Two boundaries are existing: inner boundary is defined with $r = a$, and outer boundary is defined with $r = b$, and body is subjected to temperature variation $T(r)$.

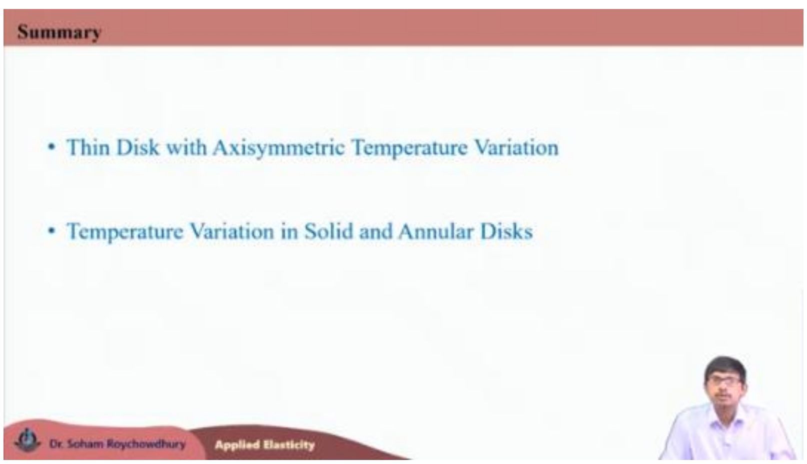
Here, the boundary condition at inner $(r = a)$ and outer $(r = b)$ boundaries is the radial stress σ_{rr} is 0. Using these two boundary conditions in the expression of σ_{rr}, both the constants, c_1 and c_2, can be obtained, and for this case, both of them would be non-zero constants. For the solid disk, c_2 was 0, c_1 was only non-zero, but for annular disk, both c_1 and c_2 would be non-zero constants, which are obtained like this. c_1 is $\frac{(1-\nu)\alpha}{(b^2-a^2)}\int_a^b Trdr$, and c_2 is $\frac{(1+\nu)\alpha a^2}{(b^2-a^2)}\int_a^b Trdr$. So, these two constants are obtained by using the surface traction boundary condition, that is, normal stress equals 0 at the inner and outer boundaries of the annular disc.

Substituting this c_1 and c_2 back in the stress components equation, radial stress σ_{rr} and hoop stress $\sigma_{\theta\theta}$ can be obtained like this, where you can clearly see the effect of the temperature field T is present.

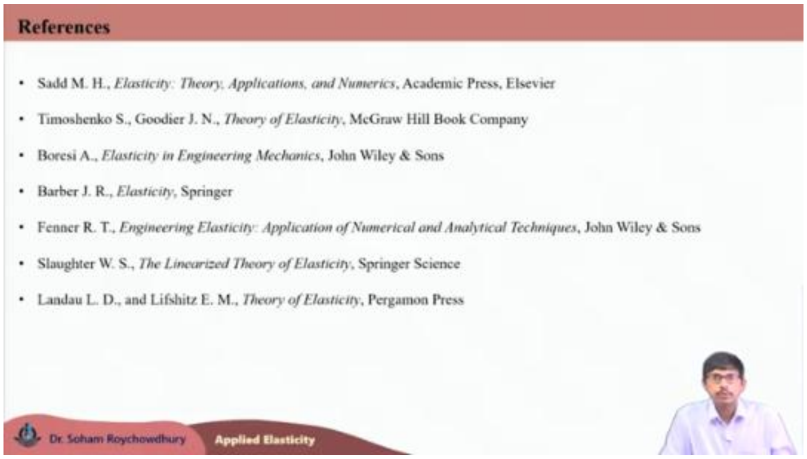
In this problem, for both the solid and annular discs, we are only considering the effect of temperature variation. Along with that, some other mechanical forces or pressures, can also be there. With this, some external pressure P_o is acting on this disc. For that particular case, you need to modify the outer boundary condition: $\sigma_{rr}(r = b)$ would be

$-P_o$, and for that case, $\sigma_{rr}(r = a)$ would be 0. If inner pressure is also there, then $\sigma_{rr}(r = a)$ should be written as $-P_i$. Like that, based on the different mechanical loads acting on the system on this axisymmetric body, you need to change the boundary condition along with the thermal loading, and for such cases, σ_{rr} and $\sigma_{\theta\theta}$ would be having some extra terms, which would be dependent on P_i and P_o, that is, the inner and the outer pressure. Those would actually be a thermo-mechanical problem where both mechanical and thermal loads are acting.

Earlier we had solved the mechanical problems, the axisymmetric problem subjected to the mechanical pressure, inner or outer pressure, for thick cylinders or compound cylinders. Now, we had solved the problem when it is subjected to temperature variation, and these can be easily combined by adding the required boundary condition, σ_{rr} at $r = a, b$, in terms of inner pressure P_i and outer pressure P_o.



In this lecture, we discussed the problem of a thin disc with axisymmetric temperature variation, and the temperature variation effect is considered for both solid disc as well as annular disc.



With this, we come to the end of this course on Applied Elasticity. Here, I present the list of books which you may refer to for further details on this course on Applied Elasticity.

At the end, I want to thank all the contributing team members, both my teaching assistants, Mr. Purusottam Nanda and Mr. Aniket Singh, who helped me in the development of this course content.

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Thank you all for attending this course. If you have any doubts or queries regarding any of the topics, you may contact me via email. Thank you once again.